FUNCTORIAL RELATIONSHIPS BETWEEN $QH^*(G/B)$ AND $QH^*(G/P)$, (II)

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Abstract. We show a canonical injective morphism from the quantum cohomology ring $QH^*(G/P)$ to the associated graded algebra of $QH^*(G/B)$, which is with respect to a nice filtration on $QH^*(G/B)$ introduced by Leung and the author. This tells us the vanishing of a lot of genus zero, three-pointed Gromov-Witten invariants of flag varieties G/P.

1. Introduction

The (small) quantum cohomology ring $QH^*(G/P)$ of a flag variety G/P is a deformation of the ring structure on the classical cohomology $H^*(G/P)$ by incorporating three-pointed, genus zero Gromov-Witten invariants of G/P. Here G denotes a simply-connected complex simple Lie group, and P denotes a parabolic subgroup of G. There has been a lot of intense studies on $QH^*(G/P)$ (see e.g. the survey [8] and references therein). In particular, there was an insight on $QH^*(G/P)$ in the unpublished work [20] of D. Peterson, which, for instance, describes a surprising connection between $QH^*(G/P)$ and the so-called Peterson subvariety. When $P = B$ is the Borel subgroup of G, Lam and Shimozono [15] proved that $QH^*(G/B)$ is isomorphic to the homology of the group of the based loops in a maximal compact Lie subgroup of G with the ring structure given by the Pontryagin product, after equivariant extension and localization (see also [20], [18]). Woodward proved a comparison formula [21] of Peterson that all genus zero, three-pointed Gromov-Witten invariants of G/P are contained in those of G/B. As a consequence, we can define a canonical (injective) map $QH^*(G/P) \hookrightarrow QH^*(G/B)$ as vector spaces. In [16], Leung and the author constructed a natural filtration $\mathcal{F}$ on $QH^*(G/B)$ which comes from a quantum analog of the Leray-Serre spectral sequence for the natural fibration $P/B \to G/B \longrightarrow G/P$. The next theorem is our main result in the present paper, precise descriptions of which will be given in Theorem 2.4.

Main Theorem. *There is a canonical injective morphism of algebras from the quantum cohomology ring $QH^*(G/P)$ to the associated graded algebra of $QH^*(G/B)$ with respect to the filtration $\mathcal{F}$.*

The above statement was proved by Leung and the author under an additional assumption on P/B. Here we do not require any constraint on P/B. That is, we prove Conjecture 5.3 of [16]. Combining the main results therein with the above theorem, we can tell a complete story as follows.

Theorem 1.1. *Let r denote the semisimple rank of the Levi subgroup of P containing a maximal torus $T \subset B$.*

(1) *There exists a* $\mathbb{Z}^{r+1}$*-filtration* $\mathcal{F}$ *on* $QH^*(G/B)$*, respecting the quantum product structure.*
(2) *There exist an ideal* $\mathcal{I}$ *of* $QH^*(G/B)$ *and a canonical algebra isomorphism*

$$QH^*(G/B)/\mathcal{I} \xrightarrow{\simeq} QH^*(P/B).$$

(3) *There exists a subalgebra* $\mathcal{A}$ *of* $QH^*(G/B)$ *together with an ideal* $\mathcal{J}$ *of* $\mathcal{A}$*, such that* $QH^*(G/P)$ *is canonically isomorphic to* $\mathcal{A}/\mathcal{J}$ *as algebras.*
(4) *There exists a canonical injective morphism of graded algebras*

$$\Psi_{r+1}: \quad QH^*(G/P) \hookrightarrow Gr^{\mathcal{F}}_{(r+1)} \subset Gr^{\mathcal{F}}(QH^*(G/B))$$

together with an isomorphism of graded algebras after localization

$$Gr^{\mathcal{F}}(QH^*(G/B)) \cong \big(\bigotimes_{j=1}^{r} QH^*(P_j/P_{j-1})\big) \bigotimes Gr^{\mathcal{F}}_{(r+1)},$$

where P_j*'s are parabolic subgroups constructed in a canonical way, forming a chain* $B := P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_{r-1} \subsetneq P_r = P \subsetneq G$*. Furthermore,* Ψ_{r+1} *is an isomorphism if and only if the next hypothesis* (Hypo1) *holds:* P_j/P_{j-1} *is a projective space for any* $1 \leq j \leq r$*.*

All the relevant ideals, subalgebras and morphisms above will be described precisely in Theorem 4.6. To get a clearer idea of them here, we use the same toy example of the variety of complete flags in $\mathbb{C}^3$ as in [16].

Example 1.2. *Let* $G = SL(3,\mathbb{C})$ *and* $B \subsetneq P \subsetneq G$*. Then we have* $G/B = \{V_1 \leqslant V_2 \leqslant \mathbb{C}^3 \mid \dim_{\mathbb{C}} V_1 = 1, \dim_{\mathbb{C}} V_2 = 2\}$*, and the natural projection* $\pi : G/B \longrightarrow G/P$ *is given by forgetting the vector subspace* V_1 *in the complete flag* $V_1 \leqslant V_2 \leqslant \mathbb{C}^3$*. In particular,* $P/B \cong \mathbb{P}^1$*,* $G/P \cong \mathbb{P}^2$*, and the semisimple rank* r *of the Levi subgroup of* P *containing a maximal torus* $T \subset B$ *equals 1. In this case, the quantum cohomology ring* $QH^*(G/B) = (H^*(G/B) \otimes \mathbb{Q}[q_1, q_2], \star)$ *has a* $\mathbb{Q}$*-basis* $\sigma^w q_1^a q_2^b$*, indexed by* $(w,(a,b)) \in W \times \mathbb{Z}^2_{\geq 0}$*, and we define a grading map* $gr(\sigma^w q_1^a q_2^b) := (2a-b, 3b) + gr(\sigma^w) \in \mathbb{Z}^2$*. Here* $W := \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$ *is the Weyl group (isomorphic to the permutation group* S_3*). The grading* $gr(\sigma^w)$ *is the usual one from the Leray-Serre spectral sequence, respectively given by* $(0,0)$*,* $(1,0)$*,* $(0,1)$*,* $(0,2)$*,* $(1,1)$*,* $(1,2)$*. Using the above gradings together with the lexicographical order on* $\mathbb{Z}^2$ *(i.e.,* $(x_1,x_2) < (y_1,y_2)$ *if and only if either* $x_1 < y_1$ *or* $(x_1 = y_1$ *and* $x_2 < y_2)$*), we have the following conclusions.*

(1) *There is a* $\mathbb{Z}^2$*-filtration* $\mathcal{F} = \{F_{\mathbf{x}}\}_{\mathbf{x}\in\mathbb{Z}^2}$ *on* $QH^*(G/B)$*, defined by* $F_{\mathbf{x}} := \bigoplus_{gr(\sigma^w q_1^a q_2^b)\leq \mathbf{x}} \mathbb{Q}\sigma^w q_1^a q_2^b \subset QH^*(G/B)$*. Furthermore,* $\mathcal{F}$ *respects the quantum product structure. That is,* $F_{\mathbf{x}} \star F_{\mathbf{y}} \subset F_{\mathbf{x}+\mathbf{y}}$*.*
(2) $\mathcal{I} := \bigoplus_{gr(\sigma^w q_1^a q_2^b)\in \mathbb{Z}\times\mathbb{Z}^+} \mathbb{Q}\sigma^w q_1^a q_2^b$ *is an ideal of* $QH^*(G/B)$*. We take the standard ring presentation* $QH^*(\mathbb{P}^1) = \mathbb{Q}[x,q]/\langle x^2 - q\rangle$*. Note* $P/B \cong \mathbb{P}^1$*. Then* $\sigma^{s_1} + \mathcal{I} \mapsto x$ *and* $q_1 + \mathcal{I} \mapsto q$ *define an isomorphism of algebras from* $QH^*(G/B)/\mathcal{I}$ *to* $QH^*(P/B)$*.*
(3) $\mathcal{A} := \sum_{k\in\mathbb{Z}} F_{(0,k)}$ *is a subalgebra of* $QH^*(G/B)$*, and* $\mathcal{J} := F_{(0,-1)}$ *is an ideal of* $\mathcal{A}$*. Write* $QH^*(\mathbb{P}^2) = \mathbb{Q}[z,t]/\langle z^3 - t\rangle$*. Note* $G/P \cong \mathbb{P}^2$*. Then* $z \mapsto \sigma^{s_2} + \mathcal{J}, z^2 \mapsto \sigma^{s_1s_2} + \mathcal{J}$ *and* $t \mapsto \sigma^{s_1}q_2 + \mathcal{J}$ *define an isomorphism of algebras from* $QH^*(G/P)$ *to* $\mathcal{A}/\mathcal{J}$*.*

(4) $Gr^{\mathcal{F}}_{(2)} := \bigoplus_{k\in\mathbb{Z}} F_{(0,k)} \big/ \sum_{\mathbf{x}<(0,k)} F_{\mathbf{x}}$ *is a graded subalgebra of* $Gr^{\mathcal{F}}(QH^*(G/B))$, *and it is canonically isomorphic to* $\mathcal{A}/\mathcal{J}$ *as algebras. Combining it with (3), we have an isomorphism of (graded) algebras* $\pi^*_{\mathbf{q}} : QH^*(G/P) \xrightarrow{\cong} Gr^{\mathcal{F}}_{(2)}$ *(which, in general, is an injective morphism only).*

In addition, by taking the classical limit, $\mathcal{F}|_{\mathbf{q}=\mathbf{0}}$ *gives the usual* $\mathbb{Z}^2$*-filtration on* $H^*(G/B)$ *from the Leray-Serre spectral sequence. The classical limit of* $\pi^*_{\mathbf{q}}$ *also coincides with the induced morphism* $\pi^* : H^*(G/P) \hookrightarrow H^*(G/B)$ *of algebras.*

In the present paper, we will prove Theorem 1.1 in a combinatorial way. It will be very interesting to explore a conceptual explanation of the theorem. Such an explanation may involve the notion of vertical quantum cohomology in [1]. As an evidence, part (2) of Theorem 1.1 turns out to coincide with equation (2.17) of [1] in the special case when $G = SL(n+1, \mathbb{C})$. In a future project, we plan to investigate the relation between our results and those from [1]. We would like to remind that a sufficient condition (Hypo2) for Ψ_{r+1} to be an isomorphism was provided in [16], which says that P/B is isomorphic to a product of complete flag varieties of type A. It is not a strong constraint, satisfied for all flag varieties G/P of type A, G_2 as well as for most of flag varieties G/P of each remaining Lie type. The necessary and sufficient condition in the above theorem is slightly more general. For instance for G of type F_4, there are 16 flag varieties G/P in total (up to isomorphism together with the two extremal cases G/B, {pt} being counted). Among them, there are 13 flag varieties satisfying both hypotheses (Hypo1) and (Hypo2), while one more flag variety satisfies (Hypo1). Precisely, for G of type F_4, (Hypo1) holds for all G/P except for the two (co)adjoint Grassmannians that respectively correspond to (the complement of) the two ending nodes of the Dynkin diagram of type F_4.

The notion of quantum cohomology was introduced by the physicists in 1990s, and it can be defined over a smooth projective variety X. It is a quite challenging problem to study the quantum cohomology ring $QH^*(X)$, partially because of the lack of functorial property. Namely, in general, a reasonable morphism between two smooth projective varieties does not induce a morphism on the level of quantum cohomogy. However, Theorem 1.1 tells us a beautiful story on the "functoriality" among the special case of the quantum cohomology of flag varieties. We may even expect nice applications of it in future research. Despite lots of interesting studies of $QH^*(G/P)$, they are mostly for the varieties of partial flags of subspaces of $\mathbb{C}^{n+1}$, i.e., when $G = SL(n+1, \mathbb{C})$. For G of general Lie type, ring presentations of the quantum cohomology are better understood for either complete flag varieties G/B [14] or most of Grassmannians, i.e., when P is maximal (cf. [5], [6] and references therein). The special case of the functorial property [16] when $P/B \cong \mathbb{P}^1$ has led to nice applications on the "quantum to classical" principle [17], as further applications of which Leung and the author obtained certain quantum Pieri rules [19] as well as alternative proofs of the main results of [4]. On the other hand, our main result could also be treated as a kind of application of the "quantum to classical" principle. As we can see later, the proof requires knowledge on the vanishing of a lot of Gromov-Witten invariants as well as explicit calculations of certain non-vanishing Gromov-Witten invariants that all turn out to be equal to 1. Although Leung and the author have showed an explicit combinatorial formula for those Gromov-Witten invariants (with sign cancelation involved) in [18], it would exceed the capacity of a computer in some cases if we use the formula directly. Instead, we will apply the "quantum to classical" principle developed in [17].

The paper is organized as follows. In section 2, we introduce a (non-recursively defined) grading map gr and state the main results of the present paper. The whole of section 3 is devoted to a proof of Main Theorem when the Dynkin diagram of the Levi subgroup of P containing a maximal torus $T \subset B$ is connected, the outline of which is given at the beginning the section. The proofs of some propositions in section 3 require arguments case by case. Details for all those cases not covered in the section are given in section 5. In section 4, we describe Theorem 1.1 in details and provide a sketch proof of it therein, in which there is no constraint on P/B. We also greatly clarify the grading map defined recursively in [16], by showing the coincidence between it and the map gr defined in section 2. Both the definition of gr and the conjecture of the coincidence between the two grading maps were due to the anonymous referee of [16]. It is quite worth to prove the coincidence, because the grading map was used to establish a nice filtration on $QH^*(G/B)$, which is the heart of the whole story of the functoriality.

2. Main results

2.1. **Notations.** We will follow most of the notations used in [16], which are repeated here for the sake of completeness. Our readers can refer to [12] and [9] for more details.

Let G be a simply-connected complex simple Lie group of rank n, B be a Borel subgroup, $T \subset B$ be a maximal complex torus with Lie algebra $\mathfrak{h} = \mathrm{Lie}(T)$, and $P \supsetneq B$ be a proper parabolic subgroup of G. Let $\Delta = \{\alpha_1, \cdots, \alpha_n\} \subset \mathfrak{h}^*$ be a basis of simple roots and $\{\alpha_1^\vee, \cdots, \alpha_n^\vee\} \subset \mathfrak{h}$ be the simple coroots. Each parabolic subgroup $\tilde{P} \supset B$ is in one-to-one correspondence with a subset $\Delta_{\tilde{P}} \subset \Delta$. Conversely, by $P_{\tilde{\Delta}}$ we mean the parabolic subgroup containing B that corresponds to a given subset $\tilde{\Delta} \subset \Delta$. Here B contains the one-parameter unipotent subgroups U_α, $\alpha \in \tilde{\Delta}$. Clearly, $P_\Delta = G$, $\Delta_B = \emptyset$ and $\Delta_P \subsetneq \Delta$. Let $\{\omega_1, \cdots, \omega_n\}$ (resp. $\{\omega_1^\vee, \cdots, \omega_n^\vee\}$) denote the fundamental (co)weights, and $\langle \cdot, \cdot \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$ denote the natural pairing. Let $\rho := \sum_{i=1}^n \omega_i$.

The Weyl group W is generated by $\{s_{\alpha_i} \mid \alpha_i \in \Delta\}$, where each simple reflection $s_i := s_{\alpha_i}$ maps $\lambda \in \mathfrak{h}$ and $\beta \in \mathfrak{h}^*$ to $s_i(\lambda) = \lambda - \langle \alpha_i, \lambda \rangle \alpha_i^\vee$ and $s_i(\beta) = \beta - \langle \beta, \alpha_i^\vee \rangle \alpha_i$ respectively. Let $\ell : W \to \mathbb{Z}_{\geq 0}$ denote the standard length function. Given a parabolic subgroup $\tilde{P} \supset B$, we denote by $W_{\tilde{P}}$ the subgroup of W generated by $\{s_\alpha \mid \alpha \in \Delta_{\tilde{P}}\}$, in which there is a unique element of maximum length, say $w_{\tilde{P}}$. Given another parabolic subgroup $\bar{P}$ with $B \subset \bar{P} \subset \tilde{P}$, we have $\Delta_{\bar{P}} \subset \Delta_{\tilde{P}}$. Each coset in $W_{\tilde{P}}/W_{\bar{P}}$ has a unique minimal length representative. The set of all these minimal length representatives is denoted by $W_{\tilde{P}}^{\bar{P}} (\subset W_{\tilde{P}} \subset W)$. Note that $W_B = \{\mathrm{id}\}$, $W_{\tilde{P}}^B = W_{\tilde{P}}$ and $W_G = W$. We simply denote $w_0 := w_G$ and $W^{\bar{P}} := W_G^{\bar{P}}$.

The root system is given by $R = W \cdot \Delta = R^+ \sqcup (-R^+)$, where $R^+ = R \cap \bigoplus_{i=1}^n \mathbb{Z}_{\geq 0} \alpha_i$ is the set of positive roots. It is a well-known fact that $\ell(w) = |\mathrm{Inv}(w)|$ where $\mathrm{Inv}(w)$ is the *inversion set* of $w \in W$ given by

$$\mathrm{Inv}(w) := \{\beta \in R^+ \mid w(\beta) \in -R^+\}.$$

Given $\gamma = w(\alpha_i) \in R$, we have the coroot $\gamma^\vee := w(\alpha_i^\vee)$ in the coroot lattice $Q^\vee := \bigoplus_{i=1}^n \mathbb{Z} \alpha_i^\vee$ and the reflection $s_\gamma := w s_i w^{-1} \in W$, which are independent of the expressions of γ. For the given P, we denote by $R_P = R_P^+ \sqcup (-R_P^+)$ the root subsystem, where $R_P^+ = R^+ \cap \bigoplus_{\alpha\in\Delta_P} \mathbb{Z}\alpha$, and denote $Q_P^\vee := \bigoplus_{\alpha_i\in\Delta_P} \mathbb{Z}\alpha_i^\vee$.

The (co)homology of the flag variety G/P has an additive basis of Schubert (co)homology classes σ_u (resp. σ^u) indexed by W^P. In particular, we can identify $H_2(G/P,\mathbb{Z}) = \bigoplus_{\alpha\in\Delta\setminus\Delta_P} \mathbb{Z}\sigma_{s_\alpha}$ with $Q^\vee/Q_P^\vee$ canonically, by mapping $\sum_{\alpha_j\in\Delta\setminus\Delta_P} a_j\sigma_{s_{\alpha_j}}$ to $\lambda_P = \sum_{\alpha_j\in\Delta\setminus\Delta_P} a_j\alpha_j^\vee + Q_P^\vee$. For each $\alpha_j \in \Delta\setminus\Delta_P$, we introduce a formal variable $q_{\alpha_j^\vee+Q_P^\vee}$. For such λ_P, we denote $q_{\lambda_P} = \prod_{\alpha_j\in\Delta\setminus\Delta_P} q_{\alpha_j^\vee+Q_P^\vee}^{a_j}$. The (**small**) **quantum cohomology ring** $QH^*(G/P) = (H^*(G/P)\otimes\mathbb{Q}[\mathbf{q}], \star_P)$ of G/P also has a natural $\mathbb{Q}[\mathbf{q}]$-basis of Schubert classes $\sigma^u = \sigma^u\otimes 1$, for which

$$\sigma^u \star_P \sigma^v = \sum_{w\in W^P,\lambda_P\in Q^\vee/Q_P^\vee} N_{u,v}^{w,\lambda_P} q_{\lambda_P}\sigma^w.$$

The quantum product $\star_P$ is associative and commutative. The *quantum Schubert structure constants* $N_{u,v}^{w,\lambda_P}$ are all *non-negative*, given by genus zero, 3-pointed Gromov-Witten invariants of G/P. When $P = B$, we have $Q_P^\vee = 0$, $W_P = \{1\}$ and $W^P = W$. In this case, we simply denote $\star = \star_P$, $\lambda = \lambda_P$ and $q_j = q_{\alpha_j^\vee}$.

2.2. **Main results.** We will assume the Dynkin diagram $Dyn(\Delta_P)$ to be connected throughout the paper except in section 4. Denote $r := |\Delta_P|$. Note $1 \le r < n$.

Recall that a natural $\mathbb{Q}$-basis of $QH^*(G/B)[q_1^{-1},\cdots,q_n^{-1}]$ is given by $q_\lambda\sigma^w$ labelled by $(w,\lambda)\in W\times Q^\vee$. Note that $q_\lambda\sigma^w \in QH^*(G/B)$ if and only if $q_\lambda\in\mathbb{Q}[\mathbf{q}]$ is a polynomial. In [16], Leung and the author introduced a grading map

$$gr : W\times Q^\vee \longrightarrow \mathbb{Z}^{r+1} = \bigoplus_{i=1}^{r+1}\mathbb{Z}\mathbf{e}_i.$$

Due to Lemma 2.12 of [16], the following subset

$$S := \{gr(w,\lambda) \mid q_\lambda\sigma^w \in QH^*(G/B)\}^1$$

is a totally-ordered sub-semigroup of $\mathbb{Z}^{r+1}$. Here we are using the **lexicographical order** on elements $\mathbf{a} = (a_1,\cdots,a_{r+1}) = \sum_{i=1}^{r+1} a_i\mathbf{e}_i$ in $\mathbb{Z}^{r+1}$. Namely $\mathbf{a} < \mathbf{b}$ if and only if there exists $1\le j\le r+1$ such that $a_j < b_j$ and $a_i = b_i$ for all $1\le i<j$. We can define a family $\mathcal{F} = \{F_\mathbf{a}\}_{\mathbf{a}\in S}$ of subspaces of $QH^*(G/B)$, in which

$$F_\mathbf{a} := \bigoplus_{gr(w,\lambda)\le\mathbf{a}} \mathbb{Q}q_\lambda\sigma^w \subset QH^*(G/B).$$

It is one of the main theorems in [16] that

Proposition 2.1 (Theorem 1.2 of [16])**.** *$QH^*(G/B)$ is an S-filtered algebra with filtration $\mathcal{F}$. Furthermore, this S-filtered algebra structure is naturally extended to a $\mathbb{Z}^{r+1}$-filtered algebra structure on $QH^*(G/B)$.*

As a consequence, we obtain the associated $\mathbb{Z}^{r+1}$-graded algebra

$$Gr^{\mathcal{F}}(QH^*(G/B)) := \bigoplus_{\mathbf{a}\in\mathbb{Z}^{r+1}} Gr_\mathbf{a}^{\mathcal{F}}, \text{ where } Gr_\mathbf{a}^{\mathcal{F}} := F_\mathbf{a}\Big/\sum_{\mathbf{b}<\mathbf{a}} F_\mathbf{b}.$$

In particular, we have a graded subalgebra

$$Gr_{(r+1)}^{\mathcal{F}} := \bigoplus_{i\in\mathbb{Z}} Gr_{i\mathbf{e}_{r+1}}^{\mathcal{F}}.$$

Recall the next Peterson-Woodward comparison formula [21] (see also [15]).

[1] (w,λ) is simply denoted as wq_λ in [16].

Proposition 2.2. *For any $\lambda_P \in Q^\vee/Q_P^\vee$, there exists a unique $\lambda_B \in Q^\vee$ such that $\lambda_P = \lambda_B + Q_P^\vee$ and $\langle\alpha, \lambda_B\rangle \in \{0, -1\}$ for all $\alpha \in R_P^+$. Furthermore for every $u, v, w \in W^P$, we have*

$$N_{u,v}^{w,\lambda_P} = N_{u,v}^{ww_Pw_{P'},\lambda_B},$$

where $\Delta_{P'} = \{\alpha \in \Delta_P \mid \langle\alpha, \lambda_B\rangle = 0\}$.

The above formula, comparing Gromov-Witten invariants for G/P and for G/B, induces an injective map

$$\psi_{\Delta,\Delta_P}: \ QH^*(G/P) \longrightarrow QH^*(G/B);$$
$$\sum a_{w,\lambda_P} q_{\lambda_P} \sigma^w \mapsto \sum a_{w,\lambda_P} q_{\lambda_B} \sigma^{ww_Pw_{P'}},$$

and we call λ_B the *Peterson-Woodward lifting* of λ_P. The next proposition is another one of the main theorems in [16] (see Proposition 3.24 and Theorem 1.4 therein).

Proposition 2.3. *Suppose that Δ_P is of A-type. Then the following map*

$$\Psi_{r+1}: \ QH^*(G/P) \hookrightarrow Gr^{\mathcal{F}}_{(r+1)};$$
$$q_{\lambda_P}\sigma^w \mapsto \overline{\psi_{\Delta,\Delta_P}(q_{\lambda_P}\sigma^w)}$$

is well-defined, and it is an isomorphism of (graded) algebras.

Conjecture 5.3 of [16] tells us the counterpart of the above proposition, and it is the main result of the present paper that such a conjecture does hold. Namely

Theorem 2.4. *Suppose that Δ_P is not of A-type. Then the map Ψ_{r+1} given in Proposition 2.3 is well-defined, and it is an injective morphism of (graded) algebras. Furthermore, Ψ_{r+1} becomes an isomorphism if and only if $r = 2$ together with either case* C1B) *or case* C9) *of Table 1 occurring.*

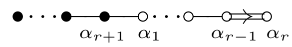
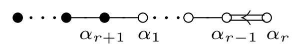
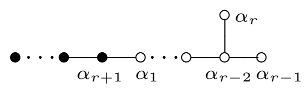
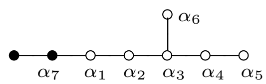
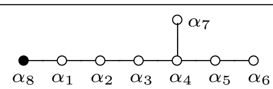
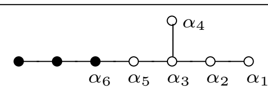
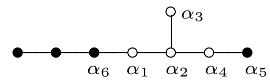
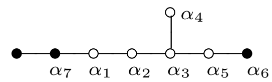
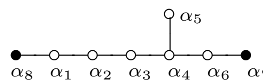
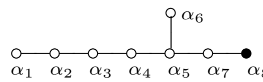
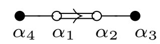
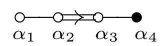
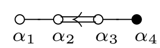
Remark 2.5. *The algebra $QH^*(G/P)$ is equipped with a natural $\mathbb{Z}$-grading: a Schubert class σ^w is of grading $\ell(w)$, and a quantum variable $q_{\alpha^\vee+Q_P^\vee}$ is of grading $\langle\sigma_{s_\alpha}, c_1(G/P)\rangle$. Once we show that Ψ_{r+1} is an morphism of algebras, the way of defining Ψ_{r+1} automatically tells us that it preserves the $\mathbb{Z}$-grading as well.*

We will provide the proof in the next section, one point of which is to compute certain Gromov-Witten invariants explicitly.

In order to define the grading map gr in [16], Leung and the author introduced an ordering on the subset Δ_P first. In our case when Δ_P is not of type A, such an ordering is equivalent to the assumption that $\Delta_P = \{\alpha_1, \cdots, \alpha_r\}$ with all the possible Dynkin diagrams $Dyn(\Delta)$ being listed in Table 1. These are precisely the cases for which Theorem 2.4 is not covered in [16]. In addition, Table 1 has exhausted all the possible cases of fiberations $G/B \to G/P$ such that $Dyn(\Delta_P)$ is connected but not of type A. Therein the cases are basically numbered according to those for $Dyn(\{\alpha_1, \cdots, \alpha_{r-1}\})$ in Table 2 of [16].

Remark 2.6. *In Table 1, we have treated bases of type E_6 and E_7 as subsets of a base of type E_8 canonically. $Dyn(\Delta_P)$ is always given by a unique case in Table 1 except when Δ is of E_6-type together with $r = 5$. In this exceptional case, both* C5) *and* C7) *occur and we can choose either of them. Note $2 \le r < n$. The case of G_2-type does not occur there.*

TABLE 1.

	Dynkin diagram of Δ
C1B)	
C1C)	
C2)	
C4)	
	
C5)	
C7)	
	
	
	
C9)	
	
C10)	

In [16], the grading map gr was defined recursively by using the Peterson-Woodward comparison formula together with the given ordering on Δ_P. Here we will define gr as below, following the suggestion of the referee of [16] (see also Remark 2.10 therein).

Definition 2.7. *Let us choose the ordering of Δ_P as given in Table 1. For each $1 \leq j \leq r$, we denote $\Delta_j := \{\alpha_1, \cdots, \alpha_j\}$. Set $\Delta_0 := \emptyset$ and $\Delta_{r+1} := \Delta$. Denote by $P_i := P_{\Delta_i}$ the parabolic subgroup corresponding to Δ_i for all $0 \leq i \leq r+1$. Recall that we have denoted by $\{\mathbf{e}_1, \cdots, \mathbf{e}_{r+1}\}$ the standard basis of $\mathbb{Z}^{r+1}$. Define a grading map gr by*

$$gr: \; W \times Q^\vee \longrightarrow \mathbb{Z}^{r+1};$$

$$(w, \lambda) \mapsto gr(w, \lambda) = \sum_{i=1}^{r+1} \Big(\big|\mathrm{Inv}(w) \cap (R^+_{P_i} \setminus R^+_{P_{i-1}})\big| + \sum_{\beta \in R^+_{P_i} \setminus R^+_{P_{i-1}}} \langle \beta, \lambda \rangle \Big) \mathbf{e}_i.$$

Say $gr(u, \eta) = \sum_{i=1}^{r+1} a_i \mathbf{e}_i$. Let $1 \leq j \leq k \leq r+1$. As usual, we define

$$|gr(u, \eta)| := \sum_{i=1}^{r+1} a_i, \qquad gr_{[j,k]}(u, \eta) := \sum_{i=j}^{k} a_i \mathbf{e}_i.$$

As a known fact, we have (see also the proof of Proposition 4.3 for detailed explanations)

$$gr(w, 0) = \sum_{j=1}^{r+1} \ell(w_j) \mathbf{e}_j,$$

where $w_j \in W^{P_{j-1}}_{P_j}$ are the unique elements such that $w = w_{r+1} w_r \cdots w_1$.

We will show the next conjecture of the referee of [16].

Theorem 2.8. *The two grading maps by Definition 2.7 above and by Definition 2.8 of* [16] *coincide with each other.*

Because of the coincidence, Proposition 2.1 holds with respect to the grading map gr. Namely for any Schubert classes σ^u, σ^v of $QH^*(G/B)$, if $q_\lambda \sigma^w$ occurs in the quantum multiplication $\sigma^u \star \sigma^v$, then

$$gr(w, \lambda) \le gr(u, 0) + gr(v, 0).$$

The proof of Theorem 2.8 will be given in section 4, which is completely independent of section 3. Due to the coincidence, the proofs of several main results in [16] may be simplified substantially. We can describe the explicit gradings of all the simple coroots as follows, which were obtained by direct calculations using Definition 4.2 of the grading map gr'. (See section 3.5 of [16] for more details on the calculations.)

Proposition 2.9. *Let* $\alpha \in \Delta$. *We simply denote* $gr(\alpha^\vee) := gr(id, \alpha^\vee)$.

(1) $gr(\alpha^\vee) = 2\mathbf{e}_{r+1}$, *if* $Dyn(\{\alpha\} \cup \Delta_P)$ *is disconnected.*
(2) $gr(\alpha^\vee) = (1+j)\mathbf{e}_j + (1-j)\mathbf{e}_{j-1}$, *if* $\alpha = \alpha_j$ *with* $j \le r-1$ *where* $0 \cdot \mathbf{e}_0 := \mathbf{0}$.
(3) $gr(\alpha^\vee)$ *is given in Table 2, if* $\alpha = \alpha_r$ *or* α_{r+1}.

TABLE 2.

	$gr(\alpha_r^\vee)$	$gr(\alpha_{r+1}^\vee)$
C1B)	$2r\mathbf{e}_r - (2r-2)\mathbf{e}_{r-1}$	$(2r+1)\mathbf{e}_{r+1} - r\mathbf{e}_r - \sum_{j=1}^{r-1} \mathbf{e}_j$
C1C)	$(r+1)\mathbf{e}_r - (r-1)\mathbf{e}_{r-1}$	$(2r+2)\mathbf{e}_{r+1} - (r+1)\mathbf{e}_r - \sum_{j=1}^{r-1} \mathbf{e}_j$
C2)	$2(r-1)\mathbf{e}_r + (2-r)(\mathbf{e}_{r-1} + \mathbf{e}_{r-2})$	$2r\mathbf{e}_{r+1} + (1-r)\mathbf{e}_r - \sum_{j=1}^{r-1} \mathbf{e}_j$
C4)	$(3r-7)\mathbf{e}_r + (3-r)\sum_{j=r-3}^{r-1} \mathbf{e}_j$	(for $r=6$) $18\mathbf{e}_7 - 11\mathbf{e}_6 - \sum_{j=1}^{5} \mathbf{e}_j$
		(for $r=7$) $29\mathbf{e}_8 - 21\mathbf{e}_7 - \sum_{j=1}^{6} \mathbf{e}_j$
C5), C7)	$2(r-1)\mathbf{e}_r + (2-r)(\mathbf{e}_{r-1} + \mathbf{e}_{r-2})$	$(\frac{r^2-r}{2} + 2)\mathbf{e}_{r+1} - \frac{r^2-r}{2}\mathbf{e}_r$
C9)	$2r\mathbf{e}_r - (2r-2)\mathbf{e}_{r-1}$	$(r^2+2)\mathbf{e}_{r+1} - r^2\mathbf{e}_r$
C10)	$4\mathbf{e}_3 - 2\mathbf{e}_2$	$8\mathbf{e}_4 - 6\mathbf{e}_3$

(4) *The remaining cases happen when there are two nodes adjacent to* $Dyn(\Delta_P)$, *namely the node* α_{r+1} *and the other node, say* α_{r+2}. *Then we have either of the followings.*
  (a) $gr(\alpha_{r+2}^\vee) = 2r\mathbf{e}_{r+1} + (1-r)\mathbf{e}_r - \sum_{j=1}^{r-1} \mathbf{e}_j$, *which holds if* C7) *occurs and* $r \le 6$;
  (b) $gr(\alpha_{r+2}^\vee) = 5\mathbf{e}_3 - 2\mathbf{e}_2 - \mathbf{e}_1$, *which holds if* C9) *occurs and* $r = 2$.

In particular, we have $|gr(\alpha^\vee)| = 2$ *for any* $\alpha \in \Delta$.

3. Proof of Theorem 2.4

Recall that we have defined a grading map $gr : W \times Q^\vee \to \mathbb{Z}^{r+1}$. For convenience, for any $q_\lambda \sigma^w \in QH^*(G/B)[q_1^{-1}, \cdots, q_n^{-1}]$, we will also use the following notation

$$gr(q_\lambda \sigma^w) := gr(w, \lambda).$$

The injective map $\psi_{\Delta,\Delta_P} : QH^*(G/P) \to QH^*(G/B)$ induces a natural map $QH^*(G/P) \to Gr^{\mathcal{F}}(QH^*(G/B))$. That is, $q_{\lambda_P}\sigma^w \mapsto \overline{\psi_{\Delta,\Delta_P}(q_{\lambda_P}\sigma^w)} \in Gr^{\mathcal{F}}_{\mathbf{a}} \subset Gr^{\mathcal{F}}(QH^*(G/B))$, where $\mathbf{a} = gr(\psi_{\Delta,\Delta_P}(q_{\lambda_P}\sigma^w))$. We state the next proposition, which extends Proposition 3.24 of [16] in the case of parabolic subgroups P such that Δ_P is not of type A.

Proposition 3.1. *For any $q_{\lambda_P}\sigma^w \in QH^*(G/P)$, we have*

$$gr_{[1,r]}(\psi_{\Delta,\Delta_P}(q_{\lambda_P}\sigma^w)) = \mathbf{0}.$$

Hence, $\mathbf{a} \in \mathbb{Z}\mathbf{e}_{r+1}$. That is, the map Ψ_{r+1} as in Proposition 2.3 is well-defined. We can further show

Proposition 3.2. *Ψ_{r+1} is an injective map of vector spaces. Furthermore, Ψ_{r+1} is surjective if and only if $r = 2$ and either case* C1B) *or case* C9) *occurs.*

We shall also show

Proposition 3.3. *Ψ_{r+1} is a morphism of algebras. That is, for any $q_{\lambda_P}, q_{\mu_P}, \sigma^{v'}, \sigma^{v''}$ in $QH^*(G/P)$, we have*

(1) $\Psi_{r+1}(\sigma^{v'} \star_P \sigma^{v''}) = \Psi_{r+1}(\sigma^{v'}) \star \Psi_{r+1}(\sigma^{v''})$;
(2) $\Psi_{r+1}(q_{\lambda_P} \star_P \sigma^{v'}) = \Psi_{r+1}(q_{\lambda_P}) \star \Psi_{r+1}(\sigma^{v'})$;
(3) $\Psi_{r+1}(q_{\lambda_P} \star_P q_{\mu_P}) = \Psi_{r+1}(q_{\lambda_P}) \star \Psi_{r+1}(q_{\mu_P})$.

To achieve the above proposition, we will need to show the vanishing of a lot of Gromov-Witten invariants occurring in certain quantum products in $QH^*(G/B)$, and will need to calculate certain Gromov-Witten invariants, which turn out to be equal to 1.

Clearly, Theorem 2.4 follows immediately from the combination of the above propositions. The rest of this section is devoted to the proofs of these propositions. Here we would like to remind our readers of the following notation conventions:

(a) *Whenever referring to an element λ_P in $Q^\vee/Q^\vee_P$, by λ_B we always mean the unique Peterson-Woodward lifting in $Q^\vee$ defined in Proposition 2.2.* Namely, $\lambda_B \in Q^\vee$ is the unique element that satisfies $\lambda_P = \lambda_B + Q^\vee_P$ and $\langle \alpha, \lambda_B \rangle \in \{0, -1\}$ for all $\alpha \in R^+_P$.
(b) *We simply denote $\tilde{P} := P_{r-1}$.* Namely, we have $\Delta_{\tilde{P}} := \{\alpha_1, \cdots, \alpha_{r-1}\}$.
(c) *Whenever an element in $\lambda \in Q^\vee$ is given first, we always denote $\lambda_P := \lambda + Q^\vee_P \in Q^\vee/Q^\vee_P$ and $\tilde{\lambda}_{\tilde{P}} := \lambda + Q^\vee_{\tilde{P}} \in Q^\vee/Q^\vee_{\tilde{P}}$.* Note that the three elements λ, λ_B and $\tilde{\lambda}_B$ (which is the Peterson-Woodward lifting of $\tilde{\lambda}_{\tilde{P}}$) are all in $Q^\vee$, and they may be distinct with each other in general.

3.1. **Proofs of Proposition 3.1 and Proposition 3.2.** In analogy with [16], we introduce the following notion with respect to the given pair (Δ, Δ_P).

Definition 3.4. *An element $\lambda \in Q^\vee$ is called a* ***virtual null coroot****, if $\langle \alpha, \lambda \rangle = 0$ for all $\alpha \in \Delta_P$. An element μ_P in $Q^\vee/Q^\vee_P$ is called a* ***virtual null coroot****, if its Peterson-Woodward lifting $\mu_B \in Q^\vee$ is a virtual null coroot.*

By the definition of gr, every virtual null coroot λ satisfies

$$gr_{[1,r]}(q_\lambda) = \mathbf{0}.$$

Example 3.5. *Suppose $\alpha \in \Delta$ satisfies that $Dyn(\{\alpha\} \cup \Delta_P)$ is disconnected. Then $\alpha \in \Delta \setminus \Delta_P$, and $\alpha^\vee$ is a virtual null coroot. Furthermore, for $\lambda_P := \alpha^\vee + Q_P^\vee \in Q^\vee/Q_P^\vee$, we have $\lambda_B = \alpha^\vee$. Therefore λ_P is also a virtual null coroot.*

Lemma 3.6. *Given $\lambda_P, \mu_P \in Q^\vee/Q_P^\vee$, we denote $\kappa_P := \lambda_P + \mu_P$. If μ_P is a virtual null coroot, then we have $\kappa_B = \lambda_B + \mu_B$. Consequently,*

$$\mathcal{L} := \{\eta_P \in Q^\vee/Q_P^\vee \mid \eta_P \textit{ is a virtual null coroot}\}$$

is a sublattice of $Q^\vee/Q_P^\vee$.

Proof. Clearly, $\kappa_P = \kappa_B + Q_P^\vee$, and we have $\langle \alpha, \kappa_B \rangle = \langle \alpha, \lambda_B \rangle \in \{0, -1\}$ for all $\alpha \in R_P^+$. Thus the statement follows from the uniqueness of the lifting. □

We will let $\mathcal{L}_B$ denote the set of virtual null coroots in $Q^\vee$:

$$\mathcal{L}_B := \{\lambda \in Q^\vee \mid \langle \alpha, \lambda \rangle = 0, \forall \alpha \in \Delta_P\}.$$

Denote by $\Lambda^\vee$ the set of coweights of G and by $\Lambda_P^\vee$ the set of coweights of the derived subgroup (L, L) of the Levi factor L of P. Denote by $\{\omega_{1,P}^\vee, \cdots, \omega_{r,P}^\vee\}$ the fundament coweights in $\Lambda_P^\vee$ dual to $\{\alpha_1, \cdots, \alpha_r\}$. Denote by $\partial\Delta_P$ the simple roots in $\Delta \setminus \Delta_P$ which are adjacent to Δ_P. The next uniform description of the quotient $\big(Q^\vee/Q_P^\vee\big)/\mathcal{L}$ is provided by the referee.

Proposition 3.7. *The quotient $\big(Q^\vee/Q_P^\vee\big)/\mathcal{L}$ is isomorphic to the subgroup of $\Lambda_P^\vee/Q_P^\vee$ generated by*

$$\{\omega_{i,P}^\vee \mid \alpha_i \textit{ is adjacent to } \partial\Delta_P\}.$$

Proof. Recall that $\Lambda_P^\vee$ is the set of integral valued linear form on $Q_P^\vee$. In particular, there is a natural morphism $R : Q^\vee \to \Lambda_P^\vee$ obtained by restriction: $R(\lambda) = \lambda|_{Q_P^\vee}$. We have $R(Q_P^\vee) = Q_P^\vee$.

Furthermore, this map factors through the quotient $Q^\vee/\mathcal{L}_B$ and the induced map is injective. In particular, the quotient $\big(Q^\vee/Q_P^\vee\big)/\mathcal{L}$ is isomorphic to the image $R(Q^\vee/Q_P^\vee)$. Since for $\alpha \in \Delta$ with $\Delta_P \cup \{\alpha\}$ disconnected we have $\alpha^\vee \in \mathcal{L}_B$, it follows that $R(Q^\vee/Q_P^\vee)$ is the subgroup of $\Lambda_P^\vee/Q_P^\vee$ generated by $R(\alpha^\vee)$ for $\alpha \in \partial\Delta_P$. But $R(\alpha^\vee) = -\omega_{i,P}^\vee$ for α_i adjacent to α and the result follows. □

Remark 3.8. *The group $\Lambda_P^\vee/Q_P^\vee$ is a finite abelian group. It is the center of simply-connected cover of (L, L), and is generated by the cominuscule coweights. One recovers this way the groups $\big(Q^\vee/Q_P^\vee\big)/\mathcal{L}$ in Table 3.*

Recall that each monomial $q_\lambda = q_1^{a_1} \cdots q_n^{a_n}$ corresponds to a coroot $\lambda = \sum_{j=1}^n a_j \alpha_j^\vee$. Given a sequence $I = [i_1, i_2, \cdots, i_m]$, we simply denote by $s_{i_1 i_2 \cdots i_m}$ or s_I the product $s_{i_1} s_{i_2} \cdots s_{i_m}$, and define $|I| := m$.

Proposition 3.9. *The virtual coroot lattice $\mathcal{L}_B$ is generated by the virtual null roots $\mu_B \in Q^\vee$ given in Table 3.*

For each case in Table 4, the corresponding coroot λ satisfies $\langle \alpha_k, \lambda \rangle = -1$ for the given number k in the table, and $\langle \alpha_j, \lambda \rangle = 0$ for all $j \in \{1, \cdots, r\} \setminus \{k\}$.

Furthermore, we have $\psi_{\Delta, \Delta_P}(q_{\lambda_P}) = q_\lambda \sigma^u$ with q_λ and u being shown in Table 4 as well (which implies $\lambda = \lambda_B$). In particular, each u is of the form $s_I s_{r-1} s_{r-2} \cdots s_1$, $s_I s_J$ or s_I where I (resp. J) is a sequence of integers ending with r (resp. $r - 1$)

TABLE 3.

		μ_B	$(Q^\vee/Q_P^\vee)/\mathcal{L}$
C1B)		$2\alpha_{r+1}^\vee + \big(\alpha_r^\vee + 2\sum_{j=1}^{r-1}\alpha_j^\vee\big)$	$\mathbb{Z}/2\mathbb{Z}$
C1C)		$\alpha_{r+1}^\vee + \big(\sum_{j=1}^{r}\alpha_j^\vee\big)$	$\{\mathrm{id}\}$
C2)		$2\alpha_{r+1}^\vee + \big(\alpha_r^\vee + \alpha_{r-1}^\vee + 2\sum_{j=1}^{r-2}\alpha_j^\vee\big)$	$\mathbb{Z}/2\mathbb{Z}$
C4)	$r=6$	$3\alpha_7^\vee + \big(4\alpha_1^\vee + 5\alpha_2^\vee + 6\alpha_3^\vee + 4\alpha_4^\vee + 2\alpha_5^\vee + 3\alpha_6^\vee\big)$	$\mathbb{Z}/3\mathbb{Z}$
	$r=7$	$2\alpha_8^\vee + \big(3\alpha_1^\vee + 4\alpha_2^\vee + 5\alpha_3^\vee + 6\alpha_4^\vee + 4\alpha_5^\vee + 2\alpha_6^\vee + 3\alpha_7^\vee\big)$	$\mathbb{Z}/2\mathbb{Z}$
C5)		$4\alpha_6^\vee + \big(5\alpha_5^\vee + 6\alpha_3^\vee + 4\alpha_2^\vee + 2\alpha_1^\vee + 3\alpha_4^\vee\big)$	$\mathbb{Z}/4\mathbb{Z}$
C7)	$r=4$	$2\alpha_5^\vee + \big(\alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee + 2\alpha_4^\vee\big)$	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$
		$2\alpha_6^\vee + \big(2\alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee\big)$	
	$r=5$	$2\alpha_6^\vee + \alpha_7^\vee + \big(2\alpha_1^\vee + 3\alpha_2^\vee + 4\alpha_3^\vee + 2\alpha_4^\vee + 3\alpha_5^\vee\big)$	$\mathbb{Z}/4\mathbb{Z}$
		$2\alpha_7^\vee + \big(2\alpha_1^\vee + 2\alpha_2^\vee + 2\alpha_3^\vee + \alpha_4^\vee + \alpha_5^\vee\big)$	
	$r=6$	$2\alpha_7^\vee + \big(\alpha_1^\vee + 2\alpha_2^\vee + 3\alpha_3^\vee + 4\alpha_4^\vee + 2\alpha_5^\vee + 3\alpha_6^\vee\big)$	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$
		$2\alpha_8^\vee + \big(2\alpha_1^\vee + 2\alpha_2^\vee + 2\alpha_3^\vee + 2\alpha_4^\vee + \alpha_5^\vee + \alpha_6^\vee\big)$	
	$r=7$	$4\alpha_8^\vee + \big(2\alpha_1^\vee + 4\alpha_2^\vee + 6\alpha_3^\vee + 8\alpha_4^\vee + 10\alpha_5^\vee + 5\alpha_6^\vee + 7\alpha_7^\vee\big)$	$\mathbb{Z}/4\mathbb{Z}$
C9)	$r=2$	$2\alpha_4^\vee + \big(2\alpha_1^\vee + \alpha_2^\vee\big)$	$\mathbb{Z}/2\mathbb{Z}$
		$\alpha_3^\vee + \big(\alpha_1^\vee + \alpha_2^\vee\big)$	
	$r=3$	$2\alpha_4^\vee + \big(2\alpha_1^\vee + 4\alpha_2^\vee + 3\alpha_3^\vee\big)$	
C10)		$2\alpha_4^\vee + \big(\alpha_1^\vee + 2\alpha_2^\vee + 3\alpha_3^\vee\big)$	

in the table. The grading $gr(\sigma^u)$ is then given by $|I|\mathbf{e}_r + \sum_{i=1}^{r-1}\mathbf{e}_i$, $|I|\mathbf{e}_r + |J|\mathbf{e}_{r-1}$ and $|J|\mathbf{e}_r$ respectively.

Proof. Assume that case C1B) occurs, then we have a unique $\mu_B = 2\alpha_{r+1}^\vee + (\alpha_r^\vee + 2\sum_{j=1}^{r-1}\alpha_j^\vee)$ and a unique $\lambda = \alpha_{r+1}^\vee$ in the tables. Clearly, $\langle\alpha, \mu_B\rangle = 0$ for all $\alpha \in \Delta_P$. Thus μ_B is a virtual null coroot, and it is the expected Peterson-Woodward lifting of $\mu_P := \mu_B + Q_P^\vee = 2\alpha_{r+1}^\vee + Q_P^\vee$. Hence, μ_P is a virtual null coroot in $Q^\vee/Q_P^\vee$ by definition. It follows from Example 3.5 that all the elements in the sublattice $\mathcal{L}'$ generated by $\{\mu_P\} \cup \{\alpha \in \Delta \mid Dyn(\{\alpha\} \cup \Delta_P) \text{ is disconnected}\}$ are virtual null coroots. Clearly, $\big(Q^\vee/Q_P^\vee\big)/\mathcal{L}' \cong \mathbb{Z}/2\mathbb{Z}$. Since $\mathcal{L}' \subset \mathcal{L} \subset Q^\vee/Q_P^\vee$, we have a surjective morphism $\big(Q^\vee/Q_P^\vee\big)/\mathcal{L}' \to \big(Q^\vee/Q_P^\vee\big)/\mathcal{L} \cong \mathbb{Z}/2\mathbb{Z}$. Hence, this is an isomorphism, and $\mathcal{L} = \mathcal{L}'$.

It is clear that for $k = 1$, we have $\langle\alpha_k, \lambda\rangle = -1$ and $\langle\alpha_j, \lambda\rangle = 0$ for all $j \in \{1, \cdots, r\} \setminus \{k\}$. Note that Δ_P is of B_r-type, and that any positive root $\gamma \in R_P^+$ is of the from $\varepsilon\alpha_1 + \sum_{j=2}^{r} c_j\alpha_j$ where $\varepsilon \in \{0, 1\}$ (see e.g. [3]). Thus $\langle\gamma, \lambda\rangle \in \{0, -1\}$. Hence, $\lambda = \lambda_B$ is the Peterson-Woodward lifting of λ_P. Consequently, λ_P is not a virtual null coroot, as λ_B is not. That is, our claim holds.

By definition, $\psi_{\Delta,\Delta_P}(q_{\lambda_P}) = q_\lambda\sigma^{w_P w_{P'}}$ where $\Delta_{P'} = \Delta_P \setminus \{\alpha_k\}$ in this case. Note that $w_P w_{P'}$ is the unique element of maximal length in $W_P^{P'}$, whose length is equal to $|R_P^+| - |R_{P'}^+|$. In order to show $u = s_1 s_2 \cdots s_r \cdot s_{r-1}s_{r-2}\cdots s_1$ coincides with $w_P w_{P'}$, it suffices to show: (1) the above expression of u is reduced of expected length; (2) $u \in W_P^{P'}$, i.e., $u(\alpha) \in R^+$ for all $\alpha \in \Delta_{P'}$. Indeed, in the case of C1B),

TABLE 4.

		q_λ	u	k
C1B)		q_{r+1}	$s_{12\cdots r}s_{r-1}s_{r-2}\cdots s_1$	1
C2)		q_{r+1}	$s_{12\cdots(r-2)r}s_{r-1}s_{r-2}\cdots s_1$	1
C4)	$r=6$	q_7	$s_{54362132436}s_5s_4\cdots s_1$	1
		$q_7^2q_1^2q_2^2q_3^2q_4q_6$	$s_{12346325436}s_{12345}$	5
	$r=7$	q_8	$s_{12347543654723451234 7}s_6s_5s_4s_3s_2s_1$	1
C5)		q_6	$s_{4352132435}$	5
		$q_6^2q_5^2q_3^2q_4q_2$	$s_{1235}s_4s_3s_2s_1$	1
		$q_6^3q_5^3q_3^3q_4q_2^2q_1$	$s_{532435}s_{1234}$	4
C7)	$r=4$	q_5	s_{423124}	4
		q_6	$s_{124}s_3s_2s_1$	1
		$q_5q_6q_1q_2q_4$	$s_{324}s_{123}$	3
	$r=5$	q_6	$s_{4352134235}$	5
		q_7	$s_{1235}s_4s_3s_2s_1$	1
		$q_6q_7q_1q_2q_3q_5$	$s_{534235}s_{1234}$	4
	$r=6$	q_7	$s_{645342132643546}$	6
		q_8	$s_{12346}s_5s_4s_3s_2s_1$	1
		$q_7q_8q_1q_2q_3q_4q_6$	$s_{5463243546}s_{12345}$	5
	$r=7$	q_8	$s_{657456345723456123457}$	7
		$q_8^2q_2q_3^2q_4^3q_5^4q_6^2q_7^3$	$s_{123457}s_6s_5s_4s_3s_2s_1$	1
		$q_8^3q_1q_2^2q_3^3q_4^4q_5^5q_6^2q_7^4$	$s_{756457345623457}s_{123456}$	6
C9)	$r=2$	q_4	$s_{12}s_1$	1
	$r=3$	$q_4q_2q_3$	$s_{123}s_2s_1$	1
C10)		q_4	s_{323123}	3

$\Delta_{P'}=\{\alpha_2,\cdots,\alpha_r\}$ is of B_{r-1}-type. For $1\leq j\leq r$, $s_1\cdots s_{j-1}(\alpha_j)=\alpha_1+\cdots+\alpha_j\in R^+$. For $r-1\geq i\geq 1$, $s_1\cdots s_rs_{r-1}\cdots s_{i+1}(\alpha_i)=\alpha_1+\cdots+\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_r\in R^+$. Thus the expression of u is reduced, and $\ell(u)=r+r-1=r^2-(r-1)^2=|R_P^+|-|R_{P'}^+|$. For all $2\leq j\leq r$, we note $u(\alpha_j)=\alpha_j\in R^+$. Therefore both (1) and (2) hold.

The expression $u=s_Is_{r-1}s_{r-2}\cdots s_1$, where $I=[1,2,\cdots,r]$, is reduced. Thus the subexpression $s_I=s_1\cdots s_r$ is also reduced. Clearly, $s_j\in W_{P_j}^{P_{j-1}}$, and $s_I(\alpha_j)\in R^+$ for all $\alpha_j\in\Delta_{P_{r-1}}=\{\alpha_1,\cdots,\alpha_{r-1}\}$, which implies $s_I\in W_P^{P_{r-1}}$. Hence, $gr(\sigma^u)=\ell(s_I)\mathbf{e}_r+\sum_{i=1}^{r-1}\ell(s_i)\mathbf{e}_i=|I|\mathbf{e}_r+\sum_{i=1}^{r-1}\mathbf{e}_i$.

The arguments for the remaining cases are all the same. □

Remark 3.10. *We obtain both tables using case by case analysis, which gives an alternative proof of Proposition 3.7 by studying the quotient* $(Q^\vee/Q_P^\vee)/\mathcal{L}'$ *first.*

Lemma 3.11. *Let $\lambda_P, \mu_P \in Q^\vee/Q_P^\vee$. Write $\psi_{\Delta,\Delta_P}(q_{\lambda_P}) = q_{\lambda_B}\sigma^u$. If μ_P is a virtual null coroot, then we have*

$$\psi_{\Delta,\Delta_P}(q_{\mu_P}) = q_{\mu_B} \quad \textit{and} \quad \psi_{\Delta,\Delta_P}(q_{\lambda_P+\mu_P}) = q_{\lambda_B+\mu_B}\sigma^u.$$

Consequently, we have

$$gr_{[1,r]}(\psi_{\Delta,\Delta_P}(q_{\mu_P})) = \mathbf{0} \quad \textit{and} \quad gr_{[1,r]}(\psi_{\Delta,\Delta_P}(q_{\lambda_P+\mu_P})) = gr_{[1,r]}(\psi_{\Delta,\Delta_P}(q_{\lambda_P})).$$

Proof. For $\kappa_P \in Q^\vee/Q_P^\vee$, by definition we have $\psi_{\Delta,\Delta_P}(q_{\kappa_P}) = q_{\kappa_B}\sigma^{w_P w_{P'}}$ with $\Delta_{P'} = \{\alpha \in \Delta_P \mid \langle \alpha, \kappa_B\rangle = 0\}$. If $\kappa_P = \mu_P$, then $\Delta_{P'} = \Delta_P$ since μ_P is a virtual null coroot. Thus $w_P w_{P'} = \mathrm{id}$ and consequently $\psi_{\Delta,\Delta_P}(q_{\mu_P}) = q_{\mu_B}$. If $\kappa_P = \lambda_P + \mu_P$, then $\kappa_B = \lambda_B + \mu_B$. Write $\Delta_{P'} = \{\alpha \in \Delta_P \mid \langle \alpha, \kappa_B\rangle = 0\} = \{\alpha \in \Delta_P \mid \langle \alpha, \lambda_B\rangle = 0\}$. That is, we have $u = w_P w_{P'}$ and $\psi_{\Delta,\Delta_P}(q_{\kappa_P}) = q_{\kappa_B}\sigma^u$. The two identities on the gradings are then a direct consequence. □

Proof of Proposition 3.1. The initial proof used case by case analysis with Table 4. Here we provide a uniform proof from the referee.

To prove the statement using Lemma 3.11, we only need to prove that

$$gr_{[1,r]}(\psi_{\Delta,\Delta_P}(q_{\alpha^\vee+Q_P^\vee})) = \mathbf{0}$$

for $\alpha \in \partial\Delta_P$.

Let $\alpha \in \partial\Delta_P$ and let α_i be the unique element in Δ_P adjacent to α. We have $\langle \alpha_i, \alpha^\vee\rangle \neq 0$. By definition we have $gr_{[1,r]}(\psi_{\Delta,\Delta_P}(q_{\alpha^\vee+Q_P^\vee})) = gr_{[1,r]}(w_P^{P'}, \alpha^\vee)$ and

$$gr_{[1,r]}(w_P^{P'}, \alpha^\vee) = \sum_{j=1}^{r}\Big(|\mathrm{Inv}(w_P^{P'}) \cap (R_j^+ \setminus R_{j-1}^+)| + \sum_{\beta\in R_j^+\setminus R_{j-1}^+}\langle \beta, \alpha^\vee\rangle\Big)\mathbf{e}_j.$$

We first remark that the above grading does only depend on the restriction of $\alpha^\vee$ to Δ_P so on $R(\alpha^\vee) = -\omega_{i,P}^\vee$ as defined in the proof of Proposition 3.7. For $w \in W_P$ and $\lambda \in \Lambda_P^\vee$ we define

$$gr_{[1,r]}(w, \lambda) = \sum_{j=1}^{r}\Big(|\mathrm{Inv}(w_P^{P'}) \cap (R_j^+ \setminus R_{j-1}^+)| + \sum_{\beta\in R_j^+\setminus R_{j-1}^+}\langle \beta, \lambda\rangle\Big)\mathbf{e}_j.$$

For $\mathbf{a} = \sum_{j=1}^r a_j\mathbf{e}_j$, define

$$\|\mathbf{a}\| := \sum_{j=1}^{r}|a_j|.$$

Next remak (see Corollary 3.13 of [7]) that for $w \in W_P$ and $\lambda \in \Lambda_P^\vee$, we have

$$\ell(wt_\lambda) = \|gr_{[1,r]}(w, \lambda)\|$$

where ℓ denotes the length function on $\widetilde{W}_{\mathrm{aff}}$ the extended affine Weyl group and where we consider the element wt_λ as an element of the extended affine Weyl group $\widetilde{W}_{\mathrm{aff}}$ (see Definition 3.9 of [7]).

Now for P' defined by $\Delta_{P'} = \{\beta \in \Delta_P \mid \langle \beta, \omega_{i,P}^\vee\rangle = 0\}$, the element

$$\tau_i := w_P^{P'} t_{-\omega_{i,P}^\vee}$$

is the element τ_i defined on page 9 of [7]. In particular this element satisfies $\ell(\tau_i) = 0$ (since this element is in the stabiliser of the fundamental alcove, see also page 5 of [15]). As a consequence we get

$$\|gr_{[1,r]}(\psi_{\Delta,\Delta_P}(q_{\alpha^\vee+Q_P^\vee}))\| = 0 \quad \text{and} \quad gr_{[1,r]}(\psi_{\Delta,\Delta_P}(q_{\alpha^\vee+Q_P^\vee})) = \mathbf{0}.$$

□

To prove Proposition 3.2, we need the next lemma.

Lemma 3.12 (Lemma 4.1 (1) of [16])**.** *For any* $\mathbf{d} = \sum_{i=1}^{r-1} d_i\mathbf{e}_i \in \mathbb{Z}^{r-1} \times \{(0,0)\} \subset \mathbb{Z}^{r+1}$*, there exists a unique* $(w,\eta) \in W \times Q^\vee$ *such that* $gr(q_\eta\sigma^w) = \mathbf{d}$.

Proof of Proposition 3.2. Since ψ_{Δ,Δ_P} is injective, so is Ψ_{r+1}.

For a nonzero element $\overline{q_\mu\sigma^w} \in Gr^{\mathcal{F}}_{(r+1)}$, we write $w = vu$ where $v \in W^P$ and $u \in W_P$, and write $\mu = \mu' + \mu''$, where $\mu' \in \bigoplus_{i=r+1}^n \mathbb{Z}_{\geq 0}\alpha_i^\vee$ and $\mu'' \in \bigoplus_{i=1}^r \mathbb{Z}_{\geq 0}\alpha_i^\vee$. Note $gr_{[1,r]}(q_{\mu'}\sigma^v) \in \bigoplus_{i=1}^r \mathbb{Z}_{\leq 0}\mathbf{e}_i$ and $\mathbf{0} = gr_{[1,r]}(q_\mu\sigma^w) = gr_{[1,r]}(q_{\mu''}\sigma^u) + gr_{[1,r]}(q_{\mu'}\sigma^v)$. Thus we have $gr_{[1,r]}(q_{\mu''}\sigma^u) \in \bigoplus_{i=1}^r \mathbb{Z}_{\geq 0}\mathbf{e}_i$. Setting $\lambda_P := \mu + Q_P^\vee$, we have $\psi_{\Delta,\Delta_P}(q_{\lambda_P}\sigma^v) = q_{\lambda_B}\sigma^{vw_Pw_{P'}}$ with $\lambda_B = \mu' + \lambda''$ for some $\lambda'' \in \bigoplus_{i=1}^r \mathbb{Z}_{\geq 0}\alpha_i^\vee$. Note $gr_{[1,r]}(q_{\lambda_B}\sigma^{vw_Pw_{P'}}) = gr_{[1,r]}(q_{\lambda''}\sigma^{w_Pw_{P'}}) + gr_{[1,r]}(q_{\mu'}\sigma^v)$, and it is equal to $\mathbf{0}$ by Proposition 3.1. Hence, $\mathbf{d} := gr_{[1,r]}(q_{\lambda''}\sigma^{w_Pw_{P'}}) = gr_{[1,r]}(q_{\mu''}\sigma^u) \in \bigoplus_{i=1}^r \mathbb{Z}_{\geq 0}\mathbf{e}_i$. Thus the map Ψ_{r+1} is surjective as soon as there is a unique element $q_{\mu''}\sigma^u$ of grading $\mathbf{d}$.

Suppose $r = 2$ and either case C1B) or case C9) occurs. Note $gr(q_1) = 2\mathbf{e}_1$, $gr(q_2) = 4\mathbf{e}_2 - 2\mathbf{e}_1$, and $u = u_2u_1$ for a unique $u_1 \in W_{P_1} = \{1, s_1\}$ and $u_2 \in W_P^{P_1} = \{1, s_2, s_1s_2, s_2s_1s_2\}$. Note for given $d_1, d_2 \geq 0$, the next equalities

$$d_1\mathbf{e}_1 + d_2\mathbf{e}_2 = gr_{[1,r]}(q_{a_1\alpha_1^\vee + a_2\alpha_2^\vee}\sigma^u) = a_1 \cdot gr(q_1) + a_2 \cdot gr(q_2) + \ell(u_1)\mathbf{e}_1 + \ell(u_2)\mathbf{e}_2$$

determine a unique $(a_1, a_2, \ell(u_1), \ell(u_2)) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \{0,1\} \times \{0,1,2,3\}$. The pair $(\ell(u_1), \ell(u_2))$ further determines a unique $(u_1, u_2) \in W_{P_1} \times W_P^{P_1}$. Hence, $q_{\mu''}\sigma^u = q_{\lambda''}\sigma^{w_Pw_{P'}}$ follows from the uniqueness.

In order to show Ψ_{r+1} is not surjective for the remaining cases, it suffices to consider the virtual null roots μ_P in Proposition 3.7, for which we note $\Psi_{r+1}(q_{\mu_P}) = \overline{q_{\mu_B}}$. The point is to show $gr_{[r,r]}(q_r) \leq \ell(w_Pw_{P_{r-1}})\mathbf{e}_r$. Once this is done, we show the existence of $q_r^{a_r-1}\sigma^{u_r}$ satisfying $gr_{[r,r]}(q_r^{a_r-1}\sigma^{u_r}) = gr_{[r,r]}(q_{\lambda_B}\sigma^{w_Pw_{P'}})$, where $u_r \in W_P^{P_{r-1}}$, and a_r denotes the power of q_r in the monomial q_{λ_B}. Then we apply Lemma 3.12 to construct an element in $W_{P_{r-1}} \times \bigoplus_{i=1}^{r-1} \mathbb{Z}_{\geq 0}\alpha_i^\vee$ with grading $gr_{[1,r-1]}(q_{\lambda_B}\sigma^{w_Pw_{P'}}) - gr_{[1,r-1]}(q_r^{a_r-1}\sigma^{u_r})$. In this way, we obtain an element of the same grading as $gr(q_{\lambda_B}\sigma^{w_Pw_{P'}})$ that is not in the image of Ψ_{r+1}. Precise arguments are given as follows.

For case C1C), we have $\mu_P = \alpha_{r+1}^\vee + Q_P^\vee$ and $\mu_B = \alpha_1^\vee + \cdots + \alpha_{r+1}^\vee$. Note $gr_{[r,r]}(q_r) = (r+1)\mathbf{e}_r$, and $w_Pw_{P_{r-1}}$ is the longest element in $W_P^{P_{r-1}}$, which is of length $\ell(w_Pw_{P_{r-1}}) = |R_P^+| - |R_{P_{r-1}}^+| = r^2 - \frac{(r-1)r}{2} = \frac{r^2+r}{2} \geq r+1$. Hence, there exists $u_r \in W_P^{P_{r-1}}$ of length $r+1$. Note for each $1 \leq j \leq r-1$, $u_j := s_j \in W_{P_j}^{P_{j-1}}$. Thus $gr(q_{r+1}\sigma^{u_r\cdots u_2s_1}) = (2r+2)\mathbf{e}_{r+1} = gr(q_1\cdots q_{r+1})$. However, $q_{r+1}\sigma^{u_r\cdots u_2u_1} \notin \psi_{\Delta,\Delta_P}(QH^*(G/P))$.

For case C1B) with $r \geq 3$, we have $\mu_P = 2\alpha_{r+1}^\vee + Q_P^\vee$ and $\mu_B = 2\alpha_{r+1}^\vee + \alpha_r^\vee + 2\sum_{j=1}^{r-1}\alpha_j^\vee$. Note $gr_{[r,r]}(q_r) = 2r\mathbf{e}_r$, and $\ell(w_Pw_{P_{r-1}}) = \frac{r(r+1)}{2} \geq 2r$. Hence, there exists $u_r \in W_P^{P_{r-1}}$ of length $2r$. Set $u_j = s_{j-1}s_j$ for $2 \leq j \leq r-1$. Then $gr_{[1,r]}(q_{r+1}^2 q_1 \sigma^{u_r u_{r-1}\cdots u_2}) = \mathbf{0}$ so that $\overline{q_{r+1}^2 q_1 \sigma^{u_r u_{r-1}\cdots u_2}} \in Gr^{\mathcal{F}}_{(r+1)}$. However, $q_{r+1}^2 q_1 \sigma^{u_r u_{r-1}\cdots u_2} \notin \psi_{\Delta,\Delta_P}(QH^*(G/P))$.

The arguments for the remaining cases are also easy and similar. □

3.2. **Proof of Proposition 3.3 (1).** For $v', v'' \in W^P$, we note

$$\Psi_{r+1}(\sigma^{v'}) \star \Psi_{r+1}(\sigma^{v''}) - \Psi_{r+1}(\sigma^{v'} \star_P \sigma^{v''}) = \sum N_{v',v''}^{w,\lambda} \overline{q_\lambda \sigma^w},$$

the summation over those $q_\lambda \sigma^w \in QH^*(G/B)$ satisfying $gr_{[1,r]}(q_\lambda\sigma^w) = \mathbf{0}$ and $q_\lambda \sigma^w \notin \psi_{\Delta,\Delta_P}(QH^*(G/P))$. It suffices to show the vanishing of all the coefficients $N_{v',v''}^{w,\lambda}$ (if any). In particular, it is already done, if Ψ_{r+1} is an isomorphism of vector spaces. Therefore, if $r=2$, then both C1B) and C9) could be excluded in the rest of this subsection.

To do this, we will use the same idea occurring in section 3.5 of [16]. Namely, we consider the fibration $G/B \to G/\tilde{P}$ where $\Delta_{\tilde{P}} = \{\alpha_1, \cdots, \alpha_{r-1}\}$. Set $\varsigma := r-1$, and note that $\Delta_{\tilde{P}}$ is of A-type satisfying the assumption on the ordering as in [16]. Using Definition 2.7 with respect to $(\Delta, \Delta_{\tilde{P}})$, we have a grading map

$$\tilde{gr} : W \times Q^\vee \longrightarrow \mathbb{Z}^{\varsigma+1} = \bigoplus_{i=1}^{r} \mathbb{Z}\mathbf{e}_i \hookrightarrow \mathbb{Z}^{r+1},$$

which satisfies the next obvious property

$$\tilde{gr}_{[1,r-1]} = gr_{[1,r-1]}.$$

Consequently, we obtain a filtration $\tilde{\mathcal{F}}$ on $QH^*(G/B)$ and a (well-defined) induced map $\tilde{\Psi}_{\varsigma+1} : QH^*(G/\tilde{P}) \to Gr^{\tilde{\mathcal{F}}}_{(\varsigma+1)} \subset Gr^{\tilde{\mathcal{F}}}(QH^*(G/B))$ as well. Furthermore, all the results of [16] hold with respect to the fibration $G/B \to G/\tilde{P}$. In particular, we have the next proposition (which follows immediately from Theorem 1.6 of [16]).

Proposition 3.13. *Let $\tilde{u}, \tilde{v} \in W^{\tilde{P}}$ and $\tilde{w} \in W_{\tilde{P}}$. In $Gr^{\tilde{\mathcal{F}}}(QH^*(G/B))$, we have*

$$(1)\ \overline{\sigma^{\tilde{u}}} \star \overline{\sigma^{\tilde{v}}} = \overline{\psi_{\Delta,\Delta_{\tilde{P}}}(\sigma^{\tilde{u}} \star_{\tilde{P}} \sigma^{\tilde{v}})}\ ; \qquad (2)\ \overline{\sigma^{\tilde{u}}} \star \overline{\sigma^{\tilde{w}}} = \overline{\sigma^{\tilde{u}\tilde{w}}}.$$

Lemma 3.14. *For any $u, v \in W^P$, we have in $QH^*(G/B)$ that*

$$\sigma^u \star \sigma^v = \sum N_{u,v}^{w,\lambda} q_\lambda \sigma^w + \sum N_{u,v}^{w',\lambda'} q_{\lambda'} \sigma^{w'} + \sum N_{u,v}^{w'',\lambda''} q_{\lambda''}\sigma^{w''},$$

where the first summation is over those $q_\lambda\sigma^w \in \psi_{\Delta,\Delta_P}(QH^*(G/P))$, the second summation is over those $q_{\lambda'}\sigma^{w'} \in \psi_{\Delta,\Delta_{\tilde{P}}}(QH^*(G/\tilde{P})) \setminus \psi_{\Delta,\Delta_P}(QH^*(G/P))$, and the third summation is over those $q_{\lambda''}\sigma^{w''}$ satisfying $gr_{[1,r-1]}(q_{\lambda''}\sigma^{w''}) < \mathbf{0}$.

Proof. Since $\Delta_{\tilde{P}} \subset \Delta_P$, we have $u, v \in W^{\tilde{P}}$. By Proposition 3.13(1), we have

$$\sigma^u \star \sigma^v = \sum_{q_\lambda\sigma^w \in \psi_{\Delta,\Delta_{\tilde{P}}}(QH^*(G/\tilde{P}))} N_{u,v}^{w,\lambda} q_\lambda \sigma^w + \sum_{\tilde{gr}_{[1,r-1]}(q_{\lambda''}\sigma^{w''})<\mathbf{0}} N_{u,v}^{w'',\lambda''} q_{\lambda''}\sigma^{w''}.$$

If $q_\lambda \sigma^w \in \psi_{\Delta,\Delta_P} QH^*(G/P)$, then $\lambda = \lambda_B$ is the Peterson-Woodward lifting of $\lambda_P := \lambda + Q_P^\vee$ and $w = w_1 w_P w_{P'}$ with w_1 being the minimal length representative of the coset wW_P. Since $R^+_{\tilde{P}} \subset R^+_P$, $\lambda_B = \tilde{\lambda}_B$ is also the lifting of $\tilde{\lambda}_{\tilde{P}} := \lambda + Q^\vee_{\tilde{P}}$. Note that $\Delta_{\tilde{P}'} = \{\alpha \in \Delta_{\tilde{P}} \mid \langle \alpha, \tilde{\lambda}_B\rangle = 0\} \subset \{\alpha \in \Delta_P \mid \langle\alpha, \lambda_B\rangle = 0\} = \Delta_{P'}$. Thus $\text{Inv}(w_P w_{P'}) = R^+_P \setminus R^+_{P'} = \big(R^+_{\tilde{P}} \setminus R^+_{\tilde{P}'}\big) \bigsqcup \big((R^+_P \setminus R^+_{\tilde{P}}) \setminus (R_{P'} \setminus R^+_{\tilde{P}'})\big)$. Hence, we have $w_P w_{P'} = w_2 w_{\tilde{P}} w_{\tilde{P}'}$ where w_2 is the minimal length representative of the coset $w_Pw_{P'}W_{\tilde P}$ (for which we have $\mathrm{Inv}(w_2) = (R^+_P \setminus R^+_{\tilde P}) \setminus (R_{P'} \setminus R^+_{\tilde P'})$). Note $w_1w_2 \in W^{\tilde P}$. Thus we have $q_\lambda\sigma^w = \psi_{\Delta,\Delta_{\tilde P}}(q_{\tilde\lambda_{\tilde P}}\sigma^{w_1w_2}) \in \psi_{\Delta,\Delta_{\tilde P}}(QH^*(G/\tilde P))$. Therefore the statement follows by noting $\tilde{gr}_{[1,r-1]} = gr_{[1,r-1]}$. □

Due to the above lemma, it remains to show that for any element $q_{\lambda'}\sigma^{w'}$ in $\psi_{\Delta,\Delta_{\tilde P}}(QH^*(G/P_{r-1})) \setminus \psi_{\Delta,\Delta_P}(QH^*(G/P))$, either $N^{w',\lambda'}_{u,v} = 0$ or $gr(q_{\lambda'}\sigma^{w'}) < 0$ holds. The latter claim could be further simplified as $gr_{[r,r]}(q_{\lambda'}\sigma^{w'}) < \mathbf{0}$, by noting $gr_{[1,r-1]}(q_{\lambda'}\sigma^{w'}) = \mathbf{0}$. For this purpose, we need the next main result of [17], which is in fact an application of [16] in the special case of $P/B \cong \mathbb{P}^1$. For each $\alpha \in \Delta$, we define a map $\mathrm{sgn}_\alpha : W \to \{0,1\}$ by $\mathrm{sgn}_\alpha(w) := 1$ if $\ell(w) - \ell(ws_\alpha) > 0$, and 0 otherwise.

Proposition 3.15 (Theorem 1.1 of [17])**.** *Given $u, v, w \in W$ and $\lambda \in Q^\vee$, we have*

(1) $N^{w,\lambda}_{u,v} = 0$ *unless* $\mathrm{sgn}_\alpha(w) + \langle\alpha,\lambda\rangle \le \mathrm{sgn}_\alpha(u) + \mathrm{sgn}_\alpha(v)$ *for all* $\alpha \in \Delta$.

(2) *Suppose* $\mathrm{sgn}_\alpha(w) + \langle\alpha,\lambda\rangle = \mathrm{sgn}_\alpha(u) + \mathrm{sgn}_\alpha(v) = 2$ *for some* $\alpha \in \Delta$, *then*

$$N^{w,\lambda}_{u,v} = N^{w,\lambda-\alpha^\vee}_{us_\alpha,vs_\alpha} = \begin{cases} N^{ws_\alpha,\lambda-\alpha^\vee}_{u,vs_\alpha}, & \text{if } \mathrm{sgn}_\alpha(w) = 0 \\ N^{ws_\alpha,\lambda}_{u,vs_\alpha}, & \text{if } \mathrm{sgn}_\alpha(w) = 1 \end{cases}.$$

Corollary 3.16. *Let $u, v \in W^P$. Suppose $N^{w,\lambda}_{u,v} \ne 0$ for some $w \in W$ and $\lambda \in Q^\vee$. Then we have*

(1) $\langle\alpha,\lambda\rangle \le 0$ *for all* $\alpha \in \Delta_P$;

(2) *Set $\lambda_P := \lambda + Q^\vee_P$ and denote by w_1 the minimal length representative of the coset wW_P. If $\lambda = \lambda_B$ and $gr_{[1,r]}(q_\lambda\sigma^w) = \mathbf{0}$, then $q_\lambda\sigma^w = \psi_{\Delta,\Delta_P}(q_{\lambda_P}\sigma^{w_1})$.*

Proof. Assume $\langle\alpha,\lambda\rangle > 0$ for some $\alpha \in \Delta_P$, then we have $\mathrm{sgn}_\alpha(u) + \mathrm{sgn}_\alpha(v) = 0 < \langle\alpha,\lambda\rangle \le \mathrm{sgn}_\alpha(w) + \langle\alpha,\lambda\rangle$. Thus $N^{w,\lambda}_{u,v} = 0$ by Proposition 3.15 (1), contradicting with the hypothesis.

Since $N^{w,\lambda}_{u,v} \ne 0$, we have $\mathrm{sgn}_\alpha(w) = 0$ for any $\alpha \in \Delta_{P'} = \{\beta \in \Delta_P \mid \langle\beta,\lambda_B\rangle = 0\}$, following from Proposition 3.15 (1) again; that is, $w(\alpha) \in R^+$. Thus $w \in W^{P'}$ and consequently $w = w_1w_2$ for a unique $w_2 \in W^{P'}_P$. Since $gr_{[1,r]}(q_\lambda\sigma^{w_Pw_{P'}}) = gr_{[1,r]}(\psi_{\Delta,\Delta_P}(q_{\lambda_P})) = \mathbf{0} = gr_{[1,r]}(q_\lambda\sigma^{w_1w_2}) = gr_{[1,r]}(q_\lambda\sigma^{w_2})$, we have $gr_{[1,r]}(w_Pw_{P'}) = gr_{[1,r]}(w_2)$. Since $w_2, w_Pw_{P'} \in W_P$, $gr_{[r+1,r+1]}(w_Pw_{P'}) = \mathbf{0} = gr_{[r+1,r+1]}(w_2)$. Therefore $\ell(w_2) = |gr(w_2)| = |gr(w_Pw_{P'})| = \ell(w_Pw_{P'})$. Hence, $w_2 = w_Pw_{P'}$ by the uniqueness of elements of maximal length in $W^{P'}_P$. Thus the statement follows. □

Lemma 3.17. *Let $u, v \in W^P$. Suppose $N^{w,\lambda}_{u,v} \ne 0$ for some $w \in W$ and $\lambda \in Q^\vee$. Assume $gr_{[1,r]}(q_\lambda\sigma^w) = \mathbf{0}$ and $\lambda \ne \lambda_B$ where $\lambda_P := \lambda + Q^\vee_P$. Then we have*

$$gr_{[r,r]}(q_\lambda) < (|R^+_{\tilde P} \cup R^+_{\hat P}| - |R^+_P|)\mathbf{e}_r$$

where $\Delta_{\hat P} := \{\alpha \in \Delta_P \mid \langle\alpha,\lambda\rangle = 0\}$.

Proof. Write $\lambda = \sum_{j=1}^n a_j\alpha_j^\vee$, $gr_{[r,r]}(q_r) = x\mathbf{e}_r$ and $gr_{[r,r]}(q_{r+1}) = y\mathbf{e}_r$. Whenever $r+2 \le n$, we denote $gr_{[r,r]}(q_{r+2}) = z\mathbf{e}_r$. Note $gr_{[r,r]}(q_\lambda) = (xa_r + ya_{r+1} + za_{r+2})\mathbf{e}_r$ (where $z = 0$ unless case C7) occurs with $r \le 6$). Let $\varepsilon_j = -\langle\alpha_j,\lambda\rangle$, $j = 1, \cdots, r$.

Note that Proposition 3.3 holds with respect to $QH^*(G/\tilde P)$ and $\tilde{gr}_{[1,\varsigma]}(q_\lambda\sigma^w) = gr_{[1,r-1]}(q_\lambda\sigma^w) = \mathbf{0}$. Hence, λ is the unique Peterson-Woodward lifting of $\lambda + Q^\vee_{\tilde P} \in Q^\vee/Q^\vee_{\tilde P}$ to $Q^\vee$. Thus $\langle\gamma,\lambda\rangle \in \{0,-1\}$ for all $\gamma \in R^+_{\tilde P}$. Consequently, $\varepsilon_j = 0$ for all j

in $\{1, \cdots, r-1\}$ with at most one exception, and if there exists such an exception, say k, then $\varepsilon_k = 1$. Furthermore, we have $\varepsilon_r \geq 0$, by noting $N_{u,v}^{w,\lambda} \neq 0$ and using Corollary 3.16.

Assume case C1B) (resp. case C1C)) occurs, then we have $-2y = x = 2r$ (resp. $-y = x = r+1$) and $z = 0$. In this case, we note $a_{r+1} + \frac{x}{y}a_r = \sum_{j=1}^{r} j\varepsilon_j$ (resp. $\frac{r}{2}\varepsilon_r + \sum_{j=1}^{r-1} j\varepsilon_j$ where $\varepsilon_r = 2a_{r-1} - 2a_r$ is even). If $\varepsilon_r > 0$, then we have $-(ya_{r+1} + xa_r) \geq -yr > \frac{r(r+1)}{2} = r^2 - \frac{(r-1)r}{2} = |R_P^+| - |R_{\tilde{P}}^+| \geq |R_P^+| - |R_{\tilde{P}}^+ \cup R_{\hat{P}}^+|$. If $\varepsilon_r = 0$, then there exists such an exception k with $2 \leq k \leq r-1$ (resp. $1 \leq k \leq r-1$). (For case C1B), each positive root in R_P^+ is of the form $\gamma = \varepsilon\alpha_1 + \sum_{j=2}^{r} b_j\alpha_j$ where $\varepsilon = 0$ or 1. If $k = 1$, it would imply that $\lambda = \lambda_B$ is the Peterson-Woodward lifting of λ_P, contradicting with the hypothesis.) Consequently, we have $|R_P^+| - |R_{\tilde{P}}^+ \cup R_{\hat{P}}^+| = |R_P^+| - |R_{\tilde{P}}^+| - |R^+_{P_{\Delta_P \setminus \{\alpha_k\}}}| + |R^+_{P_{\Delta_{\tilde{P}} \setminus \{\alpha_k\}}}| = r^2 - \frac{(r-1)r}{2} - \big(\frac{(k-1)k}{2} + (r-k)^2\big) + \big(\frac{(k-1)k}{2} + \frac{(r-k-1)(r-k)}{2}\big) = kr - \frac{k(k-1)}{2} < -yk = -(ya_{r+1} + xa_r)$.

Assume case C2) occurs, then we have $-2y = x = 2(r-1)$ and $z = 0$. Note $a_{r+1} + \frac{x}{y}a_r = \frac{r}{2}\varepsilon_r + \frac{r-2}{2}\varepsilon_{r-1} + \sum_{j=1}^{r-2} j\varepsilon_j$, and $\varepsilon_r - \varepsilon_{r-1} = 2a_{r-1} - 2a_r \equiv 0($ mod 2). If $\varepsilon_r > 0$, then $-(ya_{r+1} + xa_r) \geq (r-1)(\frac{r}{2}\varepsilon_r + \frac{r-2}{2}\varepsilon_{r-1} + 0) \geq (r-1)^2 > \frac{r(r-1)}{2} = |R_P^+| - |R_{\tilde{P}}^+| \geq |R_P^+| - |R_{\tilde{P}}^+ \cup R_{\hat{P}}^+|$. If $\varepsilon_r = 0$, then there exists such an exception k with $2 \leq k \leq r-2$ (since $\lambda \neq \lambda_B$). Consequently, we have $|R_P^+| - |R_{\tilde{P}}^+ \cup R_{\hat{P}}^+| = kr - \frac{k(k+1)}{2} < k(r-1) = -(ya_{r+1} + xa_r)$.

For the remaining cases, the arguments are all similar, and the details will be given in section 5.1. □

Proof of Proposition 3.3 (1). Since $QH^*(G/B)$ is an S-filtered algebra, we have $gr_{[1,r]}(q_\lambda\sigma^w) \leq gr_{[1,r]}(\sigma^{v'}) + gr_{[1,r]}(\sigma^{v''}) = \mathbf{0}$ if $N_{v',v''}^{w,\lambda} \neq 0$. Due to Lemma 3.14, it is sufficient to show $gr_{[1,r]}(q_\lambda\sigma^w) < \mathbf{0}$ whenever both $N_{v',v''}^{w,\lambda} \neq 0$ and $q_\lambda\sigma^w \in \psi_{\Delta,\Delta_{\tilde{P}}}(QH^*(G/\tilde{P})) \setminus \psi_{\Delta,\Delta_P}(QH^*(G/P))$ hold. For the latter hypothesis, we only need to check that either of the following holds: (a) $gr_{[1,r]}(q_\lambda\sigma^w) = \mathbf{0}$, $\lambda \neq \lambda_B$; (b) $gr_{[1,r]}(q_\lambda\sigma^w) = \mathbf{0}$, $\lambda = \lambda_B$, $w \neq w_1w_Pw_{P'}$ where w_1 is the minimal length representative of the coset wW_P. If (b) holds, then it is done by Corollary 3.16 (2). Write $w = w_1w_2$ where $w_1 \in W^P$ and $w_2 \in W_P$. By Proposition 3.15 (1), we conclude $w_2(\alpha) \in R^+$ whenever $\alpha \in \Delta_{\hat{P}} = \{\beta \in \Delta_P \mid \langle\beta, \lambda\rangle = 0\}$. Thus $w_2 \in W_P^{\hat{P}}$. Hence, $gr_{[r,r]}(\sigma^{w_2}) = |\mathrm{Inv}(w_2) \cap (R_P^+ \setminus R_{\hat{P}}^+)|\mathbf{e}_r \leq |(R_P^+ \setminus R_{\hat{P}}^+) \cap (R_P^+ \setminus R_{\tilde{P}}^+)|\mathbf{e}_r = (|R_P^+| - |R_{\tilde{P}}^+ \cup R_{\hat{P}}^+|)\mathbf{e}_r$. Thus if (a) holds, then the statement follows as well, by noting $gr_{[r,r]}(q_\lambda\sigma^w) = gr_{[r,r]}(q_\lambda) + gr_{[r,r]}(\sigma^{w_2})$ and using Lemma 3.17. □

3.3. **Proof of Proposition 3.3 (2).** The statement to prove is a direct consequence of the next proposition.

Proposition 3.18. *Let $u \in W_P$ and $v \in W^P$. In $QH^*(G/B)$, we have*

$$\sigma^v \star \sigma^u = \sigma^{vu} + \sum_{w,\lambda} b_{w,\lambda} q_\lambda \sigma^w$$

with $gr(q_\lambda\sigma^w) < gr(\sigma^{vu})$ whenever $b_{w,\lambda} \neq 0$.

Remark 3.19. *Proposition 3.18 here extends Proposition 3.23 of* [16] *in the case of parabolic subgroups P such that Δ_P is not of type A. In Proposition 3.23 of*

[16], *the same property for* $\sigma^v \star \sigma^{s_j}$ *was discussed, under the assumptions that* Δ_P *is of type* A*,* $s_j \in W_P$ *and* $v \in W^P$*. By modifying the proof therein slightly, the assumption "*$v \in W^P$*" could be generalized to "*$v \in W$ *with* $gr_{[j,j]}(v) < j\mathbf{e}_j$*".*

Proof of Proposition 3.3 (2). This follows immediately from Proposition 3.18:

$$\Psi_{r+1}(q_{\lambda_P} \star_P \sigma^{v'}) = \overline{q_{\lambda_B}\sigma^{v'w_Pw_{P'}}} = \overline{q_{\lambda_B}} \star \overline{\sigma^{v'}} \star \overline{\sigma^{w_Pw_{P'}}} = \Psi_{r+1}(q_{\lambda_P}) \star \Psi_{r+1}(\sigma^{v'}).$$

□

To show Proposition 3.18, we prove some lemmas first.

Lemma 3.20. *Let* $v \in W^P$ *and* $u \in W_P$*. Take any* $w \in W$ *and* $\lambda \in Q^\vee$ *satisfying* $gr(q_\lambda\sigma^w) = gr(\sigma^u) + gr(\sigma^v)$*. If* λ *is a virtual null coroot, then we have*

$$N_{v,u}^{w,\lambda} = \begin{cases} 1, & \text{if } (w,\lambda) = (vu, 0) \\ 0, & \text{otherwise} \end{cases}.$$

Proof. Write $w = w_1w_2$ where $w_1 \in W^P$ and $w_2 \in W_P$. Take a reduced expression $w_2 = s_{i_1}\cdots s_{i_m}$ (i.e., $\ell(w_2) = m$). Since $v \in W^P$ and λ is a virtual null coroot, we have $\mathrm{sgn}_\alpha(v) = 0 = \langle\alpha, \lambda\rangle$ for all $\alpha \in \Delta_P$. Note $\alpha_{i_j} \in \Delta_P$ and $\mathrm{sgn}_{\alpha_{i_j}}(w_1s_{i_1}\cdots s_{i_j}) = 1$ for all $1 \le j \le m$. Applying the tuple $(u, v, w, \lambda, \alpha)$ of Proposition 3.15 (2) to the case $(us_{i_m}\cdots s_{i_{j+1}}, vs_{i_j}, w_1s_{i_1}\cdots s_{i_j}, \lambda + \alpha_{i_j}^\vee, \alpha_{i_j})$, we have $N_{v,us_{i_m}\cdots s_{i_{j+1}}}^{w_1s_{i_1}\cdots s_{i_j},\lambda} = N_{v,us_{i_m}\cdots s_{i_{j+1}}s_{i_j}}^{w_1s_{i_1}\cdots s_{i_{j-1}},\lambda}$ if $\ell(us_{i_m}\cdots s_{i_{j+1}}s_{i_j}) = \ell(us_{i_m}\cdots s_{i_{j+1}}) - 1$, or 0 otherwise. Hence, we have $N_{v,u}^{w,\lambda} = N_{v,u\cdot w_2^{-1}}^{w_1,\lambda}$ if $\ell(u \cdot w_2^{-1}) = \ell(u) - \ell(w_2)$, or 0 otherwise. Note $\ell(u) = |gr(u)| = |gr_{[1,r]}(u) + gr_{[1,r]}(v)| = |gr_{[1,r]}(w)| = |gr(w_2)| = \ell(w_2)$. When $\ell(u \cdot w_2^{-1}) = 0$, we have $u \cdot w_2^{-1} =$id, and consequently $N_{v,\mathrm{id}}^{w_1,\lambda} = 1$ if $(w_1, \lambda) = (v, 0)$, or 0 otherwise. Thus the statement follows. □

Recall $\Delta_{\tilde{P}} = \{\alpha_1, \cdots, \alpha_{r-1}\}$, whose Dynkin diagram is of type A_{r-1}. It is easy to see the next combinatorial fact (see e.g. Lemma 2.8 and Remark 2.9 of [17]).

Lemma 3.21. *Let* $\lambda \in Q^\vee$ *be a nonzero effective coroot, i.e.,* $\lambda = \sum_{i=1}^n a_i\alpha^\vee \neq 0$ *satisfies* $a_i \ge 0$ *for all* i*. Then there exists* $\alpha \in \Delta$ *such that* $\langle\alpha, \lambda\rangle > 0$*. Furthermore if* $a_i = 0$ *for* $i = r, r+1, \cdots, n$*, and if there exists only one such* α*, then* $\langle\alpha, \lambda\rangle > 1$*.*

Lemma 3.22. *Let* $\lambda \in Q^\vee$*, and* λ_B *be the Peterson-Woodward lifting of* $\lambda + Q_P^\vee$*. If* λ *is the Peterson-Woodward lifting of* $\lambda + Q_{\tilde{P}}^\vee$*, then either* $\lambda - \lambda_B$ *or* $\lambda_B - \lambda$ *is effective. Furthermore if* $\lambda - \lambda_B = \sum_{j=1}^r c_j\alpha_j^\vee \neq 0$*, then the coefficient* $c_r \neq 0$*.*

Proof. If follows from the definition of a Peterson-Woodward lifting that $\langle\alpha, \lambda_B\rangle = 0$ (resp. $\langle\alpha, \lambda\rangle = 0$) for all $\alpha \in \Delta_P$ (resp. $\Delta_{\tilde{P}}$) with at most one exception, and if such exception α_k (resp. $\alpha_{k'}$) exists, then $\langle\alpha, \lambda_B\rangle = -1$ (resp. $\langle\alpha_{k'}, \lambda\rangle = 1$). When there does not exist such exception, we denote $k = k' = n+1$ for notation conventions.

We may assume $\langle\alpha_r, \lambda - \lambda_B\rangle \ge 0$ (otherwise we consider $\lambda_B - \lambda$). Then $\lambda - \lambda_B \in Q_P^\vee$ is given by the difference between a dominate coweights and a fundamental coweight in $\Lambda_P^\vee$. Therefore it is well known that $\lambda - \lambda_B$ is either a nonpositive combination or a nonnegative combination of $\alpha_1^\vee, \cdots, \alpha_r^\vee$. (For instance, we can prove this by direct calculations using Table 1 of [11]).

Now we assume $c_j \ge 0$ for all j (otherwise we consider $\lambda_B - \lambda$). Since $\langle\alpha_i, \lambda - \lambda_B\rangle$, $i = 1, \cdots, r-1$, are all nonpositive with at most an exception of value 1, we conclude $c_r > 0$. Otherwise, it would make a contradiction with the second half of the statement of Lemma 3.21. □

Recall that $\partial\Delta_P$ denotes the set of simple roots in $\Delta \setminus \Delta_P$ which are adjacent to Δ_P.

Lemma 3.23. *Let $v \in W^P, u \in W_P$ and $w \in W$. Let $\lambda \in Q^\vee$ be effective, and λ_B be the Peterson-Woodward lifting of $\lambda + Q_P^\vee$. If $\lambda - \lambda_B = \sum_{j=1}^r c_j\alpha_j^\vee$ satisfies $c_r < 0$ and $c_j \le 0$ for all j, then $N_{v,u}^{w,\lambda} = 0$.*

Proof. Let $\alpha \in \Delta \setminus (\Delta_P \cup \partial\Delta_P)$. Then $\mathrm{sgn}_\alpha(u) = 0$ and $vs_\alpha \in W^P$. If $\langle\alpha, \lambda\rangle > 0$, then the coefficient of $\alpha^\vee$ in λ must be positive. By Proposition (1), we have $N_{v,u}^{w,\lambda} = 0$ unless $\mathrm{sgn}_\alpha(v) = \langle\alpha, \lambda\rangle = 1 - \mathrm{sgn}_\alpha(w) = 1$. Furthermore when this holds, we have $N_{v,u}^{w,\lambda} = N_{vs_\alpha,u}^{ws_\alpha,\lambda-\alpha^\vee}$ by Proposition 3.15 (2). Clearly, $\lambda - \lambda_B = (\lambda - \alpha^\vee) - (\lambda - \alpha^\vee)_B$. Therefore by induction, we can assume $\langle\alpha, \lambda\rangle \le 0$ for all $\alpha \in \Delta \setminus (\Delta_P \cup \partial\Delta_P)$.

The boundary $\partial\Delta_P$ consists one or two nodes. We assume $\partial\Delta_P = \{\alpha_{r+1}\}$ first. Then by Lemma 3.21, we have $\langle\alpha_{r+1}, \lambda_B\rangle \ge 1$.

Assume that α_r is adjacent to α_{r+1}, which happens in cases C5), C7) with $r = 7$, C9) with $r = 3$, and C10). Then $\langle\alpha_{r+1}, \lambda\rangle = \langle\alpha_{r+1}, \lambda_B\rangle + c_r\langle\alpha_{r+1}, \alpha_r^\vee\rangle \ge 2$. Since $\mathrm{sgn}_{\alpha_{r+1}}(u) = 0$, we have $N_{v,u}^{w,\lambda} = 0$ by Proposition 3.15 (1).

Assume that α_1 is adjacent to α_{r+1}. This happens in cases C1B), C1C), C2) and C4). If λ_B is of the form $a\mu_B + \alpha_{r+1}^\vee + \sum_{j=r+2}^n a_j\alpha_j^\vee$, then by the hypotheses $\langle\alpha_j, \lambda_B\rangle \le 0$ for all $j \ge r+2$, and the precise description of μ_B in Table 3, we can easily conclude that $\langle\alpha_{r+1}, \lambda_B - \alpha_{r+1}^\vee\rangle \ge 0$. Hence, we obtain $N_{v,u}^{w,\lambda} = 0$ again by the same arguments above. If λ_B is not of the aforementioned form, then λ_B is the combination of a virtual null coroot and a non-simple coroot in Table 3 (or zero coroot). Such λ_B satisfies $\langle\alpha_i, \lambda_B\rangle = 0$ for all $1 \le i \le r_a$, where α_{r_a} is the simple root adjacent to α_r. Let m be the minimum of the set $\{i \mid 1 \le i \le r_a, c_{i+1} < 0\}$ if nonempty, or $m := r_a$ otherwise. Then $\langle\alpha_m, \lambda\rangle = \langle\alpha_m, \sum_{i=1}^r c_i\alpha_i^\vee\rangle > 0$. Since $\mathrm{sgn}_{\alpha_m}(v) = 0$, $N_{v,u}^{w,\lambda} = 0$ unless $\mathrm{sgn}_{\alpha_m}(u) = \langle\alpha_m, \lambda\rangle = 1 - \mathrm{sgn}_{\alpha_m}(w) = 1$. When this holds, we have $N_{v,u}^{w,\lambda} = N_{v,us_m}^{ws_m,\lambda-\alpha_m^\vee}$ with $us_m \in W_P$ and $\lambda - \alpha_m^\vee = \lambda_B + (-1)\cdot\alpha_m^\vee + \sum_{j=1}^r c_j\alpha_j^\vee$, by Proposition 3.15. Hence, by reduction, we can assume $c_1 < 0$. Consequently, we have $\langle\alpha_{r+1}, \lambda\rangle \ge 2$, and then obtain $N_{v,u}^{w,\lambda} = 0$.

Now we assume $\partial\Delta_P = \{\alpha_{r+1}, \alpha_{r+2}\}$. That is, case C7) with $4 \le r \le 6$, or case C9) with $r = 2$ occurs. If $\langle\alpha_{r+1}, \lambda_B\rangle > 0$, then we are done by the same arguments as above, since α_r is adjacent to $\alpha_{r+1}^\vee$ and $c_r < 0$. If $\langle\alpha_{r+1}, \lambda_B\rangle \le 0$, then $\langle\alpha_{r+2}, \lambda_B\rangle > 0$. If $\lambda_B = \tau + \alpha_{r+2}^\vee$ with τ a virtual null coroot, then we conclude $\langle\alpha_{r+2}, \tau\rangle \ge 0$. (For instance when case 7) with $r = 6$ occurs, we have $\tau = a\mu_B^{(1)} + b\mu_B^{(2)}$, where $\mu_B^{(1)}, \mu_B^{(2)}$ denote the corresponding two coroots in Table 3, and $a, b \ge 0$. We have $\langle\alpha_7, \lambda_B\rangle = a - b \le 0$ and $\langle\alpha_8, \lambda_B\rangle = 2b - a + 2 > 0$. This implies $2b - a \ge 0$. The arguments for the remaining cases are similar.) If λ_B is not of the aforementioned form, then by Table 4 we conclude that $\langle\alpha_i, \lambda_B\rangle = 0$ for all $1 \le i \le r_a$, where α_{r_a} is the simple root of Δ_P adjacent to α_r. Therefore, we are done by the same arguments as above. □

Proof of Proposition 3.18. Since $QH^*(G/B)$ is an S-filtered algebra, by Lemma 3.20, we have

$$\sigma^v \star \sigma^u = \sigma^{vu} + \sum N_{v,u}^{w,\lambda} q_\lambda \sigma^w + \sum b_{w',\lambda'} q_{\lambda'} \sigma^{w'}.$$

Here $gr(q_{\lambda'}\sigma^{w'}) < gr(\sigma^v) + gr(\sigma^{\tilde u})$. The first summation is over those $q_\lambda\sigma^w$ satisfying both

$$\text{(i) } \lambda = \sum\nolimits_{j=1}^{n} a_j\alpha_j^\vee \text{ is not a virtual null coroot, where } a_j \geq 0 \text{ for all } j,$$

and (ii) $gr(q_\lambda\sigma^w) = gr(\sigma^v) + gr(\sigma^u)$. The hypothesis (ii) is equivalent to

$$\text{(ii)}' \ gr_{[1,r]}(q_\lambda\sigma^w) = gr_{[1,r]}(\sigma^u),$$

following from the dimension constraint of Gromov-Witten invariants $N_{v,u}^{w,\lambda}$ (see also Lemma 3.11 of [16]) and the assumption that $v \in W^P$.

By Proposition 3.13 (1), we conclude that elements in the first summation also satisfy

$$\text{(iii) } \sigma^w q_\lambda = \psi_{\Delta,\Delta_{\tilde P}}(q_{\tilde\lambda_{\tilde P}}) \text{ where } \tilde\lambda_{\tilde P} := \lambda + Q^\vee_{\tilde P}.$$

Therefore, it suffices to show $N_{v,\tilde u}^{w,\lambda} = 0$ whenever all (i), (ii)′ and (iii) hold.

Let λ_B denote the Peterson-Woodward lifting of $\lambda + Q_P^\vee$. By Lemma 3.22, the coefficients c_j of $\lambda - \lambda_B = \sum_{j=1}^r c_j\alpha_j^\vee$ are all nonpositive or all nonnegative, and $c_r \neq 0$ due to (i). If $c_r < 0$, then we are done by Lemma 3.23. Therefore we assume $c_r > 0$ in the following.

Since all $c_j \geq 0$, we write $\lambda = \lambda_B + \sum_{i=1}^t \beta_i^\vee$. The set $\Delta_{P'} = \{\alpha \in \Delta_P \mid \langle\alpha, \lambda_B\rangle = 0\}$ coincides with either Δ_P or $\Delta_P\setminus\{\alpha_k\}$ for a unique $\alpha_k \in \Delta_P$ with $\langle\alpha_k, \lambda_B\rangle = -1$. Therefore we can further assume that $\beta_i \in \Delta_P$, $i = 1, 2, \cdots, t$, satisfy $\langle\beta_j, \lambda_B\rangle + \langle\beta_j, \sum_{i=j}^t \beta_i^\vee\rangle > 0$ for all $1 \leq j \leq t$. (This can be done: if $\langle\alpha, \lambda - \lambda_B\rangle > 0$ holds for some α in Δ_P distinct from α_k, then we simply choose $\beta_1 = \alpha$. If not, then $\alpha = \alpha_k$, and the coefficient of α_k in the highest root of the root subsystem R_P is equal to 1. Hence we conclude $\langle\alpha, \lambda - \lambda_B\rangle \geq 2$.)

Since $v \in W^P$, $\mathrm{sgn}_\alpha(v) = 0$ for all $\alpha \in \Delta_P$. For each $1 \leq j \leq t$, we have $N_{v,us_{\beta_1}\cdots s_{\beta_{j-1}}}^{ws_{\beta_1}\cdots s_{\beta_{j-1}},\lambda_B+\sum_{i=j}^t\beta_j^\vee} = 0$ unless $\ell(us_{\beta_1}\cdots s_{\beta_j}) = \ell(us_{\beta_1}\cdots s_{\beta_{j-1}}) - 1$, $\ell(ws_{\beta_1}\cdots s_{\beta_j}) = \ell(us_{\beta_1}\cdots s_{\beta_{j-1}}) + 1$ and $\langle\beta_j, \lambda_B\rangle + \langle\beta_j, \sum_{i=j}^t \beta_i^\vee\rangle = 1$ all hold, by Proposition 3.15 (1). Furthermore when all these hypotheses hold, we have

$$N_{v,us_{\beta_1}\cdots s_{\beta_{j-1}}}^{ws_{\beta_1}\cdots s_{\beta_{j-1}},\lambda_B+\sum_{i=j}^t\beta_j^\vee} = N_{v,us_{\beta_1}\cdots s_{\beta_j}}^{ws_{\beta_1}\cdots s_{\beta_{j-1}}s_{\beta_j},\lambda_B+\sum_{i=j+1}^t\beta_j^\vee}$$

by applying the tuple $(u, v, w, \lambda, \alpha)$ of Proposition 3.15 (2) to the tuple $(\tilde u s_{\beta_1}\cdots s_{\beta_{j-1}}, vs_{\beta_j}, ws_{\beta_1}\cdots s_{\beta_{j-1}}s_{\beta_j}, \lambda_B + \sum_{i=j}^t \beta_j^\vee, \beta_j)$. Denote $u' := us_{\beta_1}\cdots s_{\beta_t}$. Combining all these, we have

$$N_{v,u}^{w,\lambda} = N_{v,u'}^{ws_{\beta_1}\cdots s_{\beta_t},\lambda_B}$$

if all the hypotheses (‡) hold:

$$\ell(ws_{\beta_1}\cdots s_{\beta_t}) = \ell(w) + t, \ell(u') = \ell(u) - t, \langle\beta_j, \lambda_B\rangle + \langle\beta_j, \sum_{i=j}^t \beta_i^\vee\rangle = 1, j = 1, \cdots, t,$$

or 0 otherwise. In particular if $\lambda_B = 0$, then we are done since the hypotheses on the step $j = t$ cannot hold.

It suffices to show $N_{v,u'}^{ws_{\beta_1}\cdots s_{\beta_t},\lambda_B} = 0$ under the hypotheses (‡) and $\lambda_B \neq 0$. If $\ell(u') = 0$, then $u' = \mathrm{id}$, and we are done. Assume $\ell(u') > 0$ now. For any $\eta \in Q_P^\vee$, we have $|gr_{[1,r]}(q_\eta)| = |gr(q_\eta)| = \langle 2\rho, \eta\rangle$, following from the definition. Due to (ii)′,

$$|gr_{[1,r]}(w)| + |gr_{[1,r]}(q_{\lambda_B})| + 2t = |gr_{[1,r]}(q_\lambda\sigma^w)| = |gr_{[1,r]}(u)| = \ell(u).$$

By Proposition 3.1, $-|gr_{[1,r]}(q_{\lambda_B})| = |gr_{[1,r]}(w_P w_{P'})| = \ell(w_P w_{P'}) = |R_P^+| - |R_{P'}^+|$. Combining both, we have

$$|gr_{[1,r]}(ws_{\beta_1}\cdots s_{\beta_t})| = |gr_{[1,r]}(w)| + t = \ell(u') - |gr_{[1,r]}(q_{\lambda_B})| = \ell(u') + |R_P^+| - |R_{P'}^+|.$$

Thus there is $\alpha \in \Delta_{P'}$ such that $\mathrm{sgn}_\alpha(ws_{\beta_1}\cdots s_{\beta_t}) = 1$ (otherwise, $ws_{\beta_1}\cdots s_{\beta_t}(\alpha) \in R^+$ for all $\alpha \in \Delta_{P'}$, which would imply $|gr_{[1,r]}(ws_{\beta_1}\cdots s_{\beta_t})| \le |R_P^+| - |R_{P'}^+|$). Since $v \in W^P$, $\mathrm{sgn}_\alpha(v) = 0$. By Proposition 3.15 (1), we have $N_{v,u'}^{ws_{\beta_1}\cdots s_{\beta_t},\lambda_B} = 0$ unless $\mathrm{sgn}_\alpha(u') = 1$, i.e., $\ell(u's_\alpha) = \ell(u') - 1$. Furthermore when this holds, we have $N_{v,u'}^{ws_{\beta_1}\cdots s_{\beta_t},\lambda_B} = N_{v,u's_\alpha}^{ws_{\beta_1}\cdots s_{\beta_t}s_\alpha,\lambda_B}$ (by applying the tuple $(u, v, w, \lambda, \alpha)$ of Proposition 3.15 (2) to $(vs_\alpha, u', ws_{\beta_1}\cdots s_{\beta_t}s_\alpha, \lambda_B + \alpha^\vee, \alpha)$). By induction, we conclude $N_{v,u'}^{ws_{\beta_1}\cdots s_{\beta_t},\lambda_B} = 0$ unless both $u' \in W_{P'}$ and $\ell(ws_{\beta_1}\cdots s_{\beta_t}u'^{-1}) = \ell(ws_{\beta_1}\cdots s_{\beta_t}) - \ell(u')$ hold. Furthermore when both hypotheses hold, we have $N_{v,u'}^{ws_{\beta_1}\cdots s_{\beta_t},\lambda_B} = N_{v,\mathrm{id}}^{ws_{\beta_1}\cdots s_{\beta_t}u'^{-1},\lambda_B} = 0$ since $\lambda_B \neq 0$. □

3.4. **Proof of Proposition 3.3 (3).** The statement tells us that the elements $\Psi_{r+1}(q_{\lambda_P})$ in $Gr^{\mathcal{F}}_{(r+1)}$ do behave like monomials. Due to Lemma 3.11, it suffices to show those $q_\lambda\sigma^u$ behave like the non-identity elements of the finite abelian group $(Q^\vee/Q_P^\vee)/\mathcal{L}$ as in Table 4. For any one of the cases C1B), C1C), C2) and C9), we use the first virtual null coroot μ_B in Table 3 and the unique element $q_\lambda\sigma^u$ in Table 4. Namely for the only exceptional case when C9) with $r = 2$ occurs, there are two virtual null coroots, and we will use the one $\mu_B = 2\alpha_4^\vee + 2\alpha_1^\vee + \alpha_2^\vee$. For any one of these cases, we only need to use check one quantum multiplication as in the next proposition, which we assume first. The remaining cases require verifications of more quantum multiplications, which will be discussed in section 5.3.

Proposition 3.24. *Assume* C1B), C2) *or* C9) *occurs. In $QH^*(G/B)$, we have*

$$q_\lambda\sigma^u \star q_\lambda\sigma^u = q_{\mu_B} + \sum b_{w',\lambda'} q_{\lambda'}\sigma^{w'}$$

with $gr(q_{\lambda'}\sigma^{w'}) < gr(q_{\mu_B})$ whenever $b_{w',\lambda'} \neq 0$.

Proof of Proposition 3.3 (3). Let $q_{\kappa_P}, q_{\kappa'_P} \in QH^*(G/P)$. If case C1$C$) occurs, then we note $\psi_{\Delta,\Delta_P}(q_{\kappa_P}) \star \psi_{\Delta,\Delta_P}(q_{\kappa'_P}) = q_{\kappa_B} \star q_{\kappa'_B} = q_{\kappa_B+\kappa'_B} = \psi_{\Delta,\Delta_P}(q_{\kappa_P+\kappa'_P})$. Therefore, $\Psi_{r+1}(q_{\kappa_P}) \star \Psi_{r+1}(q_{\kappa'_P}) = \Psi_{r+1}(q_{\kappa_P+\kappa'_P})$. Assume that case C1$B$), C9) or C2) occurs now. If either κ_P or κ'_P is a virtual null coroot, then we are done, by using Lemma 3.11. Otherwise, by Proposition 3.9 we have $\kappa_P = \tau_P + \lambda_P$ and $\kappa'_P = \tau'_P + \lambda_P$ for some virtual null coroots τ_P, τ'_P, and consequently $\kappa_P + \kappa'_P = \tau_P + \tau'_P + (\mu_B + Q_P^\vee)$. Here μ_B and $\psi_{\Delta,\Delta_P}(q_{\lambda_P}) = q_\lambda\sigma^u$ are given in Table 3 and Table 4 respectively. Hence, we have $\Psi_{r+1}(q_{\kappa_P}) = \Psi_{r+1}(q_{\tau_P}) \star \Psi_{r+1}(q_{\lambda_P})$ and $\Psi_{r+1}(q_{\kappa'_P}) = \Psi_{r+1}(q_{\tau'_P}) \star \Psi_{r+1}(q_{\lambda_P})$, by Lemma 3.11. Using Proposition 3.24, we have $\Psi_{r+1}(q_{\lambda_P}) \star \Psi_{r+1}(q_{\lambda_P}) = \overline{q_\lambda\sigma^u} \star \overline{q_\lambda\sigma^u} = \overline{q_{\mu_P}}$. Hence, we have $\Psi_{r+1}(q_{\kappa_P}) \star \Psi_{r+1}(q_{\kappa'_P}) = \overline{q_{\tau_B}} \star \overline{q_{\tau'_P}} \star \overline{q_{\mu_B}} = \overline{q_{\tau_B+\tau'_B+\mu_B}} = \Psi_{r+1}(q_{\kappa_P+\kappa'_P})$. For the remaining cases in Table 1, the statements follows from the arguments given in section 5.2. Thus we are done. □

Now we prepare some lemmas in order to prove Proposition 3.24. The reduced expressions of the longest element w_P in W_P are not unique. There is a conceptual approach to construct w_P of the form $w^{\frac{h}{2}}$ whenever h is even (see e.g. Chapter 3 of [13]). Here h denotes the Coxeter number of W_P, and it is equal to $2r$ (resp. $2r-2$) for Δ_P of type B_r (resp. D_r). The next lemma provides a special choice of the above $w \in W_P$.

Lemma 3.25. *For Δ_P of type B_r or D_r, $(s_1 \cdots s_r)^{\frac{h}{2}}$ is a reduced expression of the longest element w_P.*

Proof. It is easy to check that the given element maps all simple roots in Δ_P to negative roots, and note $\ell(w_P) = r^2$ (resp. $r(r-1)$). Thus the statement follows. □

Recall that for u in Table 4, $\tilde{u}$ denotes the minimal length representative of $uW_{\tilde{P}}$.

Lemma 3.26. *Let $v = s_{\beta_1} \cdots s_{\beta_p} \in W_P$ be a reduced expression. Assume* C1B), C2) *or* C9) *occurs, then $\tilde{u}^{-1} \leqslant v$ if and only if there exists a subsequence $[i_1, \cdots, i_{\frac{h}{2}}]$ of $[1, \cdots, p]$ such that $[\beta_{i_1}, \cdots, \beta_{i_{\frac{h}{2}}}] = [\alpha_r, \alpha_{\frac{h}{2}-1}, \cdots, \alpha_2, \alpha_1]$.*

Proof. Note $\ell(\tilde{u}^{-1}) = \frac{h}{2}$. It is a general fact that $\tilde{u}^{-1} \leqslant v$ if and only if there exists a subsequence $[i_1, \cdots, i_{\frac{h}{2}}]$ of $[1, \cdots, p]$ such that $\tilde{u}^{-1} = s_{\beta_{i_1}} \cdots s_{\beta_{i_{\frac{h}{2}}}}$. Since the simple reflections in $\tilde{u}^{-1} = s_r s_{\frac{h}{2}-1} \cdots s_2 s_1$ are distinct, we conclude that the two sets $\{\alpha_r, \alpha_{\frac{h}{2}-1}, \cdots, \alpha_2, \alpha_1\}$ and $\{\beta_{i_1}, \cdots, \beta_{i_{\frac{h}{2}}}\}$ coincide with each other. Then the coincidence of the corresponding two ordered sequences follows immediately from the obvious observation that $s_r s_{\frac{h}{2}-1} \cdots s_{j+1} s_j(\alpha_j) \in -R^+$ for all j. □

The next well-known fact works for arbitrary Δ_P (see e.g. Theorem 3.17 (iv) of [2][2]).

Lemma 3.27. *Let $w, v \in W_P$. If $w^{-1} \not\leqslant v^{-1}w_P$, then $\sigma^w \cup \sigma^v = 0$ in $H^*(P/B)$.*

Corollary 3.28. *For case* C1B), C2) *or* C9), *we have $\sigma^{\tilde{u}} \cup \sigma^{\tilde{u}} = 0$ in $H^*(P/B)$.*

Proof. By Lemma 3.25, $\tilde{u}^{-1}w_P$ is equal to $(s_1 s_2 \cdots s_r)^{r-1}$ if case C1B) or C9) occurs, or equal to $s_{r-1}(s_1 s_2 \cdots s_r)^{r-2}$ if case C2) occurs (since $s_r s_{r-1} = s_{r-1} s_r$). Clearly, there does not exist a subsequence $[i_1, \cdots, i_{\frac{h}{2}}]$ satisfying $[\alpha_{i_1}, \cdots, \alpha_{i_{\frac{h}{2}}}] = [\alpha_r, \alpha_{\frac{h}{2}-1}, \cdots, \alpha_2, \alpha_1]$. Thus $\tilde{u}^{-1} \not\leqslant \tilde{u}^{-1}w_P$ by Lemma 3.26. Hence, the statement follows from Lemma 3.27. □

Proof of Proposition 3.24. Due to the filtered algebra structure of $QH^*(G/B)$, we have $q_\lambda \sigma^u \star q_\lambda \sigma^u = \sum_{w,\eta} N_{u,u}^{w,\eta} q_{\eta+2\lambda} \sigma^w + \sum b_{w',\lambda'} q_{\lambda'} \sigma^{w'}$, where $gr(q_{\eta+2\lambda}\sigma^w) = 2gr(q_\lambda \sigma^u)$ and $gr(q_{\lambda'}\sigma^{w'}) < 2gr(q_\lambda \sigma^u)$. Since $gr_{[r+1,r+1]}(\sigma^u) = \mathbf{0}$, we conclude $w \in W_P$ and $\eta = \sum_{i=1}^r b_i \alpha_i^\vee$ where $b_i \geq 0$. Note $gr_{[r,r]}(q_r) = h\mathbf{e}_r$ by Table 2. Write $w = w_1 w_2$ where $w_1 \in W_P^{\tilde{P}}$ and $w_2 \in W_{\tilde{P}}$. Using $gr_{[r,r]}$, we conclude $\ell(w_1) + b_r h = h$. If $b_r = 0$, then $gr(q_\eta \sigma^{w_2}) = gr_{[1,r-1]}(q_\eta \sigma^{w_1 w_2}) = 2gr_{[1,r-1]}(u) = 2\sum_{i=1}^{r-1} \mathbf{e}_i = gr(q_1 \sigma^{v_{r-1}\cdots v_2})$ where $v_j := s_{j-1}s_j$ for each $2 \leq j \leq r-1$. Thus we have $q_\eta \sigma^{w_2} = q_1 \sigma^{v_{r-1}\cdots v_2}$ by Lemma 3.12. In $Gr^{\tilde{\mathcal{F}}}(QH^*(G/B))$, we have $\overline{\sigma^u \star \sigma^u} = \overline{N_{u,u}^{w,\eta} q_1 \sigma^{w_1 v_{r-1}\cdots v_1}} +$ other terms. On the other hand, by Proposition 3.13, we have $\overline{\sigma^u \star \sigma^u} = (\overline{\sigma^{\tilde{u}}} \star \overline{\sigma^{s_{r-1}\cdots s_1}})^2 = \big(\overline{N_{\tilde{u},\tilde{u}}^{w_1,0} \sigma^{w_1} + \text{term 1}}\big) \star \big(\overline{\text{term 2}}\big)$. Here term 1

[2]The Schubert cohomology classes σ^v are denoted as $P_{v^{-1}}$ in [2].

is a nonnegative combination of the form $\psi_{\Delta,\Delta_{\tilde{P}}}(q_{\eta'_{\tilde{P}}}\sigma^{w'_1})$ with either $\eta'_{\tilde{P}} \neq 0$ or $w_1 \neq w'_1 \in W^{\tilde{P}}$, and term 2 is a combination of elements $q_{\lambda''}\sigma^{v''}$ with $(v'',\lambda'') \in W_{\tilde{P}} \times Q^\vee_{\tilde{P}}$. Hence, $N^{w,\eta}_{u,u} \neq 0$ only if $N^{w_1,0}_{\tilde{u},\tilde{u}} \neq 0$, the latter of which is the coefficient of σ^{w_1} in $\sigma^{\tilde{u}} \cup \sigma^{\tilde{u}} \in H^*(G/B) \subset QH^*(G/B)$. It is a general fact (following from the surjection $H^*(G/B) \to H^*(P/B)$) that $N^{w_1,0}_{\tilde{u},\tilde{u}}$ coincides with the coefficient of σ^{w_1} in $\sigma^{\tilde{u}} \cup \sigma^{\tilde{u}} \in H^*(P/B)$, and therefore it is equal to 0 by Corollary 3.28. Thus $N^{w,\eta}_{u,u} = 0$ whenever $b_r = 0$. It remains to deal with the case $b_r = 1$. By Lemma 3.12, there is exactly one such term, which turns out to be $q_\eta\sigma^w = q_{\mu_B - 2\lambda}$. Thus it suffices to show $N^{\mathrm{id},\mu_B-2\lambda}_{u,u} = 1$. Using Proposition 3.15 (2) repeatedly, we conclude the followings.

(1) If case C1B) or C9) occurs, then $\eta = \mu_B - 2\lambda = \alpha_r^\vee + 2\sum_{i=1}^{r-1}\alpha_i^\vee$, and we have

$$N^{\mathrm{id},\eta}_{u,u} = N^{s_1\cdots s_{r-1}s_1\cdots s_{r-2},q_{r-1}q_r}_{\tilde{u},\tilde{u}s_{r-1}} = N^{s_1\cdots s_{r-1}s_1\cdots s_{r-2}s_r,q_{r-1}}_{s_1\cdots s_{r-1},s_1\cdots s_r s_{r-1}} = N^{s_1\cdots s_{r-1}s_r s_1\cdots s_{r-2},0}_{s_1\cdots s_{r-2},s_1\cdots s_r}.$$

(2) If C2) occurs, then $\eta = \mu_B - 2\lambda = \alpha_r^\vee + \alpha_{r-1}^\vee + 2\sum_{i=1}^{r-2}\alpha_i^\vee$, and we have

$$N^{\mathrm{id},\eta}_{u,u} = N^{s_1\cdots s_{r-2}s_1\cdots s_{r-3},q_{r-2}q_{r-1}q_r}_{\tilde{u}s_{r-1},\tilde{u}s_{r-1}s_{r-2}} = N^{s_1\cdots s_{r-2}s_1\cdots s_{r-3}s_r s_{r-1},q_{r-2}}_{s_1\cdots s_{r-2},\tilde{u}s_{r-1}s_{r-2}} = N^{s_1\cdots s_r s_1\cdots s_{r-3},0}_{s_1\cdots s_{r-3},s_1\cdots s_r}.$$

Namely, we always have $N^{\mathrm{id},\mu_B-2\lambda}_{u,u} = N^{v''u'',0}_{u'',v''}$ with $u'' = s_1\cdots s_{\frac{h}{2}-2} \in W_{P''}$ and $v'' = s_1\cdots s_r \in W^{P''}$, where $\Delta_{P''} := \{\alpha_1,\cdots,\alpha_{\frac{h}{2}-2}\}$. Thus it is equal to 1 by Lemma 3.20. □

4. Conclusions for general Δ_P

In this section, we allow P/B to be reducible, namely the Dynkin diagram $Dyn(\Delta_P)$ could be disconnected. We will first show the coincidence between the grading map gr defined in section 2.2 and the one introduced in [16]. Then we will refine the statement of Theorem 5.2 of [16], and will sketch the proof of it.

Whenever referring to the subset $\Delta_P = \{\alpha_1,\cdots,\alpha_r\}$, in fact, we have already given an ordering on the r simple roots in Δ_P, in terms of α_i's. As we can see in Definition 2.7, the grading map $gr: W \times Q^\vee \to \mathbb{Z}^{r+1}$ depends only on such an ordering of α_i's in Δ_P, which has nothing to do with the connectedness of $Dyn(\Delta_P)$. Therefore we can use the same definition even if $Dyn(\Delta_P)$ is disconnected. We want to show gr coincides with the grading map given by Definition 2.8 (resp. 5.1) of [16] when $Dyn(\Delta_P)$ is connected (resp. disconnected).

Recall $\Delta_0 := \emptyset, \Delta_{r+1} := \Delta$, $\Delta_i := \{\alpha_1,\cdots,\alpha_i\}$ for $1 \le i \le r$, and $P_j := P_{\Delta_j}$ for all j. Denote $\rho_j := \frac{1}{2}\sum_{\beta\in R^+_{P_j}}\beta$ where $\rho_0 := 0$. Then for any $\lambda \in Q^\vee$, we have $gr(\mathrm{id},\lambda) = \sum_{j=1}^{r+1}\langle 2\rho_j - 2\rho_{j-1},\lambda\rangle\mathbf{e}_j$ by Definition 2.7.

Lemma 4.1. *For any $\alpha \in \Delta_j$, we have $gr_{[j+1,r+1]}(id,\alpha^\vee) = \mathbf{0}$ and $|gr(id,\alpha^\vee)| = 2$.*

Proof. It is well-known that ρ_k equals the sum of fundamental weights in the root subsystem R_{P_k}. That is, we have $\langle\rho_k,\alpha^\vee\rangle = 1$ for any $\alpha \in \Delta_k$. Hence, for $j \le k \le r+1$, we have $|gr_{[1,k]}(\mathrm{id},\alpha^\vee)| = \sum_{i=1}^k\langle 2\rho_i - 2\rho_{i-1},\alpha^\vee\rangle = \langle 2\rho_k,\alpha^\vee\rangle = 2$. Thus if $i > j$, then $|gr_{[i,i]}(\mathrm{id},\alpha^\vee)| = |gr_{[1,i]}(\mathrm{id},\alpha^\vee)| - |gr_{[1,i-1]}(\mathrm{id},\alpha^\vee)| = 2 - 2 = 0$. □

By abuse of notation, we still denote by $\psi_{\Delta_{j+1},\Delta_j}$ the injective map $\psi_{\Delta_{j+1},\Delta_j}: W^{P_j}_{P_{j+1}} \times Q^\vee_{P_{j+1}}/Q^\vee_{P_j} \longrightarrow W \times Q^\vee$ induced from the Peterson-Woodward comparison formula. We recall Definition 2.8 of [16] as follows.

Definition 4.2. *Define a grading map* $gr' : W \times Q^\vee \longrightarrow \mathbb{Z}^{r+1}$ *associated to* $\Delta_P = (\alpha_1, \cdots, \alpha_r)$ *as follows.*

(1) *For* $w \in W$*, we take its (unique) decomposition* $w = v_{r+1} \cdots v_1$ *where* $v_j \in W_{P_j}^{P_{j-1}}$*. Then we define* $gr'(w, 0) = \sum_{i=1}^{r+1} \ell(v_i)\mathbf{e}_i$*.*

(2) *For* $\alpha \in \Delta$*, we can define all* $gr'(id, \alpha^\vee)$ *recursively in the following way. Define* $gr'(id, \alpha_1^\vee) = 2\mathbf{e}_1$*; for any* $\alpha \in \Delta_{j+1} \setminus \Delta_j$*, we define*

$$gr'(id, \alpha^\vee) = \big(\ell(w_{P_j} w_{P'_j}) + 2 + \sum\nolimits_{i=1}^{j} 2a_i\big)\mathbf{e}_{j+1} - gr'(w_{P_j} w_{P'_j}, 0) - \sum\nolimits_{i=1}^{j} a_i gr'(id, \alpha_i^\vee),$$

where $w_{P_j} w_{P'_j}$ *and* a_i*'s are defined by the image* $\psi_{\Delta_{j+1}, \Delta_j}(id, \alpha^\vee + Q_{P_j}^\vee) = (w_{P_j} w_{P'_j}, \alpha^\vee + \sum\limits_{i=1}^{j} a_i \alpha_i^\vee)$*.*

(3) *In general, we define* $gr'(w, \sum_{k=1}^n b_k \alpha_k^\vee) = gr'(w, 0) + \sum_{k=1}^n b_k gr'(id, \alpha_k^\vee)$*.*

One of the main results of [16], i.e., Proposition 2.1, tells us that the grading gr' respects the quantum multiplication. Precisely for any Schubert classes σ^u, σ^v of $QH^*(G/B)$, if $q_\lambda \sigma^w$ occurs in the quantum multiplication $\sigma^u \star \sigma^v$, then

$$gr'(w, \lambda) \le gr'(u, 0) + gr'(v, 0).$$

Proposition 4.3. *If* $Dyn(\Delta_P)$ *is connected, then* $gr = gr'$*.*

Proof. While it is a general fact that $gr|_{W \times \{0\}} = gr'|_{W \times \{0\}}$, we illustrate a little bit details here. For each j, $v_j \cdots v_1 \in W_{P_j}$ preserves R_{P_j}, and $v_{r+1} v_r \cdots v_{j+1} \in W^{P_j}$ maps $R_{P_j}^+$ (resp. $-R_{P_j}^+$) to R^+ (resp. $-R^+$). Thus for any $\beta \in R_{P_j}^+$, $w(\beta) \in -R^+$ if and only if $v_j \cdots v_1(\beta) \in -R_{P_j}^+ \subset -R^+$. That is, we have $\ell(v_j \cdots v_1) = |\mathrm{Inv}(v_j \cdots v_1)| = |\mathrm{Inv}(w) \cap R_{P_j}^+|$. Hence, $\ell(v_j) = \ell(v_j \cdots v_1) - \ell(v_{j-1} \cdots v_1) = |\mathrm{Inv}(w) \cap R_{P_j}^+| - |\mathrm{Inv}(w) \cap R_{P_{j-1}}^+| = |\mathrm{Inv}(w) \cap (R_{P_j}^+ \setminus R_{P_{j-1}}^+)|$.

Note $\Delta_1 = \{\alpha_1\}$ and $gr(\mathrm{id}, \alpha_1^\vee) = 2\mathbf{e}_1$ (by Lemma 4.1). Assume the statement follows for simple roots in Δ_k. For $\alpha \in \Delta_{k+1} \setminus \Delta_k$, say $\psi_{\Delta_{k+1}, \Delta_k}(\mathrm{id}, \alpha^\vee + Q_{P_k}^\vee) = (w_{P_k} w_{P'_k}, \lambda)$ where $\lambda = \alpha^\vee + \sum_{i=1}^k a_i \alpha_i^\vee$. Then $gr'_{[1,k]}(\mathrm{id}, \alpha^\vee) = -gr'_{[1,k]}(w_{P_k} w_{P'_k}, 0) - \sum_{i=1}^k a_i gr'_{[1,k]}(\mathrm{id}, \alpha_i^\vee) = -gr'_{[1,k]}(w_{P_k} w_{P'_k}, 0) - \sum_{i=1}^k a_i \sum_{j=1}^k \langle 2\rho_j - 2\rho_{j-1}, \alpha_i^\vee \rangle \mathbf{e}_j = -gr'_{[1,k]}(w_{P_k} w_{P'_k}, 0) + gr_{[1,k]}(\mathrm{id}, \alpha^\vee) - \sum_{j=1}^k \langle 2\rho_j - 2\rho_{j-1}, \lambda \rangle \mathbf{e}_j$. Note $\langle \gamma, \lambda \rangle \in \{0, , -1\}$ for any $\gamma \in R_{P_k}^+$, and $\Delta_{P'_k} = \{\beta \in \Delta_k \mid \langle \beta, \lambda \rangle = 0\}$. Thus we have $\langle \gamma, \lambda \rangle = -1$ if $\gamma \in R_{P_k}^+ \setminus R_{P'_k}^+$, or 0 if $\gamma \in R_{P'_k}^+$. Hence, for any $1 \le i \le k$, we have $-\langle 2\rho_i - 2\rho_{i-1}, \lambda \rangle = -\sum\limits_{\gamma \in R_{P_i}^+ \setminus R_{P_{i-1}}^+} \langle \gamma, \lambda \rangle = -\sum\limits_{\gamma \in (R_{P_i}^+ \setminus R_{P_{i-1}}^+) \bigcap (R_{P_k}^+ \setminus R_{P'_k}^+)} \langle \gamma, \lambda \rangle = -\sum\limits_{\gamma \in (R_{P_i}^+ \setminus R_{P_{i-1}}^+) \bigcap (R_{P_k}^+ \setminus R_{P'_k}^+)} -1 = \big|(R_{P_i}^+ \setminus R_{P_{i-1}}^+) \bigcap (R_{P_k}^+ \setminus R_{P'_k}^+)\big| = \big|(R_{P_i}^+ \setminus R_{P_{i-1}}^+) \bigcap \mathrm{Inv}(w_{P_k} w_{P'_k})\big| = |gr_{[i,i]}(w_{P_k} w_{P'_k}, 0)|$. Hence, $gr_{[1,k]}(\mathrm{id}, \alpha^\vee) = gr'_{[1,k]}(\mathrm{id}, \alpha^\vee)$. Thus we have $gr(\mathrm{id}, \alpha^\vee) = gr'(\mathrm{id}, \alpha^\vee)$, by noting $gr_{[k+2,r+1]}(\mathrm{id}, \alpha^\vee) = \mathbf{0} = gr'_{[k+2,r+1]}(\mathrm{id}, \alpha^\vee)$ and $|gr(\mathrm{id}, \alpha^\vee)| = 2 = |gr'(\mathrm{id}, \alpha^\vee)|$. Hence, the statement follows by induction on k. □

When $Dyn(\Delta_P)$ is not connected, we use the same ordering on Δ_P as in section 5 of [16]. Namely, we write $\Delta_P = \bigsqcup_{k=1}^m \Delta^{(k)}$ such that each $Dyn(\Delta_{(k)})$ is a connected component of $Dyn(\Delta_P)$. Clearly, $\Delta^{(k)}$'s are all of A-type with at most one exception, say $\Delta^{(m)}$ if it exists. We fix a canonical order on Δ_P. Namely, we say $\Delta_P = (\Delta^{(1)}, \cdots, \Delta^{(m)}) = (\alpha_1, \cdots, \alpha_r)$ such that for each k, $\Delta^{(k)} = \{\alpha_{k,1}, \cdots, \alpha_{k,r_k}\}$ satisfying (1) if $\Delta^{(k)}$ is of A-type, then $Dyn(\Delta^{(k)})$ is given by $\underset{\alpha_{k,1}}{\circ}\!-\!\underset{\alpha_{k,2}}{\circ}\cdots\underset{}{\circ}\!-\!\underset{\alpha_{k,r_k}}{\circ}$ together with the same way of denoting an ending point (by $\alpha_{k,1}$ or α_{k,r_k}) as in section 2.4 of [16]; (2) if $\Delta^{(k)}$ is not of A-type, then $Dyn(\Delta^{(k)})$ is given in the way of Table 1. We also denote the standard basis of $\mathbb{Z}^{r+1}$ as $\{\mathbf{e}_{1,1}, \cdots, \mathbf{e}_{1,r_1}, \cdots, \mathbf{e}_{m,1}, \cdots, \mathbf{e}_{m,r_m}, \mathbf{e}_{m+1,1}\}$. In order words, we have $\mathbf{e}_{k,i} = \mathbf{e}_{i+\sum_{t=1}^{k-1} r_t}$ and $\alpha_{k,i} = \alpha_{i+\sum_{t=1}^{k-1} r_t}$ in terms of our previous notations of $\mathbf{e}_j$'s and α_j's respectively.

Using Definition 4.2 (resp. 2.7) with respect to $\Delta^{(k)}$, we obtain a grading map

$$gr'_{(k)} : W \times Q^\vee \longrightarrow \mathbb{Z}^{r_k+1} = \bigoplus_{i=1}^{r_k+1} \mathbb{Z}\mathbf{e}_{k,i}$$

(resp. $gr_{(k)} : W \times Q^\vee \longrightarrow \mathbb{Z}^{r_k+1} = \bigoplus_{i=1}^{r_k+1} \mathbb{Z}\mathbf{e}_{k,i}$). Note $W_P = W_1 \times \cdots \times W_m$ where each W_k is the Weyl subgroup generated by simple reflections from $\Delta^{(k)}$. In particular for any $(w, \lambda) \in W_k \times (\bigoplus_{\alpha\in\Delta^{(k)}} \mathbb{Z}\alpha^\vee) \subset W \times Q^\vee$, we have $gr'_{(k)}(w, \lambda) \in \bigoplus_{i=1}^{r_k} \mathbb{Z}\mathbf{e}_{k,i} \hookrightarrow \mathbb{Z}^{r+1}$ which we treat as an element of $\mathbb{Z}^{r+1}$ via the natural inclusion. Now we recall Definition 5.1 of [16] for general Δ_P as follows.

Definition 4.4. *We define a grading map as follows, say again* $gr' : W \times Q^\vee \longrightarrow \mathbb{Z}^{r+1}$ *by abuse of notation.*

1. *Write* $w = v_{m+1}v_m \cdots v_1$ *(uniquely), in which* $(v_1, \cdots, v_m, v_{m+1}) \in W_1 \times \cdots \times W_m \times W^P$*. Then* $gr'(w, 0) := \ell(v_{m+1})\mathbf{e}_{m+1,1} + \sum_{k=1}^m gr'_{(k)}(v_k, 0)$*.*
2. *For each* $\alpha_{k,i} \in \Delta^{(k)}$*,* $gr'(id, q_{\alpha^\vee_{k,i}}) := gr'_{(k)}(id, q_{\alpha^\vee_{k,i}})$*. For* $\alpha \in \Delta \setminus \Delta_P$*, we write* $\psi_{\Delta,\Delta_P}(q_{\alpha^\vee+Q^\vee_P}) = w_Pw_{P'}q_{\alpha^\vee}\prod_{k=1}^m\prod_{i=1}^{r_k} q_{\alpha^\vee_{k,i}}^{a_{k,i}}$ *and then define*

$$gr'(id, \alpha^\vee) = \big(\ell(w_Pw_{P'})+2+\sum_{k=1}^m\sum_{i=1}^{r_k} 2a_{k,i}\big)\mathbf{e}_{m+1,1} - gr'(w_Pw_{P'}, 0) - \sum_{k=1}^m\sum_{i=1}^{r_k} a_{k,i}gr'(id, \alpha^\vee_{k,i}).$$

3. *In general,* $gr'(w, \sum_{\alpha\in\Delta} b_\alpha\alpha^\vee) := gr'(w, 0) + \sum_{\alpha\in\Delta} b_\alpha gr'(id, \alpha^\vee)$*.*

By abuse of notation, we denote π_k for both of the natural projections

$$\mathbb{Z}^{r_k+1} = \bigoplus_{i=1}^{r_k+1} \mathbb{Z}\mathbf{e}_{k,i} \longrightarrow \bigoplus_{i=1}^{r_k} \mathbb{Z}\mathbf{e}_{k,i} \quad \text{and} \quad \mathbb{Z}^{r+1} = \bigoplus_{j=1}^{m+1}\bigoplus_{i=1}^{r_j} \mathbb{Z}\mathbf{e}_{j,i} \longrightarrow \bigoplus_{i=1}^{r_k} \mathbb{Z}\mathbf{e}_{k,i}.$$

Lemma 4.5. *For* $1 \le k \le m$*, we have* $\pi_k \circ gr = \pi_k \circ gr_{(k)}$ *and* $\pi_k \circ gr' = \pi_k \circ gr'_{(k)}$*.*

Proof. It follows immediately from the definition that $\pi_k \circ gr'(w, \alpha^\vee) = \pi_k \circ gr'_{(k)}(w, \alpha^\vee)$ for $(w, \alpha^\vee) \in W \times \Delta^{(k)}$. For $\beta \in \Delta^{(\tilde{k})}$ where $\tilde{k} \neq k$, we note $\langle \alpha, \beta^\vee \rangle = 0$. Thus $\psi_{\Delta,\Delta^{(k)}}(q_{\beta^\vee+Q^\vee_{P_{\Delta^{(k)}}}}) = q_{\beta^\vee}$; furthermore if $\psi_{\Delta,\Delta_P}(q_{\gamma^\vee+Q^\vee_P}) = w_Pw_{P'}q_{\gamma^\vee}\prod_{k=1}^m\prod_{i=1}^{r_k} q_{\alpha^\vee_{k,i}}^{a_{k,i}}$ where $\gamma \in \Delta\setminus\Delta_P$, then $\psi_{\Delta,\Delta^{(k)}}(q_{\gamma^\vee+Q^\vee_{P_{\Delta^{(k)}}}}) = w_{\check{P}}w_{\check{P}'}q_{\gamma^\vee}\prod_{i=1}^{r_k} q_{\alpha^\vee_{k,i}}^{a_{k,i}}$ with $w_{\check{P}}w_{\check{P}'}$ given by the W_k-component of $w_Pw_{P'}$, implying $\pi_k \circ gr'(w_{\check{P}}w_{\check{P}'}) = \pi_k \circ gr'(w_Pw_{P'})$. Thus we have $gr'_{(k)}(\mathrm{id}, \beta^\vee) = 2\mathbf{e}_{k,r_k+1}$, $\pi_k \circ gr'(\mathrm{id}, \beta^\vee) = \pi_k \circ gr'_{(\tilde{k})}(\mathrm{id}, \beta^\vee) = \mathbf{0} = \pi_k \circ gr'_{(k)}(\mathrm{id}, \beta^\vee)$, and consequently $\pi_k \circ gr'(\mathrm{id}, \gamma^\vee) = \pi_k \circ gr'_{(k)}(\mathrm{id}, \gamma^\vee)$. Hence, $\pi_k \circ gr' = \pi_k \circ gr'_{(k)}$.

Due to our notation conventions, we have $\mathbf{e}_j = \mathbf{e}_{k,i}$ for $j = i+\sum_{t=1}^{k-1} r_t$. Thus $\pi_k \circ gr = \pi_k \circ gr_{(k)}$ follows immediately, by noting $R^+_{P_j} = R^+_{P_{\Delta_i^{(k)}}} \bigsqcup \big(\bigsqcup_{t=1}^{k-1} R^+_{P_{\Delta_i^{(k)}}}\big)$. $\square$

Proof of Theorem 2.8. For each $1 \le k \le m$, we have $gr_{(k)} = gr'_{(k)}$ by Proposition 4.3. Thus $\pi_k \circ gr = \pi_k \circ gr'$ by Lemma 4.5. That is, we have $gr_{[1,r]} = gr'_{[1,r]}$. Note $|gr(w,0)| = |gr'(w,0)| = \ell(w)$ and $|gr(\mathrm{id},\alpha^\vee)| = |gr'(\mathrm{id},\alpha^\vee)| = 2$ for any $\alpha \in \Delta$. Thus we have $|gr(w,\lambda)| = |gr'(w,\lambda)|$ for any $(w,\lambda) \in W \times Q^\vee$. Hence, the statement follows. □

For general Δ_P, the subset $\{gr(w,\lambda) \mid q_\lambda\sigma^w \in QH^*(G/B)\}$ of $\mathbb{Z}^{r+1}$, denoted as S by abuse of notation, turns out again to be a totally-ordered sub-semigroup of $\mathbb{Z}^{r+1}$. (The proof is similar to the one for Lemma 2.12 of [16] in the case when $Dyn(\Delta_P)$ is connected.) In the same way as in section 2.2, we obtain an S-family of subspaces of $QH^*(G/B)$; it naturally extends to a $\mathbb{Z}^{r+1}$-family, and induces graded vector subspaces. Namely, by abuse of notation, we have $\mathcal{F} = \{F_{\mathbf{a}}\}$ with $F_{\mathbf{a}} := \bigoplus_{gr(w,\lambda)\le \mathbf{a}} \mathbb{Q}q_\lambda\sigma^w$; $Gr^{\mathcal{F}}(QH^*(G/B)) := \bigoplus_{\mathbf{a}\in\mathbb{Z}^{r+1}} Gr^{\mathcal{F}}_{\mathbf{a}}$, where $Gr^{\mathcal{F}}_{\mathbf{a}} := F_{\mathbf{a}}/\sum_{\mathbf{b}<\mathbf{a}} F_{\mathbf{b}}$; for each $1 \le j \le r+1$, $Gr^{\mathcal{F}}_{(j)} := \bigoplus_{i\in\mathbb{Z}} Gr^{\mathcal{F}}_{i\mathbf{e}_j}$. In addition, we denote

$$\mathcal{I} := \bigoplus_{gr_{[r+1,r+1]}(w,\lambda)>\mathbf{0}} \mathbb{Q}q_\lambda\sigma^w \subset QH^*(G/B)$$

and

$$\mathcal{A} := \psi_{\Delta,\Delta_P}(QH^*(G/P)) \oplus \mathcal{J} \quad \text{where} \quad \mathcal{J} := F_{-\mathbf{e}_{r+1}}.$$

For each $1 \le j \le r$, $X_j := P_j/P_{j-1}$ is a Grassmannian (possibly of general type), and the quantum cohomology $QH^*(X_j)$ is therefore isomorphic to $H^*(X_j)\otimes\mathbb{Q}[t_j]$ as vector spaces. Note $X_{r+1} := P_{r+1}/P_r = G/P$.

Now we can restate Theorem 1.1 in the introduction more precisely as follows.

Theorem 4.6.

(1) *$QH^*(G/B)$ has an S-filtered algebra structure with filtration $\mathcal{F}$, which naturally extends to a $\mathbb{Z}^{r+1}$-filtered algebra structure on $QH^*(G/B)$.*

(2) *$\mathcal{I}$ is an ideal of $QH^*(G/B)$, and there is a canonical algebra isomorphism*

$$QH^*(G/B)/\mathcal{I} \xrightarrow{\simeq} QH^*(P/B).$$

(3) *$\mathcal{A}$ is a subalgebra of $QH^*(G/B)$ and $\mathcal{J}$ is an ideal of $\mathcal{A}$. Furthermore, there is a canonical algebra isomorphism (induced by ψ_{Δ,Δ_P})*

$$QH^*(G/P) \cong \mathcal{A}/\mathcal{J}.$$

(4) *There is a canonical isomorphism of $\mathbb{Z}^r \times \mathbb{Z}_{\ge 0}$-graded algebras:*

$$Gr^{\mathcal{F}}(QH^*(G/B)[q_1^{-1},\cdots,q_r^{-1}]) \xrightarrow{\cong} \Big(\bigotimes_{j=1}^{r} QH^*(X_j)[t_j^{-1}]\Big) \bigotimes Gr^{\mathcal{F}}_{(r+1)}.$$

There is also an injective morphism of graded algebras:

$$\Psi_{r+1}: \quad QH^*(G/P) \hookrightarrow Gr^{\mathcal{F}}_{(r+1)},$$

well defined by $q_{\lambda_P}\sigma^w \mapsto \overline{\psi_{\Delta,\Delta_P}(q_{\lambda_P}\sigma^w)}$. Furthermore, Ψ_{r+1} is an isomorphism if and only if either (a) *$\Delta^{(k)}$'s are all of A-type or* (b) *the only exception $\Delta^{(m)}$ is of B_2-type with α_r being a short simple root.*

Remark 4.7. *Say $\alpha_j \in \Delta^{(k)}$, i.e., $j = i + \sum_{p=1}^{k-1} r_p$ for some $1 \le i \le r_k$. Whenever $(k,i) \ne (m,r_m)$, we have $X_j \cong \mathbb{P}^i$. If $\Delta^{(m)} = \{\alpha_{r-1},\alpha_r\}$ is of type B_2 and α_r is a short simple root, then we have $X_{r-1} \cong \mathbb{P}^1$ and $X_r \cong \mathbb{P}^3$. In other words, Ψ_{r+1} is an isomorphism if and only if all X_j $(1 \le j \le r)$ are projective spaces.*

(Sketch) Proof of Theorem 4.6. The quantum cohomology ring $QH^*(G/B)$ is generated by the divisor Schubert classes $\{\sigma^{s_1}, \cdots, \sigma^{s_n}\}$. The well-known quantum Chevalley formula (see [10]) tells us $\sigma^u \star \sigma^{s_i} = \sum_\gamma \langle\omega_i, \gamma^\vee\rangle \sigma^{us_\gamma} + \sum_\gamma \langle\omega_i, \gamma^\vee\rangle q_{\gamma^\vee}\sigma^{us_\gamma}$ where $u \in W$ is arbitrary, and $\{\omega_1, \cdots, \omega_n\}$ denote the fundamental weights.

To show (1), it suffices to use induction on $\ell(u)$ and the positivity of Gromov-Witten invariants $N_{u,v}^{w,\lambda}$, together with the Key Lemma of [16] for general Δ_P. Namely, we need to show $gr(us_\gamma, 0) \leq gr(u, 0) + gr(s_i, 0)$ (resp. $gr(us_\gamma, \gamma^\vee) \leq gr(u, 0) + gr(s_i, 0)$) whenever the corresponding coefficient $\langle\omega_i, \gamma^\vee\rangle \neq 0$. Under this hypothesis, the expected inequality will hold if we replace "gr" by "$gr_{(k)}$", due to the Key Lemma of [16] which works for any $\Delta^{(k)}$. Therein the proof of the Key Lemma is most complicated part of the paper. We used the notion of virtual null coroot to do some reductions, but still had to do a big case by case analysis. Hence, the expected inequality holds if we replace "gr" by "$\pi_k \circ gr$" (for any $1 \leq k \leq m$), due to Lemma 4.5. That is, it holds when we replace "gr" by "$gr_{[1,r]}$". Thus the expected inequality holds by noting that $|gr(us_\gamma, 0)|$ (resp. $|gr(us_\gamma, \gamma^\vee)|$) is equal to $|gr(u, 0)| + |gr(s_i, 0)|$.

The proof of (2) is exactly the same as the proof of Theorem 1.3 in [16]. The quotient P/B is again a complete flag variety, and therefore teh quantum cohomology $QH^(P/B)$ is generated by the special Schubert classes σ^{s_i}, $i = 1, \cdots, r$. The prove is done by showing that $QH^*(G/B)/\mathcal{I}$ is generated by $\overline{\sigma^{s_i}}$, $i = 1, \cdots, r$, respecting the same quantum Chevalley formula.

Statement (3) is in fact a consequence of (4).

The proof of (4) is similar to the above one for (1). Namely we reduce gr to $\pi_k \circ gr = \pi_k \circ gr_{(k)}$. The expected statement will hold with respect to $gr_{(k)}$, by using either the corresponding results of [16] for $\Delta^{(k)}$ of type A or Theorem 2.4 when $\Delta^{(k)}$ is not of type A. The proof of the former case is much simpler than the latter one, although the ideas are similar. Here we also need to use the same observation that $|gr(w, \lambda)| = |gr(u, 0)| + |gr(v, 0)|$ whenever the Gromov-Witten invariant $N_{u,v}^{w,\lambda}$ in the quantum product $\sigma^u \star \sigma^v$ is nonzero. □

5. Appendix

5.1. **Proof of Lemma 3.17 (Continued).** Recall $\varepsilon_j = -\langle\alpha_j, \lambda\rangle \geq 0$, $j = 1, \cdots, r$. $gr_{[r,r]}(q_r) = x\mathbf{e}_r$, $gr_{[r,r]}(q_{r+1}) = y\mathbf{e}_r$, $gr_{[r,r]}(q_{r+2}) = z\mathbf{e}_r$. Define $(c_1, \cdots, c_r)$ by

$$\sum_{\beta \in R_{P_r} \setminus R_{P_{r-1}}} \beta = -y \sum_{i=1}^{r} c_i \alpha_i,$$

which are described in Table 5 by direct calculations. Therein we recall that the case C9) with $r = 2$ has been excluded from the discussion. By definition, we have

$$gr_{[r,r]}(q_\lambda) = (xa_r + ya_{r+1} + za_{r+2})\mathbf{e}_r = (y \sum_{j=1}^{r} c_j \varepsilon_j)\mathbf{e}_r.$$

Therefore, if $\varepsilon_r > 0$, then we have

$$-(xa_r + ya_{r+1} + za_{r+2}) = (-y) \cdot \sum_{j=1}^{r} c_j \varepsilon_j \geq (-y) \cdot c_r \cdot 1 \geq |R_P^+| - |R_{\tilde{P}}^+| \geq |R_P^+| - |R_{\tilde{P}}^+ \cup R_{\hat{P}}^+|.$$

TABLE 5.

		$(c_1,\cdots,c_r)$	(x,y,z)	$-yc_r$	$\lvert R_P^+\rvert-\lvert R_{\tilde P}^+\rvert$
C4)	$r=6$	$(1,2,3,2,1,2)$	$(11,-11,0)$	22	21
	$r=7$	$(1,2,3,4,\frac{8}{3},\frac{4}{3},\frac{7}{3})$	$(14,-21,0)$	49	42
C5)		$(\frac{2}{5},\frac{4}{5},\frac{6}{5},\frac{3}{5},1)$	$(2r-2,\frac{(1-r)r}{2},1-r)$	$\frac{r(r-1)}{2}$	$\frac{r(r-1)}{2}$
C7)	$r=4$	$(\frac{1}{2},1,\frac{1}{2},1)$			
	$r=5$	$(\frac{2}{5},\frac{4}{5},\frac{6}{5},\frac{3}{5},1)$			
	$r=6$	$(\frac{1}{3},\frac{2}{3},1,\frac{4}{3},\frac{2}{3},1)$			
	$r=7$	$(\frac{2}{7},\frac{4}{7},\frac{6}{7},\frac{8}{7},\frac{10}{7},\frac{5}{7},1)$			
C9)	$r=3$	$(\frac{1}{3},\frac{2}{3},1)$	$(6,-9,0)$	9	6
C10)		$(\frac{2}{3},\frac{4}{3},1)$	$(4,-6,0)$	6	6

If case C9) with $r=3$ occurs, we are done. If $-gr_{[r,r]}(q_\lambda)=(|R_P^+|-|R_{\tilde P}^+\cup R_{\hat P}^+|)\mathbf{e}_r$ held, then C5) or C7) occurs and all the above inequalities are equalities. This implies that $\varepsilon_r=1$ and $\varepsilon_j=0, j=1,\cdots,r-1$. Therefore we have $\langle\gamma,\lambda\rangle\in\{0,-1\}$ for any $\gamma\in R_P^+$, by noting that $\gamma=\varepsilon\alpha_r+\sum_{i=1}^{r-1}c_i\alpha_i$ (where $\varepsilon\in\{0,1\}$) for all these three cases. That is, $\lambda=\lambda_B$ is the Peterson-Woodward lifting of λ_P, contradicting with the hypothesis. Hence, the statement follows if $\varepsilon_r>0$.

Assume now $\varepsilon_r=0$. Since $\lambda\neq\lambda_B$, we have $\varepsilon_j=0$ for all $\alpha_j\in\Delta_P$ but exactly one exception, say α_k. In addition, we have

$$-(xa_r+ya_{r+1}+za_{r+2})=-yc_k\varepsilon_k=-yc_k,$$

together with the property that the coefficient θ_k in the highest root $\theta=\sum_{i=1}^r\theta_i\alpha_i$ of R_P^+ is not equal to 1. (Otherwise, λ would be the Peterson-Woodward lifting of λ_P, contradicting with the hypothesis.) Thus if $c_k\geq c_r$, then we have

$$-(xa_r+ya_{r+1}+za_{r+2})=-yc_k\geq -yc_r\geq|R_P^+|-|R_{\tilde P}^+|>|R_P^+|-|R_{\tilde P}^+\cup R_{\hat P}^+|.$$

Here the last inequality holds since $\alpha_r\in R_{\hat P}^+\setminus R_{\tilde P}^+$. If $c_k<c_r$, then all possible k, together with $-yc_k$ and the number $|R_P^+|-|R_{\tilde P}^+\cup R_{\hat P}^+|=|R_P^+|-|R_{\tilde P}^+|-|R^+_{P_{\Delta_P\setminus\{\alpha_k\}}}|+|R^+_{P_{\Delta_{\tilde P}\setminus\{\alpha_k\}}}|$, are precisely given in Table 6 by direct calculations. In particular, we also have $-(xa_r+ya_{r+1}+za_{r+2})=-yc_k>|R_P^+|-|R_{\tilde P}^+\cup R_{\hat P}^+|$.

5.2. **Proof of Proposition 3.3 (3) (Continued).** For each case, we uniformly denote those (q_λ,u) in Table 4 in order as (q_{λ_i},u_i)'s, and denote by $\tilde u_i$ the minimal length representative of $u_iW_{\tilde P}$ as before (i.e., $\tilde u_i$ is given by a subexpression s_L of u_i with the sequence ending with r). We also denote those virtual null coroot(s) μ_B in Table 3 in order as μ_1,μ_2. Namely if there is a unique μ_B, then we denote $\mu_1=\mu_2=\mu_B$ for convenience.

Due to Lemma 3.11 again, it suffices to show all the equalities in Table 7 hold in $Gr^{\mathcal F}(QH^*(G/B))$ for the corresponding cases. Note $\overline{\sigma^{u_i}}\star\overline{\sigma^{u_j}}=\overline{\sum N_{u_i,u_j}^{w,\eta}q_\eta\sigma^w}$ where $gr(q_\eta\sigma^w)=gr(\sigma^{u_i})+gr(\sigma^{u_j})$. Consequently, we have $w\in W_P$, $\eta=\sum_{k=1}^r b_k\alpha_k^\vee$ and $\ell(\tilde u_i)+\ell(\tilde u_j)=b_r|gr_{[r,r]}(q_r)|+|gr_{[r,r]}(\sigma^w)|$. Since $0\leq|gr_{[r,r]}(\sigma^w)|\leq|R_P^+|-|R_{\tilde P}^+|$, we have $b^{\max}\geq b_r\geq b^{\min}\geq 0$ for certain integers $b^{\max},b^{\min}$. Write

TABLE 6.

		k	$-yc_k$	$\lvert R_P^+\rvert - \lvert R_{\tilde{P}}^+ \cup R_{\hat{P}}^+\rvert$
C4)	$r=7$	2	42	32
		6	28	27
C5)		2	8	7
C7)	$r=5$	2	8	7
	$r=6$	2	10	9
	$r=7$	2	12	11
		3	18	15
C9)	$r=3$	2	6	5
C10)		1	4	3

$w = vw_2$ where $v \in W_P^{\tilde{P}}$ and $w_2 \in W_{\tilde{P}}$. Once b_r is given, both $\ell(v)$ and (w_2, η) are fixed by the above equalities on gradings together with Lemma 3.12. There is a unique term, say $\overline{q_\vartheta \sigma^{\check{w}}}$, on the right hand side of each expected identity in Table 7, where $\check{w} = \mathrm{id}$, u_2 or u_3. It is easy to check that $q_\vartheta \sigma^{\check{w}}$ is of expected grading with $b_r(\vartheta) = b^{\max}$. Thus if $b_r(\eta) = b^{\max}$, then we have $\eta = \vartheta$ and $w = v\check{w}_2$ with $\ell(v) = \ell(\tilde{v})$. Here $\tilde{v} \in W_P^{\tilde{P}}$ and $\check{w}_2 \in W_{\tilde{P}}$ are given by $\check{w} = \tilde{v}\check{w}_2$. In particular, we have $w = \mathrm{id}$ if $\check{w} = \mathrm{id}$. Hence, in order to conclude the expected equality, it suffices to show

1. $N_{u_i,u_j}^{v\check{w}_2,\vartheta} = 1$ if $v = \tilde{v}$, or 0 otherwise;
2. $N_{u_i,u_j}^{w,\eta} = 0$ whenever $b^{\max} > b_r(\eta) \geq b^{\min}$. Similar to the proof in section 3.4, this claim follows from the next two:
   - (a) $\sigma^{\tilde{u}_i} \cup \sigma^{\tilde{u}_j} = 0$ in $H^*(P/B)$;
   - (b) $N_{u_i,u_j}^{w,\eta} = 0$ whenever $b^{\max} > b_r(\eta) \geq \max\{1, b^{\min}\}$.

TABLE 7.

		Equalities	$N_{u_i,u_j}^{v\check{w}_2,\vartheta}$	q_η
C4)	$r=7$	$\overline{\sigma^{u_1}} \star \overline{\sigma^{u_1}} = \overline{q_{\mu_1-2\lambda_1}}$	$N_{u_1,\mathrm{id}}^{u_1^{-1},0}$	$q_7q_1^2q_2^2q_3^2q_4^2q_5$, $q_7^2q_1^2q_2^2q_3^2q_4^3q_5^2q_6$
C7)	$r=4;6$			q_4; $q_6^2q_3q_4^2q_5$
C10)				$q_3^2q_2$
C5), C7)		$\overline{\sigma^{u_2}} \star \overline{\sigma^{u_2}} = \overline{q_{\mu_2-2\lambda_2}}$	$N_{u_2,\mathrm{id}}^{u_2^{-1},0}$	$\emptyset$
		$\overline{\sigma^{u_1}} \star \overline{\sigma^{u_2}} = \overline{q_{\lambda_3-\lambda_1-\lambda_2}\sigma^{u_3}}$	$N_{u_1s_rs_{r-2}\cdots s_2s_1,s_1\cdots s_{r-1}}^{v\check{w}_2,\,0}$	
C5)		$\overline{\sigma^{u_1}} \star \overline{\sigma^{u_1}} = \overline{q_{\lambda_2-2\lambda_1}\sigma^{u_2}}$	$N_{s_{r-1}\cdots s_1,s_{r-1}\cdots s_1}^{v\check{w}_2,0}$	$\emptyset$
C7)	$r=7$			$q_7^2q_4q_5^2q_6$
	$r=5$	$\overline{\sigma^{u_1}} \star \overline{\sigma^{u_3}} = \overline{q_{\mu_1-\lambda_1-\lambda_3}}$	$N_{u_1,\mathrm{id}}^{u_3^{-1},0}$	$q_5q_3q_4$
C4)	$r=6$	$\overline{\sigma^{u_1}} \star \overline{\sigma^{u_2}} = \overline{q_{\mu_1-\lambda_1-\lambda_2}}$	$N_{u_1,\mathrm{id}}^{u_2^{-1},0}$	$q_6q_1q_2q_3q_4q_5$
		$\overline{\sigma^{u_1}} \star \overline{\sigma^{u_1}} = \overline{q_{\lambda_2-2\lambda_1}\sigma^{u_2}}$	$N_{s_{54362345},s_{54362345}}^{v\check{w}_2,0}$	$\emptyset$

To show (1), we use Proposition 3.15 (2) repeatedly. As a consequence, we can conclude that $N_{u_i,u_j}^{v\check{w}_2,\vartheta}$ coincides with a classical intersection number given in Table 7 as well, which is of the form either $N_{u_i,\mathrm{id}}^{u_j^{-1},0}$ or $N_{u',v'}^{v\check{w}_2,0}$. The formal one is equal to 1 by checking $u_i = u_j^{-1}$ easily. Denote $\Delta_{\check{P}} := \Delta_P \setminus \{\alpha_k\}$. For the latter one, it is easy to check that both u', v' are in $W_P^{\check{P}}$, where s_k denotes the last simple reflection in the reduced expression of $\check{w}_2$. Thus $N_{u',v'}^{v\check{w}_2,0} = 0$ unless $v\check{w}_2$ is in $W_P^{\check{P}}$ as well. In addition, it is easy to check that $\ell(u') + \ell(v') = \ell(v\check{w}_2) = \dim P/\check{P}$, and that u' is the minimal length representative of $W_{\check{P}} = w_P v' W_{\check{P}}$. Thus u' is dual to v' with respect to the canonical non-degenerated bilinear form on $H^*(P/\check{P})$. Hence, $N_{u',v'}^{v\check{w}_2,0} = 1$. (See e.g. section 3 of [10] for these well-known facts.) That is, (1) follows. To illustrate the above reduction more clearly, we give a little bit more details for $N_{u_1,u_1}^{v\check{w}_2,\vartheta}$ in case C4) with $r = 6$. In this case, we have $u_1 = s_{54362132436}s_5s_4s_3s_2s_1$ and $q_\vartheta = q_{\lambda_2-2\lambda_1} = q_1^2q_2^2q_3^2q_4q_6$. Proposition 3.15 (2) $u_1 = s_{54362132436}s_5s_4s_3s_2s_1$ and $q_\vartheta = q_{\lambda_2-2\lambda_1} = q_1^2q_2^2q_3^2q_4q_6$. Using Proposition 3.15 (2), we can first deduce $N_{u_1,u_1}^{v\check{w}_2,\vartheta} = N_{u_1,s_{54362345}}^{v\check{w}_2s_{12346321},0}$. Denote $u' := s_{54362345}$. Note $\Delta_{\check{P}} = \{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_6\}$, and $u'(\alpha) \in R^+$ for all $\alpha \in \Delta_{\check{P}}$. Thus we can further deduce that $N_{u_1,u'}^{v\check{w}_2s_{12346321},0} = N_{u_1s_{12364321},u'}^{v\check{w}_2,0}$. That is, we have $N_{u_1,u_1}^{v\check{w}_2,\vartheta} = N_{u',u'}^{vs_{12345},0}$. Note $\dim P/\check{P} = 36 - 20 = 16 = \ell(\check{w})$ and $\check{w} = s_{12346325436}s_{12345} \in W_P^{\check{P}}$. That is, $\check{w}$ is the (unique) longest element in $W_P^{\check{P}}$. Thus $N_{u',u'}^{vs_{12345},0} = 0$ unless $v\check{w}_2 = \check{w}$. Note $w_{\check{P}} = s_{436213243612341231 21}$, $\ell(u'w_{\check{P}}(u')^{-1}) \le 2\ell(u') + \ell(w_{\check{P}}) = 36 = |R_P^+|$, and $u'w_{\check{P}}(u')^{-1}(\alpha) \in -R^+$ for all $\alpha \in \Delta_P$. Thus $w_P = u'w_{\check{P}}(u')^{-1}$. That is, $w_Pu' = u'w_{\check{P}}$. In other words, $\sigma^{u'}$ is dual to itself in $H^*(P/\check{P})$. Hence, $N_{u',u'}^{\check{w},0} = 1$.

To show (a), we note $\tilde{u}_i \in W_P^{\tilde{P}}$. Thus in $H^*(P/B)$, we have $N_{\tilde{u}_i,\tilde{u}_j}^{w',0} = 0$ unless $w' \in W_P^{\tilde{P}}$. Consequently, we have $\sigma^{\tilde{u}_i} \cup \sigma^{\tilde{u}_j} = 0$ in $H^*(P/B)$ if $\ell(\tilde{u}_i) + \ell(\tilde{u}_j) > \dim P/\tilde{P} = |R_P^+| - |R_{\tilde{P}}^+|$. Hence, (a) follows immediately from either the above inequality or the combination of Lemma 3.27 and Corollary 3.28, except for the case when C4) with $r = 7$ occurs. In this exceptional case, we note $\ell(\tilde{u}_1) = 21 = \frac{1}{2}\dim P/\tilde{P}$. Hence, we have $\sigma^{\tilde{u}_1} \cup \sigma^{\tilde{u}_1} = N_{\tilde{u}_1,\tilde{u}_1}^{w',0}\sigma^{w'}$ in $H^*(P/B)$, where w' denotes the longest element in $W_P^{\tilde{P}}$. Equivalently, the same equality holds in $H^*(P/\tilde{P})$, once we treat the Schubert classes $\sigma^{\tilde{u}_1}, \sigma^{w'}$ as elements $H^*(P/\tilde{P})$ canonically. Then we have $N_{\tilde{u}_1,\tilde{u}_1}^{w',0}\sigma^{w'} = \int_{[P/\tilde{P}]}\sigma^{\tilde{u}_1} \cup \sigma^{\tilde{u}_1} \cup \sigma^{\mathrm{id}} = N_{\tilde{u}_1,\mathrm{id}}^{u'',0} = 1$ if $\tilde{u}_1 = u''$, or 0 otherwise. Here u'' is the minimal length representative of the coset $w_P\tilde{u}_1W_{\tilde{P}}$. Note $\tilde{u}_1$ maps $R_{\tilde{P}}^+$ to R_P^+, and w_P maps R_P^+ to $-R_P^+$. Hence, we have $w_P\tilde{u}_1 = u''w_{\tilde{P}}$, i.e., $w_P = u''w_{\tilde{P}}(\tilde{u}_1)^{-1}$. By direct calculation, we have $\tilde{u}_1w_{\tilde{P}}(\tilde{u}_1)^{-1}(\alpha_1) = \alpha_1 + \cdots + \alpha_5 \notin -R^+$. Thus $\tilde{u}_1 \neq u''$ and (a) follows. Note $\tilde{u}_1 \leqslant w_P\tilde{u}_1 = u''w_{\tilde{P}}$ if and only if $\tilde{u}_1 \leqslant u''$. Since $\ell(u'') = \ell(u''w_{\tilde{P}}) - \ell(w_{\tilde{P}}) = \ell(w_P\tilde{u}_1) - \ell(w_{\tilde{P}}) = (63-21) - 21 = 21 = \ell(\tilde{u}_1)$ and $\tilde{u}_1 \neq u''$, we can further conclude $\tilde{u}_1 \nleqslant w_P\tilde{u}_1$.

It remains to show (b). All the coroots η satisfying the hypothesis of (b) are given in terms of q_η in the last column of Table 7 if it exists, or "$\emptyset$" otherwise.

1) For case C4) with $r = 7$, we note $\mathrm{sgn}_7(u_1) = 0$. If $q_\eta = q_7^2q_1^2q_2^2q_3^2q_4^3q_5^2q_6$, then we have $\langle\alpha_7, \eta\rangle = 1$, and consequently $N_{u_1,u_1}^{w,\eta} = 0$ by Proposition 3.15 (1). Similarly, for C4) with $r = 6$ we have $N_{u_1,u_2}^{w,\eta} = 0$ by considering sgn_6.
2) For case C4) with $r = 7$ and $q_\eta = q_7q_1^2q_2^2q_3^2q_4^2q_5$, we can first conclude $N_{u_1,u_1}^{w,\eta} = N_{u_1,u'}^{ws_{1234574321},0}$ by using Proposition 3.15 (2) repeatedly. Here

$u' := u_1 s_{123474321} = s_{12347543654723456}$. Denote $\Delta_{\check{P}} := \Delta_P \setminus \{\alpha_6\}$, and note $u' \in W_P^{\check{P}}$. Thus we can further conclude $N_{u_1,u_1}^{w,\eta} = N_{u',u'}^{w,0}$. Since $2\ell(u') = 34 > 63 - 30 = \dim P/\check{P}$, we have $N_{u',u'}^{w,0} = 0$. Similarly, (b) follows if case C7) occurs.

3) For case C10), we can conclude $N_{u_1,u_1}^{w,\eta} = N_{s_{321},s_{321}}^{w,0}$. Therefore it is equal to 0, by noting $(s_{321})^{-1} w_P = s_{321} s_{321} \not\geq (s_{321})^{-1}$ and using Lemma 3.27.

Hence, (b) follows.

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