APPENDIX TO THE PAPER "FUNCTORIAL RELATIONSHIPS BETWEEN $QH^*(G/B)$ AND $QH^*(G/P)$"

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The present note is written as the appendix to [1]. That is, we want to show

Key Lemma. *Let $u \in W$ and $\gamma \in R^+$.*

b) *If $\ell(us_\gamma) = \ell(u) + 1 - \langle 2\rho, \gamma^\vee \rangle$, then we have $gr(q_{\gamma^\vee} us_\gamma) \leq gr(u) + gr(s_i)$ whenever the fundamental weight χ_i satisfies $\langle \chi_i, \gamma^\vee \rangle \neq 0$.*

holds for $\gamma \in R^+ \setminus \Delta$ when Δ_P is not of A-type and Δ is of F_4-type or E-type, together with γ satisfying the following condition:

(ii) whenever $\beta_p \in \Delta$ satisfies $\langle \beta_p, \gamma^\vee \rangle > 0$, we have $p \in \{o, \kappa + 1, \kappa + 2\}$. In this case, we note the constrain $\ell(s_\gamma) = \langle 2\rho, \gamma^\vee \rangle - 1$ on γ.

For completeness, we include the appendix in [1] and do more explanations. We will also recall the order (Δ_P, Υ) for each case that we need to discuss.

Recall the following lemma in section 3.5 of [1]:

Lemma 1. *Let $u \in W$ and $\gamma \in R^+ \setminus R_P$. Write $gr(q_{\gamma^\vee}) = \sum_{j=1}^{r+1} d_j \mathbf{e}_j$. Then Key Lemma* b) *holds, if either of the followings holds.*

a) $\sum_{j=1}^{r} d_j \leq -\ell(\omega_P \omega_{\bar{P}})$, *where $\Delta_{\bar{P}} := \{\alpha \in \Delta_P \mid \langle \alpha, \gamma^\vee \rangle = 0\}$.*

b) $\sum_{j=1}^{r} d_j \leq |\Xi_1| - |\Xi_2|$, *where $\Xi_1 := \{\alpha \in R_P^+ \mid \langle \alpha, \gamma^\vee \rangle > 0\}$ and $\Xi_2 := \{\alpha \in R_{\hat{P}}^+ \setminus R_P \mid \alpha - \gamma \in R^+, \langle \alpha, \gamma^\vee \rangle > 0\}$ with $\Delta_{\hat{P}} := \Delta_P \cup \{\alpha_i \in \Delta \mid \langle \chi_i, \gamma^\vee \rangle \neq 0\}$.*

Comments:

i) By method "(M1)", we mean Lemma 1 a) can be used. That is, given $\gamma \in R^+$, we compute $\sum_{j=1}^{r} d_j$, $\Delta_{\bar{P}}$ and consequently $\ell(\omega_P \omega_{\bar{P}})$. For such root we do have $\sum_{j=1}^{r} d_j \leq -\ell(\omega_P \omega_{\bar{P}})$.

ii) By method "(M3)", we mean Lemma 1 b) can be used. That is, given $\gamma \in R^+$, we compute $\sum_{j=1}^{r} d_j$, $|\Xi_1|$ and $|\Xi_2|$. For such root we do have $\sum_{j=1}^{r} d_j \leq |\Xi_1| - |\Xi_2|$.

In order to describe method "(M2)" precisely, we repeat some arguments, in which the induction hypothesis is involved.

Assume $\gamma \notin \Delta$. Take any simple root α_j satisfying $\langle \alpha_j, \gamma^\vee \rangle > 0$, and write $\beta = s_j(\gamma)$, $gr(q_{\beta^\vee}) = (\lambda_1, \cdots, \lambda_{r+1}), \min\{gr(s_i) \mid \langle \chi_i, \beta^\vee \rangle \neq 0\} = \mathbf{e}_c$ and

$$gr(q_j) + gr(us_j) = gr(u) + (a_1, \cdots, a_{r+1}),$$
$$gr(q_{\beta^\vee}) + gr(us_j s_\beta) = gr(us_j) + \mathbf{e}_c + (\mu_1, \cdots, \mu_{r+1}),$$
$$gr(us_j s_\beta s_j) = gr(us_j s_\beta) + (b_1, \cdots, b_{r+1}).$$

In addition, we use the notations $\tilde{c}$, $\tilde{a}_j$'s, $\tilde{b}_j$'s and $\tilde{\mu}_j$'s, whenever replacing "gr" with "$\tilde{gr}$" in the above three equalities. Write $gr(u) = (i_1, \cdots, i_{r+1}), gr(us_j) = (i'_1, \cdots, i'_{r+1}), gr(us_j s_\beta) = (k_1, \cdots, k_{r+1})$ and $gr(us_\gamma) = gr(us_j s_\beta s_j) = (k'_1, \cdots, k'_{r+1})$.

Lemma 2. *Assume we can take $\alpha_j = \alpha_r$. That is, we have $\langle \alpha_r, \gamma^\vee\rangle > 0$ and consequently $\beta := s_r(\gamma)$. Assume the induction hypothesis $(\mu_1, \cdots, \mu_{r+1}) \leq (0, \cdots, 0)$.*

$$\text{Denote} \quad m_t := \sharp\{\alpha \in R^+_{P_t} \setminus R_{P_{t-1}} \mid \langle \alpha, \beta^\vee\rangle > 0\}.$$

Then Key Lemma b) *holds for given $\gamma \in R^+ \setminus R_P$, if*

$$m_t \leq \lambda_t, \text{ for all } p \leq t \leq r-1,$$

in which $p \in \{1, \cdots, r\}$ satisfies $\langle \alpha_p, \alpha_r\rangle \neq 0$ and $\langle \alpha_k, \alpha_r^\vee\rangle = 0$ for all $1 \leq k \leq p-1$.

Proof. From the assumptions, we note that $\min\{gr(s_i)|\langle \chi_i, \beta^\vee\rangle \neq 0\} = \mathbf{e}_{r+1}$ and $\min\{\tilde{gr}(s_i)|\langle \chi_i, \beta^\vee\rangle \neq 0\} = \mathbf{e}_r$. Hence, we do not need to care about $\mathbf{e}_c$ or $\mathbf{e}_{\tilde{c}}$. That is, we have $\min\{gr(s_i)|\langle \chi_i, \gamma^\vee\rangle \neq 0\} = \min\{gr(s_i)|\langle \chi_i, \beta^\vee\rangle \neq 0\}$ and $\min\{\tilde{gr}(s_i)|\langle \chi_i, \gamma^\vee\rangle \neq 0\} = \min\{\tilde{gr}(s_i)|\langle \chi_i, \beta^\vee\rangle \neq 0\}$.

From the table on gradings in section 3.2 of [1], we note that $gr_p(q_r) = \mathbf{0}$. Thus we have $a_1 = \cdots = a_{p-1} = 0 = b_1 = \cdots = b_{p-1}$ and $a_{r+1} = b_{r+1} = 0$. Then we have $\sum_{t=p}^r a_t = 1$ and $\sum_{t=p}^r b_t = -1$, so that $\sum_{t=p}^r (a_t + b_t) = 0$. By the induction hypothesis, we conclude $\sum_{t=1}^{p-1} \mu_t \mathbf{e}_t \leq \mathbf{0}$. Therefore if $\sum_{t=1}^{p-1} \mu_t \mathbf{e}_t < \mathbf{0}$, then it is already done.

We remain to assume $\sum_{t=1}^{p-1} \mu_t \mathbf{e}_t = \mathbf{0}$. Then we have $\sum_{t=p}^r \mu_t \mathbf{e}_t \leq \mathbf{0}$ by the induction hypothesis again. In particular $\mu_p \leq 0$. Note that $\lambda_t + k_t = i'_t + \mu_t$ for each $p \leq t \leq r$. Furthermore we have the decomposition of us_r (resp. $us_r s_\beta$) associated to (Δ_P, Υ), given by $us_r = v_{r+1} v_r u^{(r-1)}_{i'_{r-1}} \cdots u^{(1)}_{i'_1}$ (resp. $us_r = \tilde{v}_{r+1}\tilde{v}_r u^{(r-1)}_{k_{r-1}} \cdots u^{(1)}_{k_1}$). In particular we note for each $p \leq t \leq r-1$, $i'_t = |A_t|$ and $k_t = |B_t|$ where $A_t := \{\alpha \in R^+_{P_t} \setminus R_{P_{t-1}} \mid us_r(\alpha) \in -R^+\}$ and $B_t := \{\alpha \in R^+_{P_t} \setminus R_{P_{t-1}} \mid us_j s_\beta(\alpha) \in -R^+\}$.

Clearly, we can write A_t as the following disjoint union

$$\left\{\alpha \in R^+_{P_t} \setminus R_{P_{t-1}} \;\middle|\; \begin{array}{l} us_r(\alpha) \in -R^+ \\ \langle \alpha, \beta^\vee\rangle \leq 0 \end{array}\right\} \bigsqcup \left\{\alpha \in R^+_{P_t} \setminus R_{P_{t-1}} \;\middle|\; \begin{array}{l} us_r(\alpha) \in -R^+ \\ \langle \alpha, \beta^\vee\rangle > 0 \end{array}\right\}$$

the sets in which we denote as C_t and D_t respectively. Obviously, $|D_t| \leq m_t$. Furthermore we have $us_r(\beta) \in -R^+$ (since $\ell(us_r s_\beta) < \ell(us_r)$). Thus $C_t \subset B_t$ and consequently we have $|C_t| \leq |B_t| = k_t$. Hence, $i'_t = |A_t| = |C_t| + |D_t| \leq k_t + m_t$. Then by the condition, we deduce that $\mu_t = \lambda_t + k_t - i'_t \geq m_t + k_t - i'_t \geq 0$ for each $p \leq t \leq r-1$. Noting that $\mu_p \leq 0$, we have $\mu_p = 0$ and consequently $\mu_{p+1} \leq 0$ by the induction hypothesis. Therefore we conclude $\mu_t = 0$ for all $p \leq t \leq r-1$ by induction. Consequently $\mu_r \leq 0$.

On the other hand by considering $\tilde{gr}$, we conclude $\sum_{t=p}^{r-1}(a_t + b_t)\mathbf{e}_= \sum_{t=p}^{r-1}(a_t + b_t + \mu_t)\mathbf{e}_t = \mathbf{0} = \sum_{t=p}^{r-1}(\tilde{a}_t + \tilde{b}_t + \tilde{\mu}_t)\mathbf{e}_t \leq \mathbf{0}$. If "<" holds, then it is done. Otherwise, "=" will hold. In particular, we have $\sum_{t=p}^{r-1}(a_t + b_t) = 0$, by noting $\sum_{t=p}^{r}(a_t + b_t) = 0$. Consequently, we have $a_r + b_r = 0$ and then $\sum_{t=1}^r (a_t + b_t + \mu_t)\mathbf{e}_t = \mu_r \mathbf{e}_r \leq \mathbf{0}$.

Hence, the statement follows. □

Comments:

iii) By mean "(M2)", we mean Lemma 2 can be used for the given $\gamma \in R^+ \setminus R^+_P$. Note that $\{\alpha_1, \cdots, \alpha_{r-1}\}$ is always of A-type. Thus it is easy to compute all m_t's for $t \leq r-1$. Furthermore, "p" can be easily determined; $\langle \alpha_r, \gamma^\vee\rangle$ and $gr(q_{\beta^\vee}) = (\lambda_1, \cdots, \lambda_{r+1})$ can also be calculated easily. Hence, (M2) can be easily carried out.

iv) Since (M1) and (M3) does not use the induction hypothesis, we would like to test the roots in the following order:

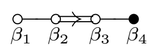
$$(M0) \longrightarrow (M1) \longrightarrow (M3) \longrightarrow (M2),$$

where by "(M0)" we mean we first test whether a given root can be reduced to the cases that have been discussed (i.e. the case when condition (i) in the proof of Key Lemma is satisfied). Since we use computer to do these calculations, we would like to include some more roots for convenience so that we need (M0). Furthermore, when $(M0)$ is used, we will point out how we reduce it to the known cases.

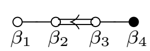
Later we will give all the outputs, which we obtain by using the software Mathematica. From the results we note that for all of the roots one of the methods (M0), (M1), (M3) and (M2) can be used, except for two special cases. In the following, we first give the tables case by case, in which the corresponding methods are listed and in particular by "**done**" we mean (M0) is used. Then we point out the two exceptional cases and show for them Key Lemma still hold. In fact, they are not "exceptional", in the sense we also using the induction hypothesis and part of the proof of Lemma 2, which are the spirit of (M2). In the end, we will show the outputs, produced by Mathematica.

Recall that $\alpha_i = \beta_{o+i}$ for each $1 \leq i \leq r$ where $o \geq 0$, and that $\kappa = o + \varsigma = o + r - 1$.

When Δ is of F_4-type, case C9) or C10) will occur.
If case C9) occurs, then we have



in which $\kappa = 2$; that is, $\alpha_r = \beta_3$, $\alpha_{r-1} = \beta_2, \cdots, \alpha_1 = \beta_{o+1}$. In fact we have $r = 2$ or $r = 3$. In this case, $\gamma^\vee$ must be either of the form $\sum_{i \leq t \leq k} \beta_t^\vee$ or equal to one of the following five coroots: $\beta_1^\vee + 2\beta_2^\vee + \beta_3^\vee$, $\beta_1^\vee + 2\beta_2^\vee + \beta_3^\vee + \beta_4^\vee$, $\beta_1^\vee + 2\beta_2^\vee + 2\beta_3^\vee + \beta_4^\vee$, $\beta_1^\vee + 3\beta_2^\vee + 2\beta_3^\vee + \beta_4^\vee$, $2\beta_1^\vee + 3\beta_2^\vee + 2\beta_3^\vee + \beta_4^\vee$, by noting $\ell(s_\gamma) = \langle 2\rho, \gamma^\vee \rangle - 1$.
If case C10) occurs, then we have



in which we also have $\kappa = 2$; that is, $\alpha_r = \beta_3$, $\alpha_{r-1} = \beta_2, \cdots, \alpha_1 = \beta_{o+1}$. $r = 2$ or $r = 3$. (The roots are obtained from case C9) by reversing the order of β_j's.)

Table for case C9)

Coroots	$r=2$	$r=3$
$\beta_2^\vee + \beta_3^\vee$	done ($\gamma \in R_P$)	
$\beta_3^\vee + \beta_4^\vee$	(M3)	
$\beta_1^\vee + \beta_2^\vee + \beta_3^\vee$	(M3)	done
$\beta_2^\vee + \beta_3^\vee + \beta_4^\vee$	(M1)	
$\beta_1^\vee + \beta_2^\vee + \beta_3^\vee + \beta_4^\vee$	(M1)	done
$\beta_1^\vee + 2\beta_2^\vee + 2\beta_3^\vee + \beta_4^\vee$	(M2)	
$2\beta_1^\vee + 3\beta_2^\vee + 2\beta_3^\vee + \beta_4^\vee$	(M1)	done

Table for case C10)

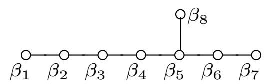
Coroots	$r=2$	$r=3$
$\beta_3^\vee + \beta_4^\vee$	(M3)	(M2)
$\beta_1^\vee + \beta_2^\vee + \beta_3^\vee$	(M1)	done
$\beta_1^\vee + \beta_2^\vee + \beta_3^\vee + \beta_4^\vee$	(M1)	done
$\beta_2^\vee + 2\beta_3^\vee + \beta_4^\vee$	(M3)	(M2)
$\beta_1^\vee + \beta_2^\vee + 2\beta_3^\vee + \beta_4^\vee$	(M2)	done
$\beta_1^\vee + 2\beta_2^\vee + 3\beta_3^\vee + \beta_4^\vee$	(M2)	
$\beta_1^\vee + 2\beta_2^\vee + 3\beta_3^\vee + 2\beta_4^\vee$	(M1)	

In order to compare it with the concrete arguments in [1], we would like to take the same example saying $\gamma^\vee = \beta_2^\vee + 2\beta_3^\vee + \beta_4^\vee$ with $r = 3$ and case C10) occurring.

In this case, we can take $\alpha_j = \beta_3 (= \alpha_3)$ so that $\beta^\vee = s_3(\gamma^\vee) = \beta_2^\vee + \beta_3^\vee + \beta_4^\vee$. Furthermore (using the notations in Lemma 2), we have $r - 1 = 2 = p$, $gr(q_{\beta_4^\vee}) = (-1, 1, -28)$ so that $\lambda_2 = 1$, and it is easy to see $m_2 = \sharp\{\alpha \in R^+_{P_2} \setminus R_{P_1} \mid \langle \alpha, \beta^\vee \rangle > 0\} = \sharp\{\beta_2\} = 1$. Hence, $m_2 = \lambda_2$. Thus Key Lemma b) holds for such γ by noting $p = r - 1 = 2$ and using Lemma 2.

Now we assume Δ is of E-type. Denote $\Xi := \{\beta_i \mid \langle \beta_i, \gamma^\vee \rangle > 0\}$. Note that any $\gamma \in R^+$ is of length $\langle 2\rho, \gamma^\vee \rangle - 1$. It suffices to assume $n = 8$. Then we remain to discuss at most the roots in the tables as below.

When case C4) occurs, we have

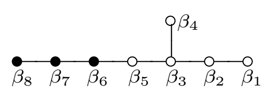


in which $r \in \{6, 7\}$, $\alpha_r = \beta_8, \alpha_{r-1} = \beta_7, \cdots, \alpha_1 = \beta_{o+1}$.

Table for case C4) with $r = 6$ or $r = 7$

Roots with $\Xi \subset \{\beta_1, \beta_2, \beta_8\}$	$r = 6$	$r = 7$
$\beta_1 + \beta_2$	done	done
$\beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_8$	(M3)	done
$\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_8$	done	(M3)
$\beta_1 + 2\beta_2 + 2\beta_3 + 2\beta_4 + 2\beta_5 + \beta_6 + \beta_8$	(M1)	done
$\beta_3 + 2\beta_4 + 3\beta_5 + 2\beta_6 + \beta_7 + 2\beta_8$	done ($\gamma \in R_P$)	
$\beta_2 + \beta_3 + 2\beta_4 + 3\beta_5 + 2\beta_6 + \beta_7 + 2\beta_8$	(M3)	done
$\beta_1 + \beta_2 + \beta_3 + 2\beta_4 + 3\beta_5 + 2\beta_6 + \beta_7 + 2\beta_8$	done	(M3)
$\beta_1 + 2\beta_2 + 2\beta_3 + 2\beta_4 + 3\beta_5 + 2\beta_6 + \beta_7 + 2\beta_8$	(M2)	done
$\beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4 + 5\beta_5 + 3\beta_6 + \beta_7 + 3\beta_8$	(M2)	(M3)
$\beta_1 + 3\beta_2 + 4\beta_3 + 5\beta_4 + 6\beta_5 + 4\beta_6 + 2\beta_7 + 3\beta_8$	(M1)	done
$2\beta_1 + 3\beta_2 + 4\beta_3 + 5\beta_4 + 6\beta_5 + 4\beta_6 + 2\beta_7 + 3\beta_8$	done	(M1)

When case C5) occurs, we have

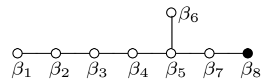


in which $r = 5$, $\alpha_5 = \beta_5, \alpha_4 = \beta_4, \cdots, \alpha_1 = \beta_1$.

Table for case C5) with $r = 5$

Roots with $\Xi \subset \{\beta_5, \beta_6\}$	Method
$\beta_5 + \beta_6$	(M3)
$\beta_2 + 2\beta_3 + \beta_4 + 2\beta_5 + \beta_6$	(M3)
$\beta_2 + 2\beta_3 + \beta_4 + 2\beta_5 + 2\beta_6 + \beta_7$	(M1)
$\beta_1 + 2\beta_2 + 3\beta_3 + \beta_4 + 3\beta_5 + 2\beta_6 + \beta_7$	(M3)
$\beta_1 + 2\beta_2 + 3\beta_3 + \beta_4 + 3\beta_5 + 3\beta_6 + 2\beta_7 + \beta_8$	(M1)
$\beta_1 + 2\beta_2 + 4\beta_3 + 2\beta_4 + 4\beta_5 + 3\beta_6 + 2\beta_7 + \beta_8$	(M2)
$2\beta_1 + 4\beta_2 + 6\beta_3 + 3\beta_4 + 5\beta_5 + 3\beta_6 + 2\beta_7 + \beta_8$	(M2)
$2\beta_1 + 4\beta_2 + 6\beta_3 + 3\beta_4 + 5\beta_5 + 4\beta_6 + 2\beta_7 + \beta_8$	(M1)

When case C7) occurs, we have



in which $0 \leq o \leq 3$ (correspondingly $7 \geq r \geq 4$), $\alpha_r = \beta_7, \alpha_{r-1} = \beta_6, \alpha_{r-2} = \beta_5, \alpha_{r-3} = \beta_4, \cdots, \alpha_1 = \beta_{o+1}$.

Table for case C7) with $0 \leq o \leq 3$

	Roots with $\left\{ \begin{array}{l} \Xi \subset \{\beta_1, \beta_2, \beta_3, \beta_7, \beta_8\} \\ \lvert\Xi \cap \{\beta_1, \beta_2, \beta_3\}\rvert \leq 1 \end{array} \right.$	Constraint	Method
1)	$\beta_3 + \beta_4 + \beta_5 + \beta_7$	$o = 3$	(M3)
2)	$\beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_7$	$o = 2$	
3)	$\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_7$	$o = 1$	
4)	$\beta_2 + 2\beta_3 + 2\beta_4 + 2\beta_5 + \beta_6 + \beta_7$	$o = 3$	(M1)
5)	$\beta_1 + 2\beta_2 + 2\beta_3 + 2\beta_4 + 2\beta_5 + \beta_6 + \beta_7$	$o = 2$	
6)	$\beta_7 + \beta_8$	$o \geq 0$	(M2,2,3,3)
7)	$\beta_3 + \beta_4 + \beta_5 + \beta_7 + \beta_8$	$o = 3$	(M1)
8)	$\beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_7 + \beta_8$	$o = 2$	
9)	$\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_7 + \beta_8$	$o = 1$	
10)	$\beta_2 + 2\beta_3 + 2\beta_4 + 2\beta_5 + \beta_6 + \beta_7 + \beta_8$	$o = 3$	
11)	$\beta_1 + 2\beta_2 + 2\beta_3 + 2\beta_4 + 2\beta_5 + \beta_6 + \beta_7 + \beta_8$	$o = 2$	
12)	$\beta_4 + 2\beta_5 + \beta_6 + 2\beta_7 + \beta_8$	$o \geq 0$	(M2,2,3,3)
13)	$\beta_3 + \beta_4 + 2\beta_5 + \beta_6 + 2\beta_7 + \beta_8$	$o = 3$	(M3)
14)	$\beta_2 + \beta_3 + \beta_4 + 2\beta_5 + \beta_6 + 2\beta_7 + \beta_8$	$o = 2$	(M2)
15)	$\beta_1 + \beta_2 + \beta_3 + \beta_4 + 2\beta_5 + \beta_6 + 2\beta_7 + \beta_8$	$o = 1$	
16)	$\beta_2 + 2\beta_3 + 2\beta_4 + 2\beta_5 + \beta_6 + 2\beta_7 + \beta_8$	$o = 3$	
17)	$\beta_1 + 2\beta_2 + 2\beta_3 + 2\beta_4 + 2\beta_5 + \beta_6 + 2\beta_7 + \beta_8$	$o = 2$	
18)	$\beta_1 + 2\beta_2 + 3\beta_3 + 3\beta_4 + 3\beta_5 + \beta_6 + 2\beta_7 + \beta_8$	$o = 3$	(M1)
19)	$\beta_2 + 2\beta_3 + 3\beta_4 + 4\beta_5 + 2\beta_6 + 3\beta_7 + \beta_8$	$o \geq 0$	(M2)
20)	$\beta_1 + \beta_2 + 2\beta_3 + 3\beta_4 + 4\beta_5 + 2\beta_6 + 3\beta_7 + \beta_8$	$o = 1$	
21)	$\beta_1 + 2\beta_2 + 2\beta_3 + 3\beta_4 + 4\beta_5 + 2\beta_6 + 3\beta_7 + \beta_8$	$o = 2$	
22)	$\beta_1 + 2\beta_2 + 3\beta_3 + 3\beta_4 + 4\beta_5 + 2\beta_6 + 3\beta_7 + \beta_8$	$o = 3$	
23)	$\beta_2 + 2\beta_3 + 3\beta_4 + 4\beta_5 + 2\beta_6 + 3\beta_7 + 2\beta_8$	$o \geq 0$	(M1)
24)	$\beta_1 + \beta_2 + 2\beta_3 + 3\beta_4 + 4\beta_5 + 2\beta_6 + 3\beta_7 + 2\beta_8$	$o = 1$	
25)	$\beta_1 + 2\beta_2 + 2\beta_3 + 3\beta_4 + 4\beta_5 + 2\beta_6 + 3\beta_7 + 2\beta_8$	$o = 2$	
26)	$\beta_1 + 2\beta_2 + 3\beta_3 + 3\beta_4 + 4\beta_5 + 2\beta_6 + 3\beta_7 + 2\beta_8$	$o = 3$	
27)	$\beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4 + 5\beta_5 + 2\beta_6 + 4\beta_7 + 2\beta_8$	$o \geq 0$	(M3,2,2,2)
28)	$\beta_1 + 2\beta_2 + 4\beta_3 + 5\beta_4 + 6\beta_5 + 3\beta_6 + 4\beta_7 + 2\beta_8$	$o = 3$	(M1)
29)	$\beta_1 + 3\beta_2 + 4\beta_3 + 5\beta_4 + 6\beta_5 + 3\beta_6 + 4\beta_7 + 2\beta_8$	$o = 2$	
30)	$2\beta_1 + 3\beta_2 + 4\beta_3 + 5\beta_4 + 6\beta_5 + 3\beta_6 + 4\beta_7 + 2\beta_8$	$o = 1$	

In the above table, by “(M2,2,3,3)” for the root $\beta_7 + \beta_8$, we mean (M2) (resp. (M2), (M3) and (M3)) is used when $o = 0$ (resp. 1, 2 and 3). Similar notations are used for the case no. 12) and no. 27).

As we said right now (or will see in the outputs), we remain to discuss the following two special cases.

(1) Case C4) occurs, $r = 6$ and $\gamma = \beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4 + 5\beta_5 + 3\beta_6 + \beta_7 + 3\beta_8$.
We use the same notations in the Lemma 2 as well as in the proof of it. Take $\alpha_j = \beta_8 = \alpha_6$. Then $\beta = s_6(\gamma)$ and $p = 3$. Furthermore from

the output, we note that that $gr(q_{\gamma^\vee}) = (0,0,0,0,-5,11,38)$, $gr(q_{\beta^\vee}) = (0,0,3,3,-2,0,38)$ so that $\lambda_3 = \lambda_4 = 3$, $m_3 = m_4 = 3$ and $m_5 = 0$. Thus from the proof of Lemma 2 we remain to assume $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0$ and $\sum_{t=1}^4 (a_t + b_t) = 0$. Consequently, we have $\mu_5 \leq 0$ by the induction hypothesis, and it suffices to show $\mu_5 = 0$. Indeed, we have $\mu_5 + i'_5 = k_5 + \lambda_5 = k_5 - 2$. Since $m_5 = 0$, we have $A_5 = C_5 \subset B_5$. We note that $gr_5(q_{\gamma^\vee} u s_\gamma) = gr_5(u) + gr_5(s_6) + \sum_{t=1}^5 (a_t + b_t + \mu_t)\mathbf{e}_t$ with $\sum_{t=1}^5 (a_t + b_t + \mu_t)\mathbf{e}_t \leq 0$. If "<" holds, it is already done. If "=" holds, then we have $\tilde{\lambda}_5 + k'_5 = i_5$, that is, $-5 + k'_5 = i_5$. Consequently, for any $\alpha \in R^+_{P_5} \setminus R_{P_4}$ we have $u(\alpha) \in R^+$ and $u s_\gamma(\alpha) \in -R^+$. Note that $s_6(\alpha) = \alpha$. In particular for $\alpha \in \{\beta_7, \beta_7 + \beta_6\} \subset R^+_{P_5} \setminus R_{P_4}$, we have $u s_6 s_\beta(\alpha) = u s_\gamma s_6(\alpha) = u s_\gamma(\alpha) \in -R^+$ and $u s_6(\alpha) = u(\alpha) \in R^+$, which implies $\alpha \in B_5 \setminus A_5$. Hence, $i'_5 = |A_5| = |B_5| - |B_5 \setminus A_5| \leq k_5 - 2$. Hence, $\mu_5 = k_5 - 2 - i'_5 \geq 0$ and consequently we do have $\mu_5 = 0$.

(2) Case C7) occurs, $\gamma = \beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4 + 5\beta_5 + 2\beta_6 + 4\beta_7 + 2\beta_8$ and $1 \leq o \leq 3$. (The remaining arguments are similar to case (1).)

From the output we see $gr(q_{\gamma^\vee}) = -\varsigma\mathbf{e}_\varsigma + \varsigma\mathbf{e}_r + \tilde{\lambda}_{r+1}\mathbf{e}_{r+1}$, $\lambda_{\varsigma-1} = \varsigma - 1$ and $\lambda_\varsigma = -1$ (where $\varsigma = r - 1$). Note that $\langle \alpha_t, \gamma^\vee \rangle = 0$ for all $1 \leq t \leq \varsigma - 1$. Hence, we conclude $\sum_{t=1}^{\varsigma-1} (a_t + b_t + \mu_t)\mathbf{e}_t = gr_{\varsigma-1}(q_{\gamma^\vee} u s_\gamma) - gr_{\varsigma-1}(u) - gr_{\varsigma-1}(s_r) = \mathbf{0}$. Furthermore we note that $\sum_{t=1}^{\varsigma} (a_t + b_t + \mu_t)\mathbf{e}_t \leq \mathbf{0}$ (by considering $\tilde{gr}$). Thus we have $a_\varsigma + b_\varsigma + \mu_\varsigma \leq 0$ and if "<" holds then it is done. We remain to assume "=" holds. Then $-\varsigma + k_\varsigma = i_\varsigma$. Thus $k'_\varsigma = \varsigma$, $i_\varsigma = 0$ and consequently for any $\alpha \in R^+_{P_\varsigma} \setminus R_{P_{\varsigma-1}}$ we have $u(\alpha) \in R^+$ and $u s_\gamma(\alpha) \in -R^+$. In particular, $u s_r(\beta_\varsigma) = u(\beta_\varsigma) \in R^+$ and $u s_r s_\beta(\beta_\varsigma) = u s_\gamma s_r(\beta_\varsigma) = u s_\gamma(\beta_\varsigma) \in -R^+$. Hence, $\beta_\varsigma \in B_\varsigma \setminus A_\varsigma$. On the other hand, we note $m_\varsigma = 0$ so that $A_\varsigma \subset B_\varsigma$. Hence, $i'_\varsigma = |A_\varsigma| = |B_\varsigma| - |B_\varsigma \setminus A_\varsigma| \leq k_\varsigma - 1$. Thus $\mu_\varsigma = k_\varsigma + \lambda_\varsigma - i_\varsigma = k_\varsigma - 1 - i'_\varsigma \geq 0$.

Similarly, we note that $\langle \alpha_t, \alpha_r^\vee \rangle = 0$ for all $1 \leq t \leq \varsigma - 2$ and $gr_{\varsigma-2}(q_r) = \mathbf{0}$. Hence, we conclude $a_t = b_t = 0$ for all $t \leq \varsigma - 2$. Thus we have $\mu_t = 0$ for $\leq t \leq \varsigma - 2$ and consequently $\mu_{\varsigma-1} \leq 0$. However, $\mu_{\varsigma-1} = \lambda_{\varsigma-1} + k_{\varsigma-1} - i'_{\varsigma-1} = \varsigma - 1 + k_{\varsigma-1} - i'_{\varsigma-1} \geq 0$. Thus $\mu_{\varsigma-1} = 0$ and consequently we have $\mu_\varsigma \leq 0$. Hence, $\mu_\varsigma = 0$ and then $a_\varsigma + b_\varsigma = a_\varsigma + b_\varsigma + \mu_\varsigma = 0$. Consequently, we have $\sum_{t=1}^{r} (a_t + b_t + \mu_t)\mathbf{e}_t = \mu_r \mathbf{e}_r \leq \mathbf{0}$.

Hence, Key Lemma b) also holds for such γ.

In the following, we give the outputs.

Out[236]//MatrixForm=

```
( 1)    case  \P9) with r=2
( 2)    case  \P9) with r=3
( 3)    case \P10) with r=2
( 4)    case \P10) with r=3
( 5)    case  \P4) with r=6
( 6)    case  \P4) with r=7
( 7)    case  \P5) with r=5
( 8)    case  \P7) with o=3
( 9)    case  \P7) with o=2
(10)    case  \P7) with o=1
(11)    case  \P7) with o=0
```

We first list all the outputs with respect to the orders as above.

After that we provide the programs to the readers for references.

"mos" stands for the roots that we are condiersing.

For each case, we first list the grading for each $q_{\beta_j^\vee}$, which we denote by q_{β_j} in our outputs. Then we list all the methods as well as relevant outputs for the corresponding coroots. For all the outputs, we call see that there are two exceptional cases for which we need to analyze the coroots more carefully.

```
(* Consider case \P9) with r=2 *)
```

possible roots=
{{0, 1, 1, 0}, {0, 0, 1, 1}, {1, 1, 1, 0}, {0, 1, 1, 1}, {1, 1, 1, 1}, {1, 2, 2, 1}, {2, 3, 2, 1}}

$$\begin{pmatrix} gr(q_{\beta_1}) & = & \{-1, -2, 5\} \\ gr(q_{\beta_2}) & = & \{2, 0, 0\} \\ gr(q_{\beta_3}) & = & \{-2, 4, 0\} \\ gr(q_{\beta_4}) & = & \{0, -4, 6\} \end{pmatrix}$$

no. 1) {0, 1, 1, 0} : It is done sine $\gamma \in \Delta_P$

no. 2) {0, 0, 1, 1} : Use Method 3

$$\begin{pmatrix} \Delta(\gamma) & gr(q_{\gamma^\vee}) & |B_2| \text{ VS } |B_3| & & -\sum_{i=1}^{r} d_i \\ \{2, \ldots, 4\} & \{-2, 0, 6\} & 2 & 1 & 2 \end{pmatrix}$$

no. 3) {1, 1, 1, 0} : Use Method 3

$$\begin{pmatrix} \Delta(\gamma) & gr(q_{\gamma^\vee}) & |B_2| \text{ VS } |B_3| & & -\sum_{i=1}^{r} d_i \\ \{1, \ldots, 3\} & \{-1, 2, 5\} & 1 & 2 & -1 \end{pmatrix}$$

no. 4) {0, 1, 1, 1} : Use Method 1

$$\begin{pmatrix} \Delta_P - \Delta_{\bar{P}} & gr(q_{\gamma^\vee}) & \sum_{i=1}^{r} d_i & length(\omega_P \omega_{\bar{P}}) \\ \{\} & \{0, 0, 6\} & 0 & 0 \end{pmatrix}$$

no. 5) {1, 1, 1, 1} : Use Method 1

$$\begin{pmatrix} \Delta_P - \Delta_{\bar{P}} & gr(q_{\gamma^\vee}) & \sum_{i=1}^{r} d_i & length(\omega_P \omega_{\bar{P}}) \\ \{\alpha_1\} & \{-1, -2, 11\} & -3 & 3 \end{pmatrix}$$

no. 6) {1, 2, 2, 1} : Use Method 2

$$\begin{pmatrix} & gr(q_{\beta^\vee}) & \{|R^{+}{}_{k} - R_{k-1}|, \ldots, |R^{+}{}_{r-1} - R_{r-2}|\} \\ \text{take } \alpha_j = \beta_3 & \{1, -2, 11\} & \{1\} \end{pmatrix}$$

no. 7) {2, 3, 2, 1} : Use Method 1

$$\begin{pmatrix} \Delta_P - \Delta_{\bar{P}} & gr(q_{\gamma^\vee}) & \sum_{i=1}^{r} d_i & length(\omega_P \omega_{\bar{P}}) \\ \{\} & \{0, 0, 16\} & 0 & 0 \end{pmatrix}$$

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(* Consider case \P9) with r=3 *)
```

possible roots= {{0, 1, 1, 0}, {0, 0, 1, 1}, {1, 1, 1, 0}, {0, 1, 1, 1}, {1, 1, 1, 1}, {1, 2, 2, 1}, {2, 3, 2, 1}}

$$\begin{pmatrix} gr(q_{\beta_1}) = \{2, 0, 0, 0\} \\ gr(q_{\beta_2}) = \{-1, 3, 0, 0\} \\ gr(q_{\beta_3}) = \{0, -4, 6, 0\} \\ gr(q_{\beta_4}) = \{0, 0, -9, 11\} \end{pmatrix}$$

no. 1) {0, 1, 1, 0} : It is done sine $\gamma \in \Delta_P$

no. 2) {0, 0, 1, 1} : Use Method 3

$\Delta(\gamma)$	$gr(q_{\gamma^\vee})$	$\lvert B_2\rvert$ VS $\lvert B_3\rvert$		$-\sum_{i=1}^{r} d_i$
{1, ..., 4}	{0, -4, -3, 11}	6	1	7

no. 3) {1, 1, 1, 0} : It's done by taking $\alpha_j = \beta_1$

no. 4) {0, 1, 1, 1} : Use Method 1

$\Delta_P - \Delta_P^-$	$gr(q_{\gamma^\vee})$	$\sum_{i=1}^{r} d_i$	$length(\omega_P \omega_P^-)$
$\{\alpha_1\}$	{-1, -1, -3, 11}	-5	5

no. 5) {1, 1, 1, 1} : It's done by taking $\alpha_j = \beta_1$

no. 6) {1, 2, 2, 1} : Use Method 2

	$gr(q_{\beta^\vee})$	$\{\lvert R^+{}_k - R_{k-1}\rvert, ..., \lvert R^+{}_{r-1} - R_{r-2}\rvert\}$
take $\alpha_j = \beta_3$	{0, 2, -3, 11}	{2}

no. 7) {2, 3, 2, 1} : It's done by taking $\alpha_j = \beta_1$

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(* Consider case \P10) with r=2 *)

possible roots=
{{0, 0, 1, 1}, {1, 1, 1, 0}, {1, 1, 1, 1}, {0, 1, 2, 1}, {1, 1, 2, 1}, {1, 2, 3, 1}, {1, 2, 3, 2}}

$$\begin{pmatrix} gr(\ q_{\beta_1}\ ) = \{-1, -3, 6\} \\ gr(\ q_{\beta_2}\ ) = \{2, 0, 0\} \\ gr(\ q_{\beta_3}\ ) = \{-1, 3, 0\} \\ gr(\ q_{\beta_4}\ ) = \{0, -3, 5\} \end{pmatrix}$$

no. 1) {0, 0, 1, 1} : Use Method 3

Δ(γ)	gr($q_{\gamma^\vee}$)	$\|B_2\|$ VS $\|B_3\|$		$-\sum_{i=1}^{r} d_i$
{2, ..., 4}	{-1, 0, 5}	2	1	1

no. 2) {1, 1, 1, 0} : Use Method 1

$\Delta_P - \Delta_{\bar{P}}$	gr($q_{\gamma^\vee}$)	$\sum_{i=1}^{r} d_i$	length($\omega_P \omega_{\bar{P}}$)
{}	{0, 0, 6}	0	0

no. 3) {1, 1, 1, 1} : Use Method 1

$\Delta_P - \Delta_{\bar{P}}$	gr($q_{\gamma^\vee}$)	$\sum_{i=1}^{r} d_i$	length($\omega_P \omega_{\bar{P}}$)
{α_2}	{0, -3, 11}	-3	3

no. 4) {0, 1, 2, 1} : Use Method 3

Δ(γ)	gr($q_{\gamma^\vee}$)	$\|B_2\|$ VS $\|B_3\|$		$-\sum_{i=1}^{r} d_i$
{2, ..., 4}	{0, 3, 5}	0	3	-3

no. 5) {1, 1, 2, 1} : Use Method 2

	gr($q_{\beta^\vee}$)	{$\|R^+{}_k - R_{k-1}\|$, ..., $\|R^+{}_{r-1} - R_{r-2}\|$}
take $\alpha_j = \beta_3$	{0, -3, 11}	{0}

no. 6) {1, 2, 3, 1} : Use Method 2

	gr($q_{\beta^\vee}$)	{$\|R^+{}_k - R_{k-1}\|$, ..., $\|R^+{}_{r-1} - R_{r-2}\|$}
take $\alpha_j = \beta_3$	{1, 0, 11}	{1}

no. 7) {1, 2, 3, 2} : Use Method 1

$\Delta_P - \Delta_{\bar{P}}$	gr($q_{\gamma^\vee}$)	$\sum_{i=1}^{r} d_i$	length($\omega_P \omega_{\bar{P}}$)
{}	{0, 0, 16}	0	0

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(* Consider case \P10) with r=3 *)

possible roots=
{{0, 0, 1, 1}, {1, 1, 1, 0}, {1, 1, 1, 1}, {0, 1, 2, 1}, {1, 1, 2, 1}, {1, 2, 3, 1}, {1, 2, 3, 2}}

$$\begin{pmatrix} gr(q_{\beta_1}) = \{2, 0, 0, 0\} \\ gr(q_{\beta_2}) = \{-1, 3, 0, 0\} \\ gr(q_{\beta_3}) = \{0, -2, 4, 0\} \\ gr(q_{\beta_4}) = \{0, 0, -6, 8\} \end{pmatrix}$$

no. 1) {0, 0, 1, 1} : Use Method 2

$$\begin{pmatrix} & gr(q_{\beta^\vee}) & \{|R^+{}_k - R_{k-1}|, \ldots, |R^+{}_{r-1} - R_{r-2}|\} \\ \text{take } \alpha_j = \beta_3 & \{0, 0, -6, 8\} & \{0\} \end{pmatrix}$$

no. 2) {1, 1, 1, 0} : It's done by taking $\alpha_j = \beta_1$

no. 3) {1, 1, 1, 1} : It's done by taking $\alpha_j = \beta_1$

no. 4) {0, 1, 2, 1} : Use Method 2

$$\begin{pmatrix} & gr(q_{\beta^\vee}) & \{|R^+{}_k - R_{k-1}|, \ldots, |R^+{}_{r-1} - R_{r-2}|\} \\ \text{take } \alpha_j = \beta_3 & \{-1, 1, -2, 8\} & \{1\} \end{pmatrix}$$

no. 5) {1, 1, 2, 1} : It's done by taking $\alpha_j = \beta_1$

no. 6) {1, 2, 3, 1} : Use Method 2

$$\begin{pmatrix} & gr(q_{\beta^\vee}) & \{|R^+{}_k - R_{k-1}|, \ldots, |R^+{}_{r-1} - R_{r-2}|\} \\ \text{take } \alpha_j = \beta_3 & \{0, 2, 2, 8\} & \{2\} \end{pmatrix}$$

no. 7) {1, 2, 3, 2} : Use Method 1

$$\begin{pmatrix} \Delta_P - \Delta_P^- & gr(q_{\gamma^\vee}) & \sum_{i=1}^{r} d_i & \text{length}(\omega_P \omega_P^-) \\ \{\} & \{0, 0, 0, 16\} & 0 & 0 \end{pmatrix}$$

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(* Consider case \P4) with r=6 *)
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possible roots= {{1, 1, 0, 0, 0, 0, 0, 0}, {0, 1, 1, 1, 1, 0, 0, 1}, {1, 1, 1, 1, 1, 0, 0, 1}, {1, 2, 2, 2, 2, 1, 0, 1}, {0, 0, 1, 2, 3, 2, 1, 2}, {0, 1, 1, 2, 3, 2, 1, 2}, {1, 1, 1, 2, 3, 2, 1, 2}, {1, 2, 2, 2, 3, 2, 1, 2}, {1, 2, 3, 4, 5, 3, 1, 3}, {1, 3, 4, 5, 6, 4, 2, 3}, {2, 3, 4, 5, 6, 4, 2, 3}}

gr(q_{β_1})	=	{0, 0, 0, 0, 0, 0, 2}
gr(q_{β_2})	=	{-1, -1, -1, -1, -1, -11, 18}
gr(q_{β_3})	=	{2, 0, 0, 0, 0, 0, 0}
gr(q_{β_4})	=	{-1, 3, 0, 0, 0, 0, 0}
gr(q_{β_5})	=	{0, -2, 4, 0, 0, 0, 0}
gr(q_{β_6})	=	{0, 0, -3, 5, 0, 0, 0}
gr(q_{β_7})	=	{0, 0, 0, -4, 6, 0, 0}
gr(q_{β_8})	=	{0, 0, -3, -3, -3, 11, 0}

no. 1) {1, 1, 0, 0, 0, 0, 0, 0} : It's done by taking $\alpha_j = \beta_1$

no. 2) {0, 1, 1, 1, 1, 0, 0, 1} : Use Method 3

$\Delta(\gamma)$	$gr(q_{\gamma^\vee})$	$\lvert B_2\rvert$ VS $\lvert B_3\rvert$		$-\sum_{i=1}^r d_i$
{2, ..., 8}	{0, 0, 0, -4, -4, 0, 18}	12	4	8

no. 3) {1, 1, 1, 1, 1, 0, 0, 1} : It's done by taking $\alpha_j = \beta_1$

no. 4) {1, 2, 2, 2, 2, 1, 0, 1} : Use Method 1

$\Delta_P - \Delta_{\bar{P}}$	$gr(q_{\gamma^\vee})$	$\sum_{i=1}^r d_i$	length$(\omega_P \omega_{\bar{P}})$
{α_5}	{0, 0, 0, 0, -5, -11, 38}	-16	16

no. 5) {0, 0, 1, 2, 3, 2, 1, 2} : It is done sine $\gamma \in \Delta_P$

no. 6) {0, 1, 1, 2, 3, 2, 1, 2} : Use Method 3

$\Delta(\gamma)$	$gr(q_{\gamma^\vee})$	$\lvert B_2\rvert$ VS $\lvert B_3\rvert$		$-\sum_{i=1}^r d_i$
{2, ..., 8}	{-1, -1, -1, -1, -1, 11, 18}	5	11	-6

no. 7) {1, 1, 1, 2, 3, 2, 1, 2} : It's done by taking $\alpha_j = \beta_1$

no. 8) {1, 2, 2, 2, 3, 2, 1, 2} : Use Method 2

	$gr(q_{\beta^\vee})$	$\{\lvert R^+_k - R_{k-1}\rvert, ..., \lvert R^+_{r-1} - R_{r-2}\rvert\}$
take $\alpha_j = \beta_8$	{0, -2, 1, 1, 1, -11, 38}	{1, 1, 1}

no. 9) {1, 2, 3, 4, 5, 3, 1, 3} : Need more analysis.

$gr(q_{\gamma^\vee})$		$gr(q_{\beta^\vee})$	$\{\lvert R^+_k - R_{k-1}\rvert, \ldots, \lvert R^+_{r-1} - R_{r-2}\rvert\}$
{0, 0, 0, 0, -5, 11, 38}	take $\alpha_j = \beta_8$	{0, 0, 3, 3, -2, 0, 38}	{3, 3, 0}

no. 10) {1, 3, 4, 5, 6, 4, 2, 3} : Use Method 1

$\Delta_P - \Delta_{\bar{P}}$	$gr(q_{\gamma^\vee})$	$\sum_{i=1}^r d_i$	$length(\omega_P \omega_{\bar{P}})$
{}	{0, 0, 0, 0, 0, 0, 56}	0	0

no. 11) {2, 3, 4, 5, 6, 4, 2, 3} : It's done by taking $\alpha_j = \beta_1$

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(* Consider case \P4) with r=7 *)

possible roots= {{1, 1, 0, 0, 0, 0, 0, 0}, {0, 1, 1, 1, 1, 0, 0, 1}, {1, 1, 1, 1, 1, 0, 0, 1}, {1, 2, 2, 2, 2, 1, 0, 1}, {0, 0, 1, 2, 3, 2, 1, 2}, {0, 1, 1, 2, 3, 2, 1, 2}, {1, 1, 1, 2, 3, 2, 1, 2}, {1, 2, 2, 2, 3, 2, 1, 2}, {1, 2, 3, 4, 5, 3, 1, 3}, {1, 3, 4, 5, 6, 4, 2, 3}, {2, 3, 4, 5, 6, 4, 2, 3}}

$gr(q_{\beta_1})$	=	{-1, -1, -1, -1, -1, -1, -57, 65}
$gr(q_{\beta_2})$	=	{2, 0, 0, 0, 0, 0, 0, 0}
$gr(q_{\beta_3})$	=	{-1, 3, 0, 0, 0, 0, 0, 0}
$gr(q_{\beta_4})$	=	{0, -2, 4, 0, 0, 0, 0, 0}
$gr(q_{\beta_5})$	=	{0, 0, -3, 5, 0, 0, 0, 0}
$gr(q_{\beta_6})$	=	{0, 0, 0, -4, 6, 0, 0, 0}
$gr(q_{\beta_7})$	=	{0, 0, 0, 0, -5, 7, 0, 0}
$gr(q_{\beta_8})$	=	{0, 0, 0, -4, -4, -4, 14, 0}

no. 1) {1, 1, 0, 0, 0, 0, 0, 0} : It's done by taking $\alpha_j = \beta_2$

no. 2) {0, 1, 1, 1, 1, 0, 0, 1} : It's done by taking $\alpha_j = \beta_2$

no. 3) {1, 1, 1, 1, 1, 0, 0, 1} : Use Method 3

$gr(q_{\gamma^\vee})$	$\lvert B_2\rvert$	VS	$\lvert B_3\rvert$	$-\sum_{i=1}^r d_i$
{0, 0, 0, 0, -5, -5, -43, 65}	23		5	53

no. 4) {1, 2, 2, 2, 2, 1, 0, 1} : It's done by taking $\alpha_j = \beta_2$

no. 5) {0, 0, 1, 2, 3, 2, 1, 2} : It is done sine $\gamma \in \Delta_P$

no. 6) {0, 1, 1, 2, 3, 2, 1, 2} : It's done by taking $\alpha_j = \beta_2$

no. 7) {1, 1, 1, 2, 3, 2, 1, 2} : Use Method 3

$gr(q_{\gamma^\vee})$	$\lvert B_2\rvert$	VS	$\lvert B_3\rvert$	$-\sum_{i=1}^{r} d_i$
{0, -2, -2, -2, -2, -2, -29, 65}	16		12	39

no. 8) {1, 2, 2, 2, 3, 2, 1, 2} : It's done by taking $\alpha_j = \beta_2$

no. 9) {1, 2, 3, 4, 5, 3, 1, 3} : Use Method 3

$gr(q_{\gamma^\vee})$	$\lvert B_2\rvert$	VS	$\lvert B_3\rvert$	$-\sum_{i=1}^{r} d_i$
{0, 0, 0, 0, 0, -6, -15, 65}	7		21	21

no. 10) {1, 3, 4, 5, 6, 4, 2, 3} : It's done by taking $\alpha_j = \beta_2$

no. 11) {2, 3, 4, 5, 6, 4, 2, 3} : Use Method 1

$\Delta_P - \Delta_{\bar{P}}$	$gr(q_{\gamma^\vee})$	$\sum_{i=1}^{r} d_i$	$length(\omega_P \omega_{\bar{P}})$
{}	{0, 0, 0, 0, 0, 0, -72, 130}	-72	0

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(* Consider case \P5) with r=5 *)
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possible roots=
{{0, 0, 0, 0, 1, 1, 0, 0}, {0, 1, 2, 1, 2, 1, 0, 0}, {0, 1, 2, 1, 2, 2, 1, 0}, {1, 2, 3, 1, 3, 2, 1, 0},
{1, 2, 3, 1, 3, 3, 2, 1}, {1, 2, 4, 2, 4, 3, 2, 1}, {2, 4, 6, 3, 5, 3, 2, 1}, {2, 4, 6, 3, 5, 4, 2, 1}}

$gr(q_{\beta_1})$	=	{2, 0, 0, 0, 0, 0}
$gr(q_{\beta_2})$	=	{-1, 3, 0, 0, 0, 0}
$gr(q_{\beta_3})$	=	{0, -2, 4, 0, 0, 0}
$gr(q_{\beta_4})$	=	{0, 0, -3, 5, 0, 0}
$gr(q_{\beta_5})$	=	{0, 0, -3, -3, 8, 0}
$gr(q_{\beta_6})$	=	{0, 0, 0, 0, -10, 12}
$gr(q_{\beta_7})$	=	{0, 0, 0, 0, 0, 2}
$gr(q_{\beta_8})$	=	{0, 0, 0, 0, 0, 2}

no. 1) {0, 0, 0, 0, 1, 1, 0, 0} : Use Method 3

$\Delta(\gamma)$	$gr(q_{\gamma^\vee})$	$\lvert B_2\rvert$	VS	$\lvert B_3\rvert$	$-\sum_{i=1}^{r} d_i$
{1, ..., 6}	{0, 0, -3, -3, -2, 12}	9		1	8

no. 2) {0, 1, 2, 1, 2, 1, 0, 0} : Use Method 3

$\Delta(\gamma)$	$gr(q_{\gamma^\vee})$	$\lvert B_2\rvert$ VS	$\lvert B_3\rvert$	$-\sum_{i=1}^{r} d_i$
{1, ..., 6}	{-1, -1, -1, -1, 6, 12}	4	6	-2

no. 3) {0, 1, 2, 1, 2, 2, 1, 0} : Use Method 1

$\Delta_P - \Delta_{\bar{P}}$	$gr(q_{\gamma^\vee})$	$\sum_{i=1}^{r} d_i$	length$(\omega_P \omega_{\bar{P}})$
$\{\alpha_1\}$	{-1, -1, -1, -1, -4, 26}	-8	8

no. 4) {1, 2, 3, 1, 3, 2, 1, 0} : Use Method 3

$\Delta(\gamma)$	$gr(q_{\gamma^\vee})$	$\lvert B_2\rvert$ VS	$\lvert B_3\rvert$	$-\sum_{i=1}^{r} d_i$
{1, ..., 7}	{0, 0, 0, -4, 4, 26}	4	4	0

no. 5) {1, 2, 3, 1, 3, 3, 2, 1} : Use Method 1

$\Delta_P - \Delta_{\bar{P}}$	$gr(q_{\gamma^\vee})$	$\sum_{i=1}^{r} d_i$	length$(\omega_P \omega_{\bar{P}})$
$\{\alpha_4\}$	{0, 0, 0, -4, -6, 42}	-10	10

no. 6) {1, 2, 4, 2, 4, 3, 2, 1} : Use Method 2

	$gr(q_{\beta^\vee})$	$\{\lvert R^+_k - R_{k-1}\rvert, ..., \lvert R^+_{r-1} - R_{r-2}\rvert\}$
take $\alpha_j = \beta_5$	{0, -2, 1, 1, -6, 42}	{1, 1}

no. 7) {2, 4, 6, 3, 5, 3, 2, 1} : Use Method 2

	$gr(q_{\beta^\vee})$	$\{\lvert R^+_k - R_{k-1}\rvert, ..., \lvert R^+_{r-1} - R_{r-2}\rvert\}$
take $\alpha_j = \beta_5$	{0, 0, 3, 3, 2, 42}	{3, 3}

no. 8) {2, 4, 6, 3, 5, 4, 2, 1} : Use Method 1

$\Delta_P - \Delta_{\bar{P}}$	$gr(q_{\gamma^\vee})$	$\sum_{i=1}^{r} d_i$	length$(\omega_P \omega_{\bar{P}})$
{}	{0, 0, 0, 0, 0, 54}	0	0

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(* Consider case \P7) with o=3 *)

possible roots= {{0, 0, 1, 1, 1, 0, 1, 0}, {0, 1, 1, 1, 1, 0, 1, 0}, {1, 1, 1, 1, 1, 0, 1, 0},
  {0, 1, 2, 2, 2, 1, 1, 0}, {1, 2, 2, 2, 2, 1, 1, 0}, {0, 0, 0, 0, 0, 0, 1, 1}, {0, 0, 1, 1, 1, 0, 1, 1},
  {0, 1, 1, 1, 1, 0, 1, 1}, {1, 1, 1, 1, 1, 0, 1, 1}, {0, 1, 2, 2, 2, 1, 1, 1}, {1, 2, 2, 2, 2, 1, 1, 1},
  {0, 0, 0, 1, 2, 1, 2, 1}, {0, 0, 1, 1, 2, 1, 2, 1}, {0, 1, 1, 1, 2, 1, 2, 1}, {1, 1, 1, 1, 2, 1, 2, 1},
  {0, 1, 2, 2, 2, 1, 2, 1}, {1, 2, 2, 2, 2, 1, 2, 1}, {1, 2, 3, 3, 3, 1, 2, 1}, {0, 1, 2, 3, 4, 2, 3, 1},
  {1, 1, 2, 3, 4, 2, 3, 1}, {1, 2, 2, 3, 4, 2, 3, 1}, {1, 2, 3, 3, 4, 2, 3, 1}, {0, 1, 2, 3, 4, 2, 3, 2},
  {1, 1, 2, 3, 4, 2, 3, 2}, {1, 2, 2, 3, 4, 2, 3, 2}, {1, 2, 3, 3, 4, 2, 3, 2}, {1, 2, 3, 4, 5, 2, 4, 2},
  {1, 2, 4, 5, 6, 3, 4, 2}, {1, 3, 4, 5, 6, 3, 4, 2}, {2, 3, 4, 5, 6, 3, 4, 2}}
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$$\begin{pmatrix} gr(q_{\beta_1}) = \{0, 0, 0, 0, 2\} \\ gr(q_{\beta_2}) = \{0, 0, 0, 0, 2\} \\ gr(q_{\beta_3}) = \{-1, -1, -1, -3, 8\} \\ gr(q_{\beta_4}) = \{2, 0, 0, 0, 0\} \\ gr(q_{\beta_5}) = \{-1, 3, 0, 0, 0\} \\ gr(q_{\beta_6}) = \{0, -2, 4, 0, 0\} \\ gr(q_{\beta_7}) = \{0, -2, -2, 6, 0\} \\ gr(q_{\beta_8}) = \{0, 0, 0, -6, 8\} \end{pmatrix}$$

no. 1) {0, 0, 1, 1, 1, 0, 1, 0} : Use Method 3

$$\begin{pmatrix} \Delta(\gamma) & gr(q_{\gamma^\vee}) & |B_2| \text{ VS } |B_3| & & -\sum_{i=1}^{r} d_i \\ \{3, \ldots, 7\} & \{0, 0, -3, 3, 8\} & 3 & 3 & 0 \end{pmatrix}$$

no. 2) {0, 1, 1, 1, 1, 0, 1, 0} : It's done by taking $\alpha_j = \beta_2$

no. 3) {1, 1, 1, 1, 1, 0, 1, 0} : It's done by taking $\alpha_j = \beta_1$

no. 4) {0, 1, 2, 2, 2, 1, 1, 0} : Use Method 1

$$\begin{pmatrix} \Delta_P - \Delta_{\bar{P}} & gr(q_{\gamma^\vee}) & \sum_{i=1}^{r} d_i & \text{length}(\omega_P \omega_{\bar{P}}) \\ \{\} & \{0, 0, 0, 0, 18\} & 0 & 0 \end{pmatrix}$$

no. 5) {1, 2, 2, 2, 2, 1, 1, 0} : It's done by taking $\alpha_j = \beta_2$

no. 6) {0, 0, 0, 0, 0, 0, 1, 1} : Use Method 3

$$\begin{pmatrix} \Delta(\gamma) & gr(q_{\gamma^\vee}) & |B_2| \text{ VS } |B_3| & & -\sum_{i=1}^{r} d_i \\ \{4, \ldots, 8\} & \{0, -2, -2, 0, 8\} & 5 & 1 & 4 \end{pmatrix}$$

no. 7) {0, 0, 1, 1, 1, 0, 1, 1} : Use Method 1

$$\begin{pmatrix} \Delta_P - \Delta_{\bar{P}} & gr(q_{\gamma^\vee}) & \sum_{i=1}^{r} d_i & \text{length}(\omega_P \omega_{\bar{P}}) \\ \{\alpha_3\} & \{0, 0, -3, -3, 16\} & -6 & 6 \end{pmatrix}$$

no. 8) {0, 1, 1, 1, 1, 0, 1, 1} : It's done by taking $\alpha_j = \beta_2$

no. 9) {1, 1, 1, 1, 1, 0, 1, 1} : It's done by taking $\alpha_j = \beta_1$

no. 10) {0, 1, 2, 2, 2, 1, 1, 1} : Use Method 1

$$\begin{pmatrix} \Delta_P - \Delta_{\bar{P}} & gr(q_{\gamma^\vee}) & \sum_{i=1}^r d_i & length(\omega_P \omega_{\bar{P}}) \\ \{\alpha_4\} & \{0, 0, 0, -6, 26\} & -6 & 6 \end{pmatrix}$$

no. 11) {1, 2, 2, 2, 2, 1, 1, 1} : It's done by taking $\alpha_j = \beta_2$

no. 12) {0, 0, 0, 1, 2, 1, 2, 1} : Use Method 3

$$\begin{pmatrix} \Delta(\gamma) & gr(q_{\gamma^\vee}) & |B_2| \text{ VS } |B_3| & -\sum_{i=1}^r d_i \\ \{4, ..., 8\} & \{0, 0, 0, 6, 8\} & 0 \quad 6 & -6 \end{pmatrix}$$

no. 13) {0, 0, 1, 1, 2, 1, 2, 1} : Use Method 3

$$\begin{pmatrix} \Delta(\gamma) & gr(q_{\gamma^\vee}) & |B_2| \text{ VS } |B_3| & -\sum_{i=1}^r d_i \\ \{3, ..., 8\} & \{-1, -1, -1, 3, 16\} & 3 \quad 3 & 0 \end{pmatrix}$$

no. 14) {0, 1, 1, 1, 2, 1, 2, 1} : It's done by taking $\alpha_j = \beta_2$

no. 15) {1, 1, 1, 1, 2, 1, 2, 1} : It's done by taking $\alpha_j = \beta_1$

no. 16) {0, 1, 2, 2, 2, 1, 2, 1} : Use Method 2

$$\begin{pmatrix} & gr(q_{\beta^\vee}) & \{|R^+{}_k - R_{k-1}|, ..., |R^+{}_{r-1} - R_{r-2}|\} \\ \text{take } \alpha_j = \beta_7 & \{0, 0, 0, -6, 26\} & \{0, 0\} \end{pmatrix}$$

no. 17) {1, 2, 2, 2, 2, 1, 2, 1} : It's done by taking $\alpha_j = \beta_2$

no. 18) {1, 2, 3, 3, 3, 1, 2, 1} : Use Method 1

$$\begin{pmatrix} \Delta_P - \Delta_{\bar{P}} & gr(q_{\gamma^\vee}) & \sum_{i=1}^r d_i & length(\omega_P \omega_{\bar{P}}) \\ \{\alpha_3\} & \{0, 0, -3, -3, 38\} & -6 & 6 \end{pmatrix}$$

no. 19) {0, 1, 2, 3, 4, 2, 3, 1} : Use Method 2

$$\begin{pmatrix} & gr(q_{\beta^\vee}) & \{|R^+{}_k - R_{k-1}|, ..., |R^+{}_{r-1} - R_{r-2}|\} \\ \text{take } \alpha_j = \beta_7 & \{0, 2, 2, 0, 26\} & \{2, 2\} \end{pmatrix}$$

no. 20) {1, 1, 2, 3, 4, 2, 3, 1} : It's done by taking $\alpha_j = \beta_1$

no. 21) {1, 2, 2, 3, 4, 2, 3, 1} : It's done by taking $\alpha_j = \beta_2$

no. 22) {1, 2, 3, 3, 4, 2, 3, 1} : Use Method 2

take $\alpha_j = \beta_7$	$gr(q_{\beta^\vee})$	$\{\lvert R^+{}_k - R_{k-1}\rvert, \ldots, \lvert R^+{}_{r-1} - R_{r-2}\rvert\}$
	{-1, 1, 1, -3, 38}	{1, 1}

no. 23) {0, 1, 2, 3, 4, 2, 3, 2} : Use Method 1

$\Delta_P - \Delta_{\bar{P}}$	$gr(q_{\gamma^\vee})$	$\sum_{i=1}^r d_i$	$length(\omega_P \omega_{\bar{P}})$
{}	{0, 0, 0, 0, 34}	0	0

no. 24) {1, 1, 2, 3, 4, 2, 3, 2} : It's done by taking $\alpha_j = \beta_1$

no. 25) {1, 2, 2, 3, 4, 2, 3, 2} : It's done by taking $\alpha_j = \beta_2$

no. 26) {1, 2, 3, 3, 4, 2, 3, 2} : Use Method 1

$\Delta_P - \Delta_{\bar{P}}$	$gr(q_{\gamma^\vee})$	$\sum_{i=1}^r d_i$	$length(\omega_P \omega_{\bar{P}})$
$\{\alpha_1\}$	{-1, -1, -1, -3, 46}	-6	6

no. 27) {1, 2, 3, 4, 5, 2, 4, 2} : Need more analysis.

$gr(q_{\gamma^\vee})$		$gr(q_{\beta^\vee})$	$\{\lvert R^+{}_k - R_{k-1}\rvert, \ldots, \lvert R^+{}_{r-1} - R_{r-2}\rvert\}$
{0, 0, -3, 3, 46}	take $\alpha_j = \beta_7$	{0, 2, -1, -3, 46}	{2, 0}

no. 28) {1, 2, 4, 5, 6, 3, 4, 2} : Use Method 1

$\Delta_P - \Delta_{\bar{P}}$	$gr(q_{\gamma^\vee})$	$\sum_{i=1}^r d_i$	$length(\omega_P \omega_{\bar{P}})$
{}	{0, 0, 0, 0, 54}	0	0

no. 29) {1, 3, 4, 5, 6, 3, 4, 2} : It's done by taking $\alpha_j = \beta_2$

no. 30) {2, 3, 4, 5, 6, 3, 4, 2} : It's done by taking $\alpha_j = \beta_1$

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(* Consider case \P7) with o=2 *)
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possible roots= {{0, 0, 1, 1, 1, 0, 1, 0}, {0, 1, 1, 1, 1, 0, 1, 0}, {1, 1, 1, 1, 1, 0, 1, 0}, {0, 1, 2, 2, 2, 1, 1, 0}, {1, 2, 2, 2, 2, 1, 1, 0}, {0, 0, 0, 0, 0, 0, 1, 1}, {0, 0, 1, 1, 1, 0, 1, 1}, {0, 1, 1, 1, 1, 0, 1, 1}, {1, 1, 1, 1, 1, 0, 1, 1}, {0, 1, 2, 2, 2, 1, 1, 1}, {1, 2, 2, 2, 2, 1, 1, 1}, {0, 0, 0, 1, 2, 1, 2, 1}, {0, 0, 1, 1, 2, 1, 2, 1}, {0, 1, 1, 1, 2, 1, 2, 1}, {1, 1, 1, 1, 2, 1, 2, 1}, {0, 1, 2, 2, 2, 1, 2, 1}, {1, 2, 2, 2, 2, 1, 2, 1}, {1, 2, 3, 3, 3, 1, 2, 1}, {0, 1, 2, 3, 4, 2, 3, 1}, {1, 1, 2, 3, 4, 2, 3, 1}, {1, 2, 2, 3, 4, 2, 3, 1}, {1, 2, 3, 3, 4, 2, 3, 1}, {0, 1, 2, 3, 4, 2, 3, 2}, {1, 1, 2, 3, 4, 2, 3, 2}, {1, 2, 2, 3, 4, 2, 3, 2}, {1, 2, 3, 3, 4, 2, 3, 2}, {1, 2, 3, 4, 5, 2, 4, 2}, {1, 2, 4, 5, 6, 3, 4, 2}, {1, 3, 4, 5, 6, 3, 4, 2}, {2, 3, 4, 5, 6, 3, 4, 2}}

$$\begin{pmatrix} gr(q_{\beta_1}) = \{0, 0, 0, 0, 0, 2\} \\ gr(q_{\beta_2}) = \{-1, -1, -1, -1, -4, 10\} \\ gr(q_{\beta_3}) = \{2, 0, 0, 0, 0, 0\} \\ gr(q_{\beta_4}) = \{-1, 3, 0, 0, 0, 0\} \\ gr(q_{\beta_5}) = \{0, -2, 4, 0, 0, 0\} \\ gr(q_{\beta_6}) = \{0, 0, -3, 5, 0, 0\} \\ gr(q_{\beta_7}) = \{0, 0, -3, -3, 8, 0\} \\ gr(q_{\beta_8}) = \{0, 0, 0, 0, -10, 12\} \end{pmatrix}$$

no. 1) {0, 0, 1, 1, 1, 0, 1, 0} : It's done by taking $\alpha_j=\beta_3$

no. 2) {0, 1, 1, 1, 1, 0, 1, 0} : Use Method 3

$\Delta(\gamma)$	$gr(q_{\gamma^\vee})$	$\lvert B_2\rvert$ VS $\lvert B_3\rvert$		$-\sum_{i=1}^{r} d_i$
{2, ..., 7}	{0, 0, 0, -4, 4, 10}	4	4	0

no. 3) {1, 1, 1, 1, 1, 0, 1, 0} : It's done by taking $\alpha_j=\beta_1$

no. 4) {0, 1, 2, 2, 2, 1, 1, 0} : It's done by taking $\alpha_j=\beta_3$

no. 5) {1, 2, 2, 2, 2, 1, 1, 0} : Use Method 1

$\Delta_P-\Delta_{\bar{P}}$	$gr(q_{\gamma^\vee})$	$\sum_{i=1}^{r} d_i$	length$(\omega_P\omega_{\bar{P}})$
{}	{0, 0, 0, 0, 0, 22}	0	0

no. 6) {0, 0, 0, 0, 0, 0, 1, 1} : Use Method 3

$\Delta(\gamma)$	$gr(q_{\gamma^\vee})$	$\lvert B_2\rvert$ VS $\lvert B_3\rvert$		$-\sum_{i=1}^{r} d_i$
{3, ..., 8}	{0, 0, -3, -3, -2, 12}	9	1	8

no. 7) {0, 0, 1, 1, 1, 0, 1, 1} : It's done by taking $\alpha_j=\beta_3$

no. 8) {0, 1, 1, 1, 1, 0, 1, 1} : Use Method 1

$$\begin{pmatrix} \Delta_P - \Delta_{\bar{P}} & gr(q_{\gamma^\vee}) & \sum_{i=1}^r d_i & length(\omega_P \omega_{\bar{P}}) \\ \{\alpha_4\} & \{0, 0, 0, -4, -6, 22\} & -10 & 10 \end{pmatrix}$$

no. 9) {1, 1, 1, 1, 1, 0, 1, 1} : It's done by taking $\alpha_j = \beta_1$

no. 10) {0, 1, 2, 2, 2, 1, 1, 1} : It's done by taking $\alpha_j = \beta_3$

no. 11) {1, 2, 2, 2, 2, 1, 1, 1} : Use Method 1

$$\begin{pmatrix} \Delta_P - \Delta_{\bar{P}} & gr(q_{\gamma^\vee}) & \sum_{i=1}^r d_i & length(\omega_P \omega_{\bar{P}}) \\ \{\alpha_5\} & \{0, 0, 0, 0, -10, 34\} & -10 & 10 \end{pmatrix}$$

no. 12) {0, 0, 0, 1, 2, 1, 2, 1} : Use Method 3

$$\begin{pmatrix} \Delta(\gamma) & gr(q_{\gamma^\vee}) & |B_2| & VS & |B_3| & -\sum_{i=1}^r d_i \\ \{3, ..., 8\} & \{-1, -1, -1, -1, 6, 12\} & 4 & & 6 & -2 \end{pmatrix}$$

no. 13) {0, 0, 1, 1, 2, 1, 2, 1} : It's done by taking $\alpha_j = \beta_3$

no. 14) {0, 1, 1, 1, 2, 1, 2, 1} : Use Method 2

$$\begin{pmatrix} & gr(q_{\beta^\vee}) & \{|R^+_k - R_{k-1}|, ..., |R^+_{r-1} - R_{r-2}|\} \\ \text{take } \alpha_j = \beta_7 & \{0, -2, 1, 1, -6, 22\} & \{1, 1\} \end{pmatrix}$$

no. 15) {1, 1, 1, 1, 2, 1, 2, 1} : It's done by taking $\alpha_j = \beta_1$

no. 16) {0, 1, 2, 2, 2, 1, 2, 1} : It's done by taking $\alpha_j = \beta_3$

no. 17) {1, 2, 2, 2, 2, 1, 2, 1} : Use Method 2

$$\begin{pmatrix} & gr(q_{\beta^\vee}) & \{|R^+_k - R_{k-1}|, ..., |R^+_{r-1} - R_{r-2}|\} \\ \text{take } \alpha_j = \beta_7 & \{0, 0, 0, 0, -10, 34\} & \{0, 0\} \end{pmatrix}$$

no. 18) {1, 2, 3, 3, 3, 1, 2, 1} : It's done by taking $\alpha_j = \beta_3$

no. 19) {0, 1, 2, 3, 4, 2, 3, 1} : Use Method 2

	gr($q_{\beta^\vee}$)	{$|R^+{}_k - R_{k-1}|, ..., |R^+{}_{r-1} - R_{r-2}|$}
take $\alpha_j = \beta_7$	{0, 0, 3, 3, 2, 22}	{3, 3}

no. 20) {1, 1, 2, 3, 4, 2, 3, 1} : It's done by taking $\alpha_j = \beta_1$

no. 21) {1, 2, 2, 3, 4, 2, 3, 1} : Use Method 2

	gr($q_{\beta^\vee}$)	{$|R^+{}_k - R_{k-1}|, ..., |R^+{}_{r-1} - R_{r-2}|$}
take $\alpha_j = \beta_7$	{-1, -1, 2, 2, -2, 34}	{2, 2}

no. 22) {1, 2, 3, 3, 4, 2, 3, 1} : It's done by taking $\alpha_j = \beta_3$

no. 23) {0, 1, 2, 3, 4, 2, 3, 2} : Use Method 1

$\Delta_P - \Delta_{\bar{P}}$	gr($q_{\gamma^\vee}$)	$\sum_{i=1}^r d_i$	length($\omega_P \omega_{\bar{P}}$)
{}	{0, 0, 0, 0, 0, 34}	0	0

no. 24) {1, 1, 2, 3, 4, 2, 3, 2} : It's done by taking $\alpha_j = \beta_1$

no. 25) {1, 2, 2, 3, 4, 2, 3, 2} : Use Method 1

$\Delta_P - \Delta_{\bar{P}}$	gr($q_{\gamma^\vee}$)	$\sum_{i=1}^r d_i$	length($\omega_P \omega_{\bar{P}}$)
{α_1}	{-1, -1, -1, -1, -4, 46}	-8	8

no. 26) {1, 2, 3, 3, 4, 2, 3, 2} : It's done by taking $\alpha_j = \beta_3$

no. 27) {1, 2, 3, 4, 5, 2, 4, 2} : Need more analysis.

gr($q_{\gamma^\vee}$)		gr($q_{\beta^\vee}$)	{$|R^+{}_k - R_{k-1}|, ..., |R^+{}_{r-1} - R_{r-2}|$}
{0, 0, 0, -4, 4, 46}	take $\alpha_j = \beta_7$	{0, 0, 3, -1, -4, 46}	{3, 0}

no. 28) {1, 2, 4, 5, 6, 3, 4, 2} : It's done by taking $\alpha_j = \beta_3$

no. 29) {1, 3, 4, 5, 6, 3, 4, 2} : Use Method 1

$\Delta_P - \Delta_{\bar{P}}$	gr($q_{\gamma^\vee}$)	$\sum_{i=1}^r d_i$	length($\omega_P \omega_{\bar{P}}$)
{}	{0, 0, 0, 0, 0, 56}	0	0

no. 30) {2, 3, 4, 5, 6, 3, 4, 2} : It's done by taking $\alpha_j = \beta_1$

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(* Consider case \P7) with o=1 *)
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possible roots= {{0, 0, 1, 1, 1, 0, 1, 0}, {0, 1, 1, 1, 1, 0, 1, 0}, {1, 1, 1, 1, 1, 0, 1, 0}, {0, 1, 2, 2, 2, 1, 1, 0}, {1, 2, 2, 2, 2, 1, 1, 0}, {0, 0, 0, 0, 0, 0, 1, 1}, {0, 0, 1, 1, 1, 0, 1, 1}, {0, 1, 1, 1, 1, 0, 1, 1}, {1, 1, 1, 1, 1, 0, 1, 1}, {0, 1, 2, 2, 2, 1, 1, 1}, {1, 2, 2, 2, 2, 1, 1, 1}, {0, 0, 0, 1, 2, 1, 2, 1}, {0, 0, 1, 1, 2, 1, 2, 1}, {0, 1, 1, 1, 2, 1, 2, 1}, {1, 1, 1, 1, 2, 1, 2, 1}, {0, 1, 2, 2, 2, 1, 2, 1}, {1, 2, 2, 2, 2, 1, 2, 1}, {1, 2, 3, 3, 3, 1, 2, 1}, {0, 1, 2, 3, 4, 2, 3, 1}, {1, 1, 2, 3, 4, 2, 3, 1}, {1, 2, 2, 3, 4, 2, 3, 1}, {1, 2, 3, 3, 4, 2, 3, 1}, {0, 1, 2, 3, 4, 2, 3, 2}, {1, 1, 2, 3, 4, 2, 3, 2}, {1, 2, 2, 3, 4, 2, 3, 2}, {1, 2, 3, 3, 4, 2, 3, 2}, {1, 2, 3, 4, 5, 2, 4, 2}, {1, 2, 4, 5, 6, 3, 4, 2}, {1, 3, 4, 5, 6, 3, 4, 2}, {2, 3, 4, 5, 6, 3, 4, 2}}

$$\begin{pmatrix} gr(q_{\beta_1}) = \{-1, -1, -1, -1, -1, -5, 12\} \\ gr(q_{\beta_2}) = \{2, 0, 0, 0, 0, 0, 0\} \\ gr(q_{\beta_3}) = \{-1, 3, 0, 0, 0, 0, 0\} \\ gr(q_{\beta_4}) = \{0, -2, 4, 0, 0, 0, 0\} \\ gr(q_{\beta_5}) = \{0, 0, -3, 5, 0, 0, 0\} \\ gr(q_{\beta_6}) = \{0, 0, 0, -4, 6, 0, 0\} \\ gr(q_{\beta_7}) = \{0, 0, 0, -4, -4, 10, 0\} \\ gr(q_{\beta_8}) = \{0, 0, 0, 0, 0, -15, 17\} \end{pmatrix}$$

no. 1) {0, 0, 1, 1, 1, 0, 1, 0} : It's done by taking $\alpha_j=\beta_3$

no. 2) {0, 1, 1, 1, 1, 0, 1, 0} : It's done by taking $\alpha_j=\beta_2$

no. 3) {1, 1, 1, 1, 1, 0, 1, 0} : Use Method 3

$$\begin{pmatrix} \Delta(\gamma) & gr(q_{\gamma^\vee}) & |B_2| \text{ VS } |B_3| & & -\sum_{i=1}^{r} d_i \\ \{1, ..., 7\} & \{0, 0, 0, 0, -5, 5, 12\} & 5 & 5 & 0 \end{pmatrix}$$

no. 4) {0, 1, 2, 2, 2, 1, 1, 0} : It's done by taking $\alpha_j=\beta_3$

no. 5) {1, 2, 2, 2, 2, 1, 1, 0} : It's done by taking $\alpha_j=\beta_2$

no. 6) {0, 0, 0, 0, 0, 0, 1, 1} : Use Method 2

$$\begin{pmatrix} & gr(q_{\beta^\vee}) & \{|R^+{}_k - R_{k-1}|, ..., |R^+{}_{r-1} - R_{r-2}|\} \\ \text{take } \alpha_j = \beta_7 & \{0, 0, 0, 0, 0, -15, 17\} & \{0, 0\} \end{pmatrix}$$

no. 7) {0, 0, 1, 1, 1, 0, 1, 1} : It's done by taking $\alpha_j=\beta_3$

no. 8) {0, 1, 1, 1, 1, 0, 1, 1} : It's done by taking $\alpha_j=\beta_2$

no. 9) {1, 1, 1, 1, 1, 0, 1, 1} : Use Method 1

$\Delta_P-\Delta_{\bar{P}}$	$gr(q_{\gamma^\vee})$	$\sum_{i=1}^{r} d_i$	$length(\omega_P\omega_{\bar{P}})$
$\{\alpha_5\}$	{0, 0, 0, 0, -5, -10, 29}	-15	15

no. 10) {0, 1, 2, 2, 2, 1, 1, 1} : It's done by taking $\alpha_j=\beta_3$

no. 11) {1, 2, 2, 2, 2, 1, 1, 1} : It's done by taking $\alpha_j=\beta_2$

no. 12) {0, 0, 0, 1, 2, 1, 2, 1} : Use Method 2

	$gr(q_{\beta^\vee})$	$\{\lvert R^+{}_k-R_{k-1}\rvert,\ldots,\lvert R^+{}_{r-1}-R_{r-2}\rvert\}$
take $\alpha_j=\beta_7$	{0, -2, -2, 2, 2, -5, 17}	{2, 2}

no. 13) {0, 0, 1, 1, 2, 1, 2, 1} : It's done by taking $\alpha_j=\beta_3$

no. 14) {0, 1, 1, 1, 2, 1, 2, 1} : It's done by taking $\alpha_j=\beta_2$

no. 15) {1, 1, 1, 1, 2, 1, 2, 1} : Use Method 2

	$gr(q_{\beta^\vee})$	$\{\lvert R^+{}_k-R_{k-1}\rvert,\ldots,\lvert R^+{}_{r-1}-R_{r-2}\rvert\}$
take $\alpha_j=\beta_7$	{0, 0, -3, 1, 1, -10, 29}	{1, 1}

no. 16) {0, 1, 2, 2, 2, 1, 2, 1} : It's done by taking $\alpha_j=\beta_3$

no. 17) {1, 2, 2, 2, 2, 1, 2, 1} : It's done by taking $\alpha_j=\beta_2$

no. 18) {1, 2, 3, 3, 3, 1, 2, 1} : It's done by taking $\alpha_j=\beta_3$

no. 19) {0, 1, 2, 3, 4, 2, 3, 1} : Use Method 2

	$gr(q_{\beta^\vee})$	$\{\lvert R^+{}_k - R_{k-1}\rvert, ..., \lvert R^+{}_{r-1} - R_{r-2}\rvert\}$
take $\alpha_j = \beta_7$	{0, 0, 0, 4, 4, 5, 17}	{4, 4}

no. 20) {1, 1, 2, 3, 4, 2, 3, 1} : Use Method 2

	$gr(q_{\beta^\vee})$	$\{\lvert R^+{}_k - R_{k-1}\rvert, ..., \lvert R^+{}_{r-1} - R_{r-2}\rvert\}$
take $\alpha_j = \beta_7$	{-1, -1, -1, 3, 3, 0, 29}	{3, 3}

no. 21) {1, 2, 2, 3, 4, 2, 3, 1} : It's done by taking $\alpha_j = \beta_2$

no. 22) {1, 2, 3, 3, 4, 2, 3, 1} : It's done by taking $\alpha_j = \beta_3$

no. 23) {0, 1, 2, 3, 4, 2, 3, 2} : Use Method 1

$\Delta_P - \Delta_P^-$	$gr(q_{\gamma^\vee})$	$\sum_{i=1}^r d_i$	$length(\omega_P \omega_P^-)$
{}	{0, 0, 0, 0, 0, 0, 34}	0	0

no. 24) {1, 1, 2, 3, 4, 2, 3, 2} : Use Method 1

$\Delta_P - \Delta_P^-$	$gr(q_{\gamma^\vee})$	$\sum_{i=1}^r d_i$	$length(\omega_P \omega_P^-)$
$\{\alpha_1\}$	{-1, -1, -1, -1, -1, -5, 46}	-10	10

no. 25) {1, 2, 2, 3, 4, 2, 3, 2} : It's done by taking $\alpha_j = \beta_2$

no. 26) {1, 2, 3, 3, 4, 2, 3, 2} : It's done by taking $\alpha_j = \beta_3$

no. 27) {1, 2, 3, 4, 5, 2, 4, 2} : Need more analysis.

$gr(q_{\gamma^\vee})$		$gr(q_{\beta^\vee})$	$\{\lvert R^+{}_k - R_{k-1}\rvert, ..., \lvert R^+{}_{r-1} - R_{r-2}\rvert\}$
{0, 0, 0, 0, -5, 5, 46}	take $\alpha_j = \beta_7$	{0, 0, 0, 4, -1, -5, 46}	{4, 0}

no. 28) {1, 2, 4, 5, 6, 3, 4, 2} : It's done by taking $\alpha_j = \beta_3$

no. 29) {1, 3, 4, 5, 6, 3, 4, 2} : It's done by taking $\alpha_j = \beta_2$

no. 30) {2, 3, 4, 5, 6, 3, 4, 2} : Use Method 1

$\Delta_P - \Delta_{\bar{P}}$	$gr(q_{\gamma^\vee})$	$\sum_{i=1}^{r} d_i$	$length(\omega_P \omega_{\bar{P}})$
{}	{0, 0, 0, 0, 0, 0, 58}	0	0

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```
(* Consider case \P7) with o=0 *)
```

possible roots= {{0, 0, 1, 1, 1, 0, 1, 0}, {0, 1, 1, 1, 1, 0, 1, 0}, {1, 1, 1, 1, 1, 0, 1, 0},
{0, 1, 2, 2, 2, 1, 1, 0}, {1, 2, 2, 2, 2, 1, 1, 0}, {0, 0, 0, 0, 0, 0, 1, 1}, {0, 0, 1, 1, 1, 0, 1, 1},
{0, 1, 1, 1, 1, 0, 1, 1}, {1, 1, 1, 1, 1, 0, 1, 1}, {0, 1, 2, 2, 2, 1, 1, 1}, {1, 2, 2, 2, 2, 1, 1, 1},
{0, 0, 0, 1, 2, 1, 2, 1}, {0, 0, 1, 1, 2, 1, 2, 1}, {0, 1, 1, 1, 2, 1, 2, 1}, {1, 1, 1, 1, 2, 1, 2, 1},
{0, 1, 2, 2, 2, 1, 2, 1}, {1, 2, 2, 2, 2, 1, 2, 1}, {1, 2, 3, 3, 3, 1, 2, 1}, {0, 1, 2, 3, 4, 2, 3, 1},
{1, 1, 2, 3, 4, 2, 3, 1}, {1, 2, 2, 3, 4, 2, 3, 1}, {1, 2, 3, 3, 4, 2, 3, 1}, {0, 1, 2, 3, 4, 2, 3, 2},
{1, 1, 2, 3, 4, 2, 3, 2}, {1, 2, 2, 3, 4, 2, 3, 2}, {1, 2, 3, 3, 4, 2, 3, 2}, {1, 2, 3, 4, 5, 2, 4, 2},
{1, 2, 4, 5, 6, 3, 4, 2}, {1, 3, 4, 5, 6, 3, 4, 2}, {2, 3, 4, 5, 6, 3, 4, 2}}

$gr(q_{\beta_1}) = \{2, 0, 0, 0, 0, 0, 0, 0\}$
$gr(q_{\beta_2}) = \{-1, 3, 0, 0, 0, 0, 0, 0\}$
$gr(q_{\beta_3}) = \{0, -2, 4, 0, 0, 0, 0, 0\}$
$gr(q_{\beta_4}) = \{0, 0, -3, 5, 0, 0, 0, 0\}$
$gr(q_{\beta_5}) = \{0, 0, 0, -4, 6, 0, 0, 0\}$
$gr(q_{\beta_6}) = \{0, 0, 0, 0, -5, 7, 0, 0\}$
$gr(q_{\beta_7}) = \{0, 0, 0, 0, -5, -5, 12, 0\}$
$gr(q_{\beta_8}) = \{0, 0, 0, 0, 0, 0, -21, 23\}$

no. 1) {0, 0, 1, 1, 1, 0, 1, 0} : It's done by taking $\alpha_j = \beta_3$

no. 2) {0, 1, 1, 1, 1, 0, 1, 0} : It's done by taking $\alpha_j = \beta_2$

no. 3) {1, 1, 1, 1, 1, 0, 1, 0} : It's done by taking $\alpha_j = \beta_1$

no. 4) {0, 1, 2, 2, 2, 1, 1, 0} : It's done by taking $\alpha_j = \beta_3$

no. 5) {1, 2, 2, 2, 2, 1, 1, 0} : It's done by taking $\alpha_j = \beta_2$

no. 6) {0, 0, 0, 0, 0, 0, 1, 1} : Use Method 2

	$gr(q_{\beta^\vee})$	$\{\lvert R^+{}_k - R_{k-1}\rvert, ..., \lvert R^+{}_{r-1} - R_{r-2}\rvert\}$
take $\alpha_j = \beta_7$	{0, 0, 0, 0, 0, 0, -21, 23}	{0, 0}

no. 7) {0, 0, 1, 1, 1, 0, 1, 1} : It's done by taking $\alpha_j = \beta_3$

no. 8) {0, 1, 1, 1, 1, 0, 1, 1} : It's done by taking $\alpha_j = \beta_2$

no. 9) {1, 1, 1, 1, 1, 0, 1, 1} : It's done by taking $\alpha_j = \beta_1$

no. 10) {0, 1, 2, 2, 2, 1, 1, 1} : It's done by taking $\alpha_j = \beta_3$

no. 11) {1, 2, 2, 2, 2, 1, 1, 1} : It's done by taking $\alpha_j = \beta_2$

no. 12) {0, 0, 0, 1, 2, 1, 2, 1} : Use Method 2

	$gr(q_{\beta^\vee})$	$\{\lvert R^+{}_k - R_{k-1}\rvert, ..., \lvert R^+{}_{r-1} - R_{r-2}\rvert\}$
take $\alpha_j = \beta_7$	{0, 0, -3, -3, 2, 2, -9, 23}	{2, 2}

no. 13) {0, 0, 1, 1, 2, 1, 2, 1} : It's done by taking $\alpha_j = \beta_3$

no. 14) {0, 1, 1, 1, 2, 1, 2, 1} : It's done by taking $\alpha_j = \beta_2$

no. 15) {1, 1, 1, 1, 2, 1, 2, 1} : It's done by taking $\alpha_j = \beta_1$

no. 16) {0, 1, 2, 2, 2, 1, 2, 1} : It's done by taking $\alpha_j = \beta_3$

no. 17) {1, 2, 2, 2, 2, 1, 2, 1} : It's done by taking $\alpha_j = \beta_2$

no. 18) {1, 2, 3, 3, 3, 1, 2, 1} : It's done by taking $\alpha_j = \beta_3$

no. 19) {0, 1, 2, 3, 4, 2, 3, 1} : Use Method 2

	$gr(q_{\beta^\vee})$	$\{\lvert R^+{}_k - R_{k-1}\rvert, \ldots, \lvert R^+{}_{r-1} - R_{r-2}\rvert\}$
take $\alpha_j = \beta_7$	{-1, -1, -1, -1, 4, 4, 3, 23}	{4, 4}

no. 20) {1, 1, 2, 3, 4, 2, 3, 1} : It's done by taking $\alpha_j = \beta_1$

no. 21) {1, 2, 2, 3, 4, 2, 3, 1} : It's done by taking $\alpha_j = \beta_2$

no. 22) {1, 2, 3, 3, 4, 2, 3, 1} : It's done by taking $\alpha_j = \beta_3$

no. 23) {0, 1, 2, 3, 4, 2, 3, 2} : Use Method 1

$\Delta_P - \Delta_{\bar{P}}$	$gr(q_{\gamma^\vee})$	$\sum_{i=1}^{r} d_i$	$length(\omega_P \omega_{\bar{P}})$
$\{\alpha_1\}$	{-1, -1, -1, -1, -1, -1, -6, 46}	-12	12

no. 24) {1, 1, 2, 3, 4, 2, 3, 2} : It's done by taking $\alpha_j = \beta_1$

no. 25) {1, 2, 2, 3, 4, 2, 3, 2} : It's done by taking $\alpha_j = \beta_2$

no. 26) {1, 2, 3, 3, 4, 2, 3, 2} : It's done by taking $\alpha_j = \beta_3$

no. 27) {1, 2, 3, 4, 5, 2, 4, 2} : Use Method 3

$gr(q_{\gamma^\vee})$	$\lvert B_2\rvert$ VS $\lvert B_3\rvert$		$-\sum_{i=1}^{r} d_i$
{0, 0, 0, 0, 0, -6, 6, 46}	6	6	0

no. 28) {1, 2, 4, 5, 6, 3, 4, 2} : It's done by taking $\alpha_j = \beta_3$

no. 29) {1, 3, 4, 5, 6, 3, 4, 2} : It's done by taking $\alpha_j = \beta_2$

no. 30) {2, 3, 4, 5, 6, 3, 4, 2} : It's done by taking $\alpha_j = \beta_1$

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(* maiNow we provide the programs.  *)

(*  This is the main program. It works
 for all cases. One can run it in the beginning.  *)
(*Whenever the program
  is given in magenta color, the corresponding part should be modified case by case *)

(* Need to input the followings the following
    (* Given a set of roots*)
           roots=...;
      (* need to define the Cartan matrix cartan=(<βi, βj∨>)  *)
           cartan=...;
    (* deltap denotes the set of \Delta_P *)
            deltap=...;        (* lenpp[[i]]=\ell(\omega_P\omega_{\bar P},
   where \Delta_{\bar P}=\Delta_P-\{\alpha_i\}, for i=1, 2, \cdots, r *)
          lenpp=...;
       (* grad[[i]]= gr(q_{βi∨})*)
          grad=...;
  *)

 (* method 0: determine whether the case can be reduced to  A-type *)
mtd0[GAM_List] := Module[
   {gam = GAM, i, j, k, aa, bb, cc, tab, n1 = 0, n2 = 0, len, m1, nn}, len = Length[deltap];
   nn = Length[roots[[1]]]; aa = IdentityMatrix[nn]; bb = {}; For[j = 1, j ≤ nn, m1 = 0;
    For[k = 1, k ≤ len, If[aa[[j]].cartan.aa[[deltap[[k]]]] == 0, m1 = m1 + 1]; k++];
    If[m1 == len, AppendTo[bb, j]]; j++];
   For[i = 1, i ≤ len - 1, If[cartan[[deltap[[i]]]].gam > 0, n1 = deltap[[i]]]; i++];
   For[i = 1, i ≤ Length[bb], If[cartan[[bb[[i]]]].gam > 0, n2 = bb[[i]]]; i++];
   If[n1 > 0, tab = {1, " It's done by taking αj=β"_n1 , " "}, If[n2 > 0, tab =
      {1, " It's done by taking  αj =β"_n2 , " "}, tab = {0, "with other method"}]]; tab];

 (* method 1:  compute gr(q_γ∨) =
 ∑_{i=1}^{r+1} di ei and  Δ_{P̄}={α∈ΔP | <α, γ∨> =0}. If  "∑_{i=1}^{r} di ≤ length(ωP ω_{P̄})", then it is done *)
mtd1[GAM_List] := Module[{gam = GAM, i,  aa, bb, cc, tab, m1 = 0, n1 = 0, len, dd, ff},
   dd = gam.grad; len = Length[deltap];
   For[i = 1, i ≤ len, If[cartan[[deltap[[i]]]].gam == 0, m1 = m1 + 1, n1 = i]; i++];
```

```
    If[m1 == len - 1, aa = lenpp[[n1]], If[m1 == len, aa = 0, aa = -1]];
    ff = Length[dd]; bb = Total[dd] - dd[[ff]]; cc = Table["α"n1, {i, 1, Sign[aa]}];
    If[(aa ≥ 0) && (bb + aa) ≤ 0, tab = {1, " Use Method 1 ",
        {{"ΔP-Δ_P", "gr(qγ∨)", "∑_{i=1}^{r} di", "length(ωPω_P)"}, {cc, dd, bb, aa}}},
     tab = {0, "with other method"}]; tab];

  (*  Δ_P={i | <χi, γ∨> ≠0}⋃ deltap determines a root sub-system, saying Rγ. Let B2:=
  {α∈ Rγ+-RP | <α, γ∨> >0,  α>γ}. Determine whether a root tt is in B2 *)
wheb2[TT_List, GAM_List] := Module[{tt = TT, gam = GAM,  aa, bb1, bb2, bb3, pos1, pos2,
     cc = 0, m1 = 0, m2 = 0, m3 = 0, ee, nn}, nn = Length[roots[[1]]]; ee = IdentityMatrix[nn];
    bb1 = {}; For[i = 1, i ≤ nn, If[Not[MemberQ[gam - ee[[i]], -1]], AppendTo[bb1, i]]; i++];
    bb2 = Union[bb1, deltap]; bb3 = Complement[bb2, deltap];
    pos1 = Table[0, {i, 1, nn}]; pos2 = pos1; For[i = 1, i ≤ nn,
     If[MemberQ[bb3, i], pos1[[i]] = 1]; If[Not[MemberQ[bb2, i]], pos2[[i]] = 1]; i++];
    If[(pos1.tt > 0) && (pos2.tt == 0), m1 = 1]; aa = tt.cartan.gam; If[aa > 0, m2 = 1];
    If[Total[tt - aa * gam] > 0, m3 = 1]; If[m1 + m2 + m3 == 3, cc = 1]; cc];
   (*   Let B3:={α∈ RP+ | <α, γ∨> >0}. Determine whether a root tt is in B3 *)
wheb3[TT_List, GAM_List] := Module[{tt = TT, gam = GAM, pos1, aa, cc = 0, m1 = 0, m2 = 0, nn},
    nn = Length[roots[[1]]]; pos1 = Table[0, {i, 1, nn}]; For[i = 1, i ≤ nn,
     If[Not[MemberQ[deltap, i]], pos1[[i]] = 1]; i++]; If[tt.pos1 == 0, m1 = 1];
    aa = tt.cartan.gam; If[aa > 0, m2 = 1]; If[m1 + m2 == 2, cc = 1]; cc];

  (* method 3: a special case that \gamma in \Delta_P is also tested  compute gr(qγ∨)=
  ∑_{i=1}^{r+1} di ei,   |B2| and |B3|.
           Then we test whether \gamma in \Delta_P or
   "|B2| ≤|B3| - ∑_{i=1}^{r} di". If either of them holds, then it is done;  *)
mtd3[GAM_List] := Module[{gam = GAM, i, j, aa, bb, cc, aa2,
     tab, m1 = 0, n1 = 0, len, dd, hh, ff, len88, ee, bb1, bb2, nn},
    len = Length[deltap]; nn = Length[roots[[1]]]; ee = IdentityMatrix[nn]; bb1 = {};
    For[i = 1, i ≤ nn, If[Not[MemberQ[gam - ee[[i]], -1]], AppendTo[bb1, i]]; i++];
    bb2 = Union[bb1, deltap]; len88 = Length[roots];
    dd = gam.grad; ff = Length[dd]; bb = Total[dd] - dd[[ff]];
     aa = Total[Table[wheb2[roots[[i]], gam], {i, 1, len88}]];
    hh = Total[Table[wheb3[roots[[i]], gam], {i, 1, len88}]];
    If[bb2 == deltap, tab = {1, " It is done sine γ∈ΔP", " " },
     If[aa - hh + bb ≤ 0, If[Sort[bb2] == Table[j, {j, 1, 8}], tab = {1, " Use Method 3 ",
         {{"gr(qγ∨)", "|B2|", "VS", "|B3|", " ", "-∑_{i=1}^{r} di"}, {dd, aa, " ", hh, " ", -bb}}},
       tab = {1, " Use Method 3 ",
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        {{"Δ(γ)", "gr(q_γ∨)", "|B2|", "VS", "|B3|", " ", "-∑_{i=1}^{r} d_i"},
          {{Min[bb2] , "...", Max[bb2]}, dd, aa, " ", hh, " ", -bb}}}],
    tab = {0, "with other method"}]]; tab];

 (* R_j denotes the root system generated by the base Δ_j={α1,..., α_j}*)
mtd2[GAM_List] :=
  Module[{gam = GAM, i, j, aa, bb, cc, tab, len, dd, ee, t1, t2, t3, m1, bet, pos , nn},
   len = Length[deltap]; aa = deltap[[len]]; nn = Length[roots[[1]]];
   ee = IdentityMatrix[nn]; dd = gam.grad; tab = {0, "<α_r, γ∨> ≤ 0; use other method."};
   If[cartan[[aa]].gam > 0, bet = gam - ee[[aa]]; bb = bet.grad; t1 = Table[0, {i, 1, len}];
    For[i = 1, i ≤ len, If[cartan[[deltap[[i]]]].ee[[aa]] ≠ 0, t1[[i]] = 1]; i++];
    pos = Position[t1, 1][[1, 1]]; t2 = Table[bb[[j]], {j, pos, len - 1}]; t3 = {};
    For[i = pos, i ≤ len - 1, cc = ee[[deltap[[i]]]]; If[cc.cartan.bet > 0, m1 = 1, m1 = 0];
     For[j = 1, j < i, cc = cc + ee[[deltap[[i - j]]]]; If[cc.cartan.bet > 0, m1 = m1 + 1]; j++];
     AppendTo[t3, m1]; i++]; If[MemberQ[Sign[t2 - t3], -1],
     tab = {1, " Need more analysis. ", {{"gr(q_γ∨)", "  " , "gr(q_β∨)",
         "{|R+_k-R_k-1|,...,|R+_r-1 - R_r-2|}"}, {dd, "take α_j =β"_aa , bb, t3 }}},
     tab = {1, " Use Method 2", {{"  " , "gr(q_β∨)", "{|R+_k-R_k-1|,...,|R+_r-1 - R_r-2|}"},
        {"take α_j =β"_aa , bb, t3}}}]]; tab];


 (* Test the method for a given coroot GAM,
in the order: (Reduction to A-type)---> (M1)--→(M3)--→(M2) *)
method[GAM_List] := Module[{gam = GAM, aa, bb, cc, i, len, ta, res},
   aa = mtd1[gam]; If[mtd0[gam][[1]] == 1, cc = Delete[mtd0[gam], 1], If[aa[[1]] == 1,
     cc = {aa[[2]], aa[[3]]}, If[mtd3[gam][[1]] == 1, cc = Delete[mtd3[gam], 1],
      If[mtd2[gam][[1]] == 1, cc = Delete[mtd2[gam], 1], cc = aa]]]]; cc];

(* Input: mos, i.e. the table of all the coroots; Output:
 the gradings of all gr(q_j)'s and the corresponding method for the given coroots *)
outform[ATMOST_List] :=
  Module[{mos = ATMOST, aa, b2, i, j, a11, a22, len}, len = Length[mos[[1]]];
    aa = {};    For[i = 1, i ≤ len, b2 = {"gr(", "q_β"_i, ")", "=", grad[[i]]};
    AppendTo[aa, b2]; i++]; Print[MatrixForm[aa] ];
   For[j = 1, j ≤ Length[mos], a11 = mos[[j]]; a22 = method[a11]; Print["no. ", j, ")   ",
     " ", a11, " : ", a22[[1]]]; Print[MatrixForm[a22[[2]]]]; Print[" "]; j++];];
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Now we input data case by case

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(* ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~ *)
(*Run  the followings  for Case \P9) with   r=2 *)

(* roots of F4 with respect the order of case 9)*)

roots = {{0, 1, 0, 0}, {0, 1, 1, 0}, {0, 1, 2, 0}, {0, 1, 1, 1}, {0, 1, 2, 2}, {0, 1, 2, 1},
   {0, 0, 0, 1}, {0, 0, 1, 0}, {1, 0, 0, 0}, {0, 0, 1, 1}, {1, 1, 0, 0}, {1, 1, 1, 0},
   {1, 1, 1, 1}, {1, 1, 2, 0}, {1, 2, 2, 0}, {1, 1, 2, 1}, {1, 2, 2, 1}, {1, 1, 2, 2},
   {1, 2, 3, 1}, {1, 2, 2, 2}, {1, 2, 3, 2}, {1, 2, 4, 2}, {1, 3, 4, 2}, {2, 3, 4, 2}};

(* Case \P9) with   r=2
     Define the Cartan matrix cartan=(<βi, βj∨>)   *)
cartan = {{2, -1, 0, 0}, {-1, 2, -2, 0}, {0, -1, 2, -1}, {0, 0, -1, 2}};

  (* deltap denotes the set of \Delta_P *)
deltap = {2, 3};
  (* lenpp[[i]]=\ell(\omega_P\omega_{\bar P},
   where \Delta_{\bar P}=\Delta_P-\{\alpha_i\}, for i=1, 2, \cdots, r *)
lenpp = {3, 3};
(* grad[[i]]= gr(q_βi∨)*)
grad = { {-1, -2, 5},
   {2, 0, 0},
   {-2, 4, 0},
   {0, -4, 6}};

(*Consider case \P9) with r=2 *)
mos =  {{0, 1, 1, 0}, {0, 0, 1, 1},
   {1, 1, 1, 0}, {0, 1, 1, 1}, {1, 1, 1, 1}, {1, 2, 2, 1}, {2, 3, 2, 1}};

Print["possible roots=", " ", mos]

outform[mos]

(*Run  it and obtain output  for Case \P9) with   r=2 *)
(* ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~ *)
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(* `````````````````````````````````` *)
(*Proceed to run the followings for  Case \P9) with   r=3 *)
 (*Neet to use the same roots*)
(* Case \P9) with   r=3
     Define the Cartan matrix cartan=(<βi, βj∨>)  *)
cartan = {{2, -1, 0, 0}, {-1, 2, -2, 0}, {0, -1, 2, -1}, {0, 0, -1, 2}};

  (* deltap denotes the set of \Delta_P *)
deltap = {1, 2, 3};
  (* lenpp[[i]]=\ell(\omega_P\omega_{\bar P},
   where \Delta_{\bar P}=\Delta_P-\{\alpha_i\}, for i=1, 2, \cdots, r *)
lenpp = {5, 7, 6};
(* grad[[i]]= gr(q_{βi∨}) *)
grad = { {2,  0, 0, 0},
   {-1, 3, 0, 0}, {0, -4, 6, 0},
   {0, 0, -9, 11}};

(*Consider case \P9) with r=3 *)
mos =  {{0, 1, 1, 0}, {0, 0, 1, 1}, {1, 1, 1, 0},
   {0, 1, 1, 1}, {1, 1, 1, 1}, {1, 2, 2, 1}, {2, 3, 2, 1}};
Print["possible roots=", " ", mos]

outform[mos]

(*Run  it and obtain output  for Case \P9) with   r=3 *)
(* ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~ *)
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(* ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~ *)
(*Run  the followings  for Case \P10) with   r=2 *)

(* roots of F4 with respect the order of case 10)*)
roots = { {0, 0, 1, 0}, {0, 1, 1, 0}, {0, 2, 1, 0}, {1, 1, 1, 0}, {2, 2, 1, 0}, {1, 2, 1, 0},
    {1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 0, 1}, {1, 1, 0, 0}, {0, 0, 1, 1}, {0, 1, 1, 1},
   {1, 1, 1, 1}, {0, 2, 1, 1}, {0, 2, 2, 1}, {1, 2, 1, 1}, {1, 2, 2, 1}, {2, 2, 1, 1},
   {1, 3, 2, 1}, {2, 2, 2, 1}, {2, 3, 2, 1}, {2, 4, 2, 1}, {2, 4, 3, 1}, {2, 4, 3, 2}};

(* Case \P10) with   r=2
     Define the Cartan matrix cartan=(<βi, βj^∨>)   *)
cartan = {{2, -1, 0, 0}, {-1, 2, -1, 0}, {0, -2, 2, -1}, {0, 0, -1, 2}};

  (* deltap denotes the set of \Delta_P *)
deltap = {2, 3};
  (* lenpp[[i]]=\ell(\omega_P\omega_{\bar P},
   where \Delta_{\bar P}=\Delta_P-\{\alpha_i\}, for i=1, 2, \cdots, r *)
lenpp = {3, 3};
(* grad[[i]]= gr(q_{βi^∨}) *)
grad = { {-1, -3, 6},
   {2, 0, 0},
   {-1, 3, 0},
   {0, -3, 5}};

(* Case \P10) with   r=2 *)
mos = {{0, 0, 1, 1}, {1, 1, 1, 0}, {1, 1, 1, 1},
   {0, 1, 2, 1}, {1, 1, 2, 1}, {1, 2, 3, 1}, {1, 2, 3, 2}};


Print["possible roots=", " ", mos]
outform[mos]


(*Run  it and obtain output  for Case \P10) with   r=2 *)
(* ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~ *)

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(* `````````````````````````````````` *)
(*Proceed to run the followings for  Case \P10) with   r=3 *)
(*Neet to use the same roots*)

(* Case \P10) with   r=3
     Define the Cartan matrix cartan=(<βi, βj∨>)  *)
cartan = {{2, -1, 0, 0}, {-1, 2, -1, 0}, {0, -2, 2, -1}, {0, 0, -1, 2}};

  (* deltap denotes the set of \Delta_P *)
deltap = {1, 2, 3};
  (* lenpp[[i]]=\ell(\omega_P\omega_{\bar P},
   where \Delta_{\bar P}=\Delta_P-\{\alpha_i\}, for i=1, 2, \cdots, r *)
lenpp = {5, 7, 6};
(* grad[[i]]= gr(qβi∨)*)
grad = { {2, 0, 0, 0},
   {-1, 3, 0, 0}, {0, -2, 4, 0},
   {0, 0, -6, 8}};

(*Consider case \P10) with r=3 *)
mos = {{0, 0, 1, 1}, {1, 1, 1, 0}, {1, 1, 1, 1},
   {0, 1, 2, 1}, {1, 1, 2, 1}, {1, 2, 3, 1}, {1, 2, 3, 2}};
Print["possible roots=", " ", mos]
outform[mos]

(*Run  it and obtain output  for Case \P10) with   r=3 *)
(* ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~ *)


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(* ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~ *)
(*Run  the followings  for Case \P4) with   r=6 *)

(* Now we consider the case of  E-type *)

(* We first produce roots of E8 with respect the order of case \P6,
and then obtain the set "roots" of positive
 roots of  E8 with respect to the order of case \P4*)
(* Define <βi, βj> for i, j=1, ..., 8 *)
E8 = {{2, -1, 0, 0, 0, 0, 0, 0}, {-1, 2, -1, 0, 0, 0, 0, 0},
   {0, -1, 2, -1, 0, 0, 0, -1}, {0, 0, -1, 2, -1, 0, 0, 0}, {0, 0, 0, -1, 2, -1, 0, 0},
```

```
   {0, 0, 0, 0, -1, 2, -1, 0}, {0, 0, 0, 0, 0, -1, 2, 0}, {0, 0, -1, 0, 0, 0, 0, 2}};
root66[TT_List] := Module[{tt = TT, len, i, j, ta, tb, tc = {}, td, te},
   len = Length[tt]; For[i = 1, i ≤ len, ta = tt[[i]]; AppendTo[tc, ta];
    tb = Sign[ta.E8]; For[j = 1, j ≤ 8, If[tb[[j]] == 1, td = {0, 0, 0, 0, 0, 0, 0, 0};
      td[[j]] = -1; AppendTo[tc, ta + td]]; j++]; i++]; te = Union[tc]; te];
aa = {{2, 4, 6, 5, 4, 3, 2, 3}};
 For[i = 1, i ≤ 120, aa = root66[aa]; i++];
rootE8case66 = Delete[aa, 1];

(* roots of E8 with respect the order of case 4)*)
roots = {};
For[i = 1, i ≤ Length[rootE8case66], cc = rootE8case66[[i]]; AppendTo[roots,
   {cc[[7]], cc[[6]], cc[[5]], cc[[4]], cc[[3]], cc[[2]], cc[[1]], cc[[8]]}]; i++];
roots;
(* Case \P4) with \Delta_P being E_6-type. i.e. r=6
     Define the Cartan matrix cartan=(<βi, βj>)   *)
cartan = {{2, -1, 0, 0, 0, 0, 0, 0}, {-1, 2, -1, 0, 0, 0, 0, 0},
   {0, -1, 2, -1, 0, 0, 0, 0}, {0, 0, -1, 2, -1, 0, 0, 0}, {0, 0, 0, -1, 2, -1, 0, -1},
   {0, 0, 0, 0, -1, 2, -1, 0}, {0, 0, 0, 0, 0, -1, 2, 0}, {0, 0, 0, 0, -1, 0, 0, 2}};

  (* deltap denotes the set of \Delta_P *)
deltap = {3, 4, 5, 6, 7, 8};
  (* lenpp[[i]]=\ell(\omega_P\omega_{\bar P},
   where \Delta_{\bar P}=\Delta_P-\{\alpha_i\}, for i=1, 2, \cdots, r *)
lenpp = {16, 25, 29, 25, 16, 21};
(* grad[[i]]= gr(qβi∨)*)
 grad = {{0, 0, 0, 0, 0, 0, 2},
   {-1, -1, -1, -1, -1, -11, 18}, {2, 0, 0, 0, 0, 0, 0},
   {-1, 3, 0, 0, 0, 0, 0},
   {0, -2, 4, 0, 0, 0, 0},
   {0, 0, -3, 5, 0, 0, 0},
   {0, 0, 0, -4, 6, 0, 0},
   {0, 0, -3, -3, -3, 11, 0}};

(* The next program output all the roots \gamma for which <αj, γ∨> ≤0 for all 1≤j≤r-1.
           It turns out that in case \P4), it produce one*)
mos = {};
For[i = 1, i ≤ Length[roots], a1 = roots[[i]];
  If[Total[a1] > 1, m1 = 0; n1 = 0; For[j = 3, j ≤ 7, b1 = cartan[[j]];
    If[a1.b1 <= 0, m1 = m1 + 1]; j++]; If[m1 == 5, AppendTo[mos, a1]]]; i++];
Print["possible roots=", " ", mos]
outform[mos]

 (*Run it and obtain output for Case \P4) with   r=6 *)
(* ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~ *)



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(* ```````````````````````````````` *)
(*Proceed to run the followings for  Case \P4) with   r=7 *)

(* Need to use the set "roots" and the same "mos"
     defined as above for case \P4) with r=6*)

(* Case \P4) with \Delta_P being E_7-type. i.e. r=7
     Define the Cartan matrix cartan=(<βi, βj>)  *)
cartan = {{2, -1, 0, 0, 0, 0, 0, 0}, {-1, 2, -1, 0, 0, 0, 0, 0},
   {0, -1, 2, -1, 0, 0, 0, 0}, {0, 0, -1, 2, -1, 0, 0, 0}, {0, 0, 0, -1, 2, -1, 0, -1},
   {0, 0, 0, 0, -1, 2, -1, 0}, {0, 0, 0, 0, 0, -1, 2, 0}, {0, 0, 0, 0, -1, 0, 0, 2}};

  (* deltap denotes the set of \Delta_P *)
deltap = {2, 3, 4, 5, 6, 7, 8};
  (* lenpp[[i]]=\ell(\omega_P\omega_{\bar P},
   where \Delta_{\bar P}=\Delta_P-\{\alpha_i\}, for i=1, 2, \cdots, r *)
lenpp = {27, 42, 50, 53, 47, 33};
(* grad[[i]]= gr(q_{βi^∨})*)
grad = {
   {-1, -1, -1, -1, -1, -1, -57, 65}, {2, 0,  0, 0, 0, 0, 0, 0},
   {-1, 3, 0, 0, 0, 0, 0, 0},
   {0, -2, 4, 0, 0, 0, 0, 0},
   {0, 0, -3, 5, 0, 0, 0, 0},
   {0, 0, 0, -4, 6, 0, 0, 0},
   {0, 0, 0, 0, -5, 7, 0, 0},
   {0, 0, 0, -4, -4, -4, 14, 0}};

Print["possible roots=", " ", mos]

outform[mos]

(*Run it and obtain output for Case \P4) with   r=7 *)
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(* ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~ *)
(*Run  the followings  for Case \P5) with   r=5 *)

(* Now we consider case \P5)*)
```

```
(* We first produce roots of E8 with respect the order of case \P6,
and then obtain the set "roots" of positive
 roots of  E8 with respect to the order of case \P5*)
(* Define <βi, βj∨> for i, j=1, ..., 8 *)
E8 = {{2, -1, 0, 0, 0, 0, 0, 0}, {-1, 2, -1, 0, 0, 0, 0, 0},
   {0, -1, 2, -1, 0, 0, 0, -1}, {0, 0, -1, 2, -1, 0, 0, 0}, {0, 0, 0, -1, 2, -1, 0, 0},
   {0, 0, 0, 0, -1, 2, -1, 0}, {0, 0, 0, 0, 0, -1, 2, 0}, {0, 0, -1, 0, 0, 0, 0, 2}};
root66[TT_List] := Module[{tt = TT, len, i, j, ta, tb, tc = {}, td, te},
   len = Length[tt]; For[i = 1, i ≤ len, ta = tt[[i]]; AppendTo[tc, ta];
    tb = Sign[ta.E8]; For[j = 1, j ≤ 8, If[tb[[j]] == 1, td = {0, 0, 0, 0, 0, 0, 0, 0};
      td[[j]] = -1; AppendTo[tc, ta + td]]; j++]; i++]; te = Union[tc]; te];
aa = {{2, 4, 6, 5, 4, 3, 2, 3}};
 For[i = 1, i ≤ 120, aa = root66[aa]; i++];
rootE8case66 = Delete[aa, 1];

(* roots of E8 with respect the order of case \P5)*)
roots = {};
For[i = 1, i ≤ Length[rootE8case66], cc = rootE8case66[[i]]; AppendTo[roots,
   {cc[[1]], cc[[2]], cc[[3]], cc[[8]], cc[[4]], cc[[5]], cc[[6]], cc[[7]]}]; i++];
roots;

 (* Case \P5) with \Delta_P being D_5-type. i.e. r=5
     Define the Cartan matrix cartan=(<βi, βj>)  *)
cartan = {{2, -1, 0, 0, 0, 0, 0, 0}, {-1, 2, -1, 0, 0, 0, 0, 0},
   {0, -1, 2, -1, -1, 0, 0, 0}, {0, 0, -1, 2, 0, 0, 0, 0}, {0, 0, -1, 0, 2, -1, 0, 0},
   {0, 0, 0, 0, -1, 2, -1, 0}, {0, 0, 0, 0, 0, -1, 2, -1}, {0, 0, 0, 0, 0, 0, -1, 2}};

  (* deltap denotes the set of \Delta_P *)
deltap = {1, 2, 3, 4, 5};
  (* lenpp[[i]]=\ell(\omega_P\omega_{\bar P},
   where \Delta_{\bar P}=\Delta_P-\{\alpha_i\}, for i=1, 2, \cdots, r *)
lenpp = {8, 13, 15, 10, 10};
(* grad[[i]]= gr(q_{βi∨})*)
 grad = {
   {2, 0,  0, 0, 0, 0},
   {-1, 3, 0, 0, 0, 0},
   {0, -2, 4, 0, 0, 0},
   {0, 0, -3, 5, 0, 0},
   {0, 0, -3, -3, 8, 0},
   {0, 0, 0, 0, -10, 12},
   {0, 0, 0, 0, 0, 2}, {0, 0, 0, 0, 0, 2}};

(*Consider case \P5) with \Delta_P being also of D_5-type *)
(* The next program output all the roots \gamma for which <βj,
γ∨> ≤0 for all 1≤j≤4 and 7≤j≤8 *)
```

```
mos = {};
For[i = 1, i ≤ Length[roots], a1 = roots[[i]]; If[Total[a1] > 1, m1 = 0;
   n1 = 0; For[j = 1, j ≤ 4, b1 = cartan[[j]]; If[a1.b1 <= 0, m1 = m1 + 1]; j++];
   For[j = 7, j ≤ 8, b1 = cartan[[j]]; If[a1.b1 <= 0, m1 = m1 + 1]; j++];
   If[(m1 == 6), AppendTo[mos, a1]]]; i++];
Print["possible roots=", " ", mos]

outform[mos]

 (*Run it and obtain output for Case \P5) with   r=5 *)
(* ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~ *)



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(* ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~ *)
(*Run  the followings  for Case \P7) with   o=3. i.e. ΔP is of D_4-type  *)

(* Now we consider case \P7) *)
(* We first produce roots of E8 with respect the order of case \P7,
and then obtain the set "roots" of positive
 roots of  E8 with respect to the order of case \P7*)
(* Define <βi, βj∨> for i, j=1, ..., 8 *)
E8 = {{2, -1, 0, 0, 0, 0, 0, 0}, {-1, 2, -1, 0, 0, 0, 0, 0},
   {0, -1, 2, -1, 0, 0, 0, -1}, {0, 0, -1, 2, -1, 0, 0, 0}, {0, 0, 0, -1, 2, -1, 0, 0},
   {0, 0, 0, 0, -1, 2, -1, 0}, {0, 0, 0, 0, 0, -1, 2, 0}, {0, 0, -1, 0, 0, 0, 0, 2}};
root66[TT_List] := Module[{tt = TT, len, i, j, ta, tb, tc = {}, td, te},
   len = Length[tt]; For[i = 1, i ≤ len, ta = tt[[i]]; AppendTo[tc, ta];
    tb = Sign[ta.E8]; For[j = 1, j ≤ 8, If[tb[[j]] == 1, td = {0, 0, 0, 0, 0, 0, 0, 0};
      td[[j]] = -1; AppendTo[tc, ta + td]]; j++]; i++]; te = Union[tc]; te];
aa = {{2, 4, 6, 5, 4, 3, 2, 3}};
 For[i = 1, i ≤ 120, aa = root66[aa]; i++];
rootE8case66 = Delete[aa, 1];


(* roots of E8 with respect the order of case 7)*)
roots = {};
For[i = 1, i ≤ Length[rootE8case66], cc = rootE8case66[[i]]; AppendTo[roots,
   {cc[[7]], cc[[6]], cc[[5]], cc[[4]], cc[[3]], cc[[8]], cc[[2]], cc[[1]]}]; i++];
roots;
(* Case \P7) with \Delta_P being D_4-type. i.e. o=3
```

```
     Define the Cartan matrix cartan=(<β_i, β_j>)   *)
cartan = {{2, -1, 0, 0, 0, 0, 0, 0}, {-1, 2, -1, 0, 0, 0, 0, 0},
   {0, -1, 2, -1, 0, 0, 0, 0}, {0, 0, -1, 2, -1, 0, 0, 0}, {0, 0, 0, -1, 2, -1, -1, 0},
   {0, 0, 0, 0, -1, 2, 0, 0}, {0, 0, 0, 0, -1, 0, 2, -1}, {0, 0, 0, 0, 0, 0, -1, 2}};

  (* deltap denotes the set of \Delta_P *)
deltap = {4, 5, 6, 7};
  (* lenpp[[i]]=\ell(\omega_P\omega_{\bar P},
   where \Delta_{\bar P}=\Delta_P-\{\alpha_i\}, for i=1, 2, \cdots, r *)
lenpp = {6, 9, 6, 6};
(* grad[[i]]= gr(q_{β_i^∨})*)
grad = { {0, 0, 0, 0, 2}, {0, 0, 0, 0, 2}, {-1, -1, -1, -3, 8},
   {2, 0, 0, 0, 0},
   {-1, 3, 0, 0, 0},
   {0, -2, 4, 0, 0},
   {0, -2, -2, 6, 0},
   {0, 0, 0, -6, 8}};

(*Consider case \P7) with \Delta_P being also of D-type *)
(* The next program output all the roots \gamma for which <β_j,
γ^∨> ≤0 for all 4≤j≤6 and at least two j in the set {1, 2, 3}.
   In other words,   <β_j, γ^∨> >0 only if j∈{7, 8} or for a unique one in {1, 2, 3}.  *)
 (*The same output "mos" is used whenever \P7) occurs. i.e. for all cases
    when o∈{0, 1, 2, 3, 4}*)
mos = {};
For[i = 1, i ≤ Length[roots], a1 = roots[[i]]; If[Total[a1] > 1, m1 = 0; n1 = 0;
     For[j = 1, j ≤ 3, b1 = cartan[[j]]; If[a1.b1 <= 0, n1 = n1 + 1]; j++];
   For[j = 4, j ≤ 6, b1 = cartan[[j]]; If[a1.b1 <= 0, m1 = m1 + 1]; j++];
   If[(m1 == 3) && (n1 ≥ 2), AppendTo[mos, a1]]]; i++];
mos;

Print["possible roots=", " ", mos]

outform[mos]

 (*Run it and obtain output for Case \P7) with   o=3 *)
(* ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~ *)



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(* ```````````````````````````````` *)
(*Proceed to run the followings for  Case \P7) with   o=2. i.e. Δ_P is of D_5-type  *)

(* Case \P7) with \Delta_P being D_5-type. i.e. o=2
     Define the Cartan matrix cartan=(<β_i, β_j>)   *)
cartan = {{2, -1, 0, 0, 0, 0, 0, 0}, {-1, 2, -1, 0, 0, 0, 0, 0},
   {0, -1, 2, -1, 0, 0, 0, 0}, {0, 0, -1, 2, -1, 0, 0, 0}, {0, 0, 0, -1, 2, -1, -1, 0},
   {0, 0, 0, 0, -1, 2, 0, 0}, {0, 0, 0, 0, -1, 0, 2, -1}, {0, 0, 0, 0, 0, 0, -1, 2}};

  (* deltap denotes the set of \Delta_P *)
deltap = {3, 4, 5, 6, 7};
  (* lenpp[[i]]=\ell(\omega_P\omega_{\bar P},
   where \Delta_{\bar P}=\Delta_P-\{\alpha_i\}, for i=1, 2, \cdots, r *)
lenpp = {8, 13, 15, 10, 10};
(* grad[[i]]= gr(q_{β_i^∨})*)
grad = { {0, 0, 0, 0, 0, 2}, {-1, -1, -1, -1, -4, 10},
   {2, 0, 0, 0, 0, 0},
   {-1, 3, 0, 0, 0, 0},
   {0, -2, 4, 0, 0, 0}, {0, 0, -3, 5, 0, 0},
   {0, 0, -3, -3, 8, 0},
   {0, 0, 0, 0, -10, 12}};

(*"mos" was defined when o=3 *)

Print["possible roots=", " ", mos]

(*Run it and obtain output for Case \P7) with   o=2 *)
(* ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~ *)
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(* ```````````````````````````````` *)
(*Proceed to run the followings for  Case \P7) with   o=1. i.e. ΔP is of D_6-type  *)

(*    Define the Cartan matrix cartan=(<βi, βj>)   *)
cartan = {{2, -1, 0, 0, 0, 0, 0, 0}, {-1, 2, -1, 0, 0, 0, 0, 0},
   {0, -1, 2, -1, 0, 0, 0, 0}, {0, 0, -1, 2, -1, 0, 0, 0}, {0, 0, 0, -1, 2, -1, -1, 0},
   {0, 0, 0, 0, -1, 2, 0, 0}, {0, 0, 0, 0, -1, 0, 2, -1}, {0, 0, 0, 0, 0, 0, -1, 2}};

  (* deltap denotes the set of \Delta_P *)
deltap = {2, 3, 4, 5, 6, 7};
  (* lenpp[[i]]=\ell(\omega_P\omega_{\bar P},
   where \Delta_{\bar P}=\Delta_P-\{\alpha_i\}, for i=1, 2, \cdots, r *)
lenpp = {10, 17, 21, 22, 15, 15};
(* grad[[i]]= gr(qβi∨)*)
grad = { {-1, -1, -1, -1, -1, -5, 12},
   {2, 0, 0, 0, 0, 0, 0},
   {-1, 3, 0, 0, 0, 0, 0},
   {0, -2, 4, 0, 0, 0, 0}, {0, 0, -3, 5, 0, 0, 0}, {0, 0, 0, -4, 6, 0, 0},
   {0, 0, 0, -4, -4, 10, 0},
   {0, 0, 0, 0, 0, -15, 17}};

(*"mos" was defined when o=3 *)


Print["possible roots=", " ", mos]

outform[mos]

 (*Run it and obtain output for Case \P7) with   o=1 *)
(* ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~ *)



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(* ````````````````````````````````` *)
(*Proceed to run the followings for  Case \P7) with   o=0. i.e. Δ_P is of D_7-type  *)

(*   Define the Cartan matrix cartan=(<β_i, β_j^∨>)   *)
cartan = {{2, -1, 0, 0, 0, 0, 0, 0}, {-1, 2, -1, 0, 0, 0, 0, 0},
   {0, -1, 2, -1, 0, 0, 0, 0}, {0, 0, -1, 2, -1, 0, 0, 0}, {0, 0, 0, -1, 2, -1, -1, 0},
   {0, 0, 0, 0, -1, 2, 0, 0}, {0, 0, 0, 0, -1, 0, 2, -1}, {0, 0, 0, 0, 0, 0, -1, 2}};

  (* deltap denotes the set of \Delta_P *)
deltap = {1, 2, 3, 4, 5, 6, 7};
  (* lenpp[[i]]=\ell(\omega_P\omega_{\bar P},
   where \Delta_{\bar P}=\Delta_P-\{\alpha_i\}, for i=1, 2, \cdots, r *)
lenpp = {12, 21, 27, 30, 30, 21, 21};
(* grad[[i]]= gr(q_{β_i^∨}) *)
grad = {
   {2, 0, 0, 0, 0, 0, 0, 0},
   {-1, 3, 0, 0, 0, 0, 0, 0},
   {0, -2, 4, 0, 0, 0, 0, 0}, {0, 0, -3, 5, 0, 0, 0, 0}, {0, 0, 0, -4, 6, 0, 0, 0},
   {0, 0, 0, 0, -5, 7, 0, 0}, {0, 0, 0, 0, -5, -5, 12, 0},
   {0, 0, 0, 0, 0, 0, -21, 23}};


 (*"mos" was defined when o=3 *)

Print["possible roots=", " ", mos]

outform[mos]

 (*Run it and obtain output for Case \P7) with   o=0 *)
(* ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~ *)
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Finish all outputs and we find two exceptional cases :

(a) case \ P4) with r = 6 and

no. 9) {1, 2, 3, 4, 5, 3, 1, 3} : Need more analysis.

$gr(q_{\gamma^\vee})$		$gr(q_{\beta^\vee})$	$\{\lvert R^+_k - R_{k-1}\rvert, \ldots, \lvert R^+_{r-1} - R_{r-2}\rvert\}$
{0, 0, 0, 0, -5, 11, 38}	take $\alpha_j = \beta_8$	{0, 0, 3, 3, -2, 0, 38}	{3, 3, 0}

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(b) case \ P7) with o = 1, 2 or 3 and no.= 27. Explicitly,

when o = 3, i.e. r = 4,

no. 27) {1, 2, 3, 4, 5, 2, 4, 2} : Need more analysis.

$$\begin{pmatrix} gr(q_{\gamma^\vee}) & & gr(q_{\beta^\vee}) & \{|R^+{}_k - R_{k-1}|, \ldots, |R^+{}_{r-1} - R_{r-2}|\} \\ \{0, 0, -3, 3, 46\} & \text{take } \alpha_j = \beta_7 & \{0, 2, -1, -3, 46\} & \{2, 0\} \end{pmatrix}$$

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when o = 2, i.e. r = 5,

no. 27) {1, 2, 3, 4, 5, 2, 4, 2} : Need more analysis.

$$\begin{pmatrix} gr(q_{\gamma^\vee}) & & gr(q_{\beta^\vee}) & \{|R^+{}_k - R_{k-1}|, \ldots, |R^+{}_{r-1} - R_{r-2}|\} \\ \{0, 0, 0, -4, 4, 46\} & \text{take } \alpha_j = \beta_7 & \{0, 0, 3, -1, -4, 46\} & \{3, 0\} \end{pmatrix}$$

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when o = 1, i.e. r = 6

no. 27) {1, 2, 3, 4, 5, 2, 4, 2} : Need more analysis.

$$\begin{pmatrix} gr(q_{\gamma^\vee}) & & gr(q_{\beta^\vee}) & \{|R^+{}_k - R_{k-1}|, \ldots, |R^+{}_{r-1} - R_{r-2}|\} \\ \{0, 0, 0, 0, -5, 5, 46\} & \text{take } \alpha_j = \beta_7 & \{0, 0, 0, 4, -1, -5, 46\} & \{4, 0\} \end{pmatrix}$$

References

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