TOWARDS SMALL QUANTUM CHERN CHARACTER

HUA-ZHONG KE, CHANGZHENG LI, AND JIAYU SONG

Abstract. We show a quantum version of Chern character homomorphism from the small quantum K-theory to the small quantum cohomology in the cases of projective spaces and incidence varieties, whose classical limit gives the classical Chern character homomorphism. We also provide a ring presentation of the small quantum K-theory of Milnor hypersurfaces.

1. Introduction

Let X be a smooth complex projective variety. Let $K(X) = K^0(X;\mathbb{Q})$ be the Grothendieck ring of topological complex vector bundles on X with rational coefficients. Consider the even part $H^{\mathrm{ev}}(X)$ of the rational cohomology $H^*(X) = H^*(X;\mathbb{Q})$. It is well-known that there is the Chern character ring homormorphism [Gr58, AH61]

$$\mathrm{ch} : K(X) \longrightarrow H^{\mathrm{ev}}(X).$$

There are quantum deformations of both sides. It is natural to ask the following.

Question 1.1. *Is there a quantum Chern character homomorhism in the quantum world?*

More precisely, the deformation $QH_{\mathrm{big}}(X) = (H^{\mathrm{ev}}(X)\otimes\mathbb{Q}[\![\mathbf{q},\mathbf{s}]\!], *_{\mathbf{s}})$ of $(H^{\mathrm{ev}}(X),\cup)$, called the (even) big quantum cohomology ring, encodes genus-zero Gromov-Witten invariants of X [RT95,BeFa97,LT98]. Its K-analogue is the big quantum K-theory $QK_{\mathrm{big}}(X) = (K(X)\otimes\mathbb{Q}[\![\mathbf{Q},\mathbf{t}]\!], \star_{\mathbf{t}})$, encoding genus-zero K-theoretic Gromov-Witten invariants [Gi00,Le04]. Their restrictions to the fiber at the origin give the small quantum cohomology/K-theory

$$QH(X) = (H^{\mathrm{ev}}(X)\otimes\mathbb{Q}[\![\mathbf{q}]\!], *), \qquad QK(X) = (K(X)\otimes\mathbb{Q}[\![\mathbf{Q}]\!], \star)$$

respectively, which are relatively more accessible than the big ones. Whenever the small quantum product is finite, we can consider the corresponding polynomial version

$$QH_{\mathrm{poly}}(X) = (H^{\mathrm{ev}}(X)\otimes\mathbb{Q}[\mathbf{q}], *), \qquad QK_{\mathrm{poly}}(X) = (K(X)\otimes\mathbb{Q}[\mathbf{Q}], \star)$$

For instance, $QH_{\mathrm{poly}}(X)$ always makes sense for any Fano manifold X, while $QK_{\mathrm{poly}}(X)$ are known to make sense for very few examples. There is a so-called fake quantum K-theory, which may be seen as building block of the big quantum K-theory. There has been the (big) quantum Chern(-Dold) character [Co03, Gi04, CG06], defined for the fake quantum K-theory. On the one hand, it gives a comparison of the genus-0 fake K-theoretic Gromov-Witten invariants with the cohomological ones in terms of symplectic geometry of loop spaces; moreover, by [IMT15, Remark 5.13], the big quantum K-theory is isomorphic to the big quantum cohomology as F-maniolds through the Hirzebruch-Riemann-Roch theorem for fake quantum K-theory [GT14]. On the other hand, in general such isomorphism does not preserve the small loci, and it looks unclear how the (big) quantum Chern character is viewed as a quantum version of the classical Chern character above.

The main aim of this paper is to explore a (small) quantum Chern character ring homomorphism qch for the small quantum K-theory, such that its classical limit gives ch, i.e. the following diagram of ring homomorphisms is commutative, where vertical maps are natural projections.

$$
\begin{tikzcd}
QK(X) \arrow[r, "\mathrm{qch}"] \arrow[d, "\bmod \mathbf{Q}"'] & QH^*(X) \arrow[d, "\bmod \mathbf{q}"] \\
QK(X)/(\mathbf{Q}) = K(X) \arrow[r, "\mathrm{ch}"] & H^*(X) = QH^*(X)/(\mathbf{q})
\end{tikzcd}
\tag{1.1}
$$

As we will see below, we succeed to find the small quantum Chern character morphism qch in the cases of projective spaces $\mathbb{P}^n$ and incidence varieties $\mathbb{F}\ell_{1,n-1;n}$.

Both $\mathbb{P}^n$ and $\mathbb{F}\ell_{1,n-1;n}$ are examples of flag varieties G/P, which are Fano manifolds and are central objects of study in enumerative geometry. There have been extensive studies of $QH_{\text{poly}}(G/P)$ (see e.g. the survey [LL17] and the references therein). There have also been studies of $QK(G/P)$ in individual cases [BM11, BCMP18, BCP23, LNS24, KLNS24, Xu24, LLSY25] from the lens of Schubert calculus, giving the quantum multiplication of certain Schubert classes. Ring presentations have recently been provided for the small quantum K-theory of type A partial flag varieities [KPSZ21, GMSZ22, GM$^+$23, GM$^+$24, HK24b, MNS25a, MNS25b, AHK$^+$25] and of type C complete flag varieties [KN24], as well as for general G/P in terms of K-theoretic version of Peterson's 'quantum = affine' statement [LLMS18, IIM20] proved in the groundbreaking works [Ka18, Ka19] (see also [ChLe20]). Moreover, the small quantum K product is in fact finite for general G/P [ACTI22] (see also [BCMP13, BCMP16] for certain Grassmannians), so that $QK_{\text{poly}}(G/P)$ is meaningful.

Let us start with the simplest case $\mathbb{P}^n$. The line bundle class $L := [\mathcal{O}_{\mathbb{P}^n}(1)]$ generates $K(\mathbb{P}^n)$ with inverse $L^{-1} = [\mathcal{O}_{\mathbb{P}^n}(-1)]$, and the hyperplane class $h := c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ (or simply $c_1(L)$) generates $H^*(\mathbb{P}^n) = H^{\text{ev}}(\mathbb{P}^n)$. We have

$$QK_{\text{poly}}(\mathbb{P}^n) = \mathbb{Q}[L^{-1}, Q]/((1 - L^{-1})^{n+1} - Q), \qquad QH_{\text{poly}}(\mathbb{P}^n) = \mathbb{Q}[h, q]/(h^{n+1} - q),$$

both of which are isomorphic to $\mathbb{Q}[x]$ as $\mathbb{Q}$-algebras. However, it is less obvious whether there is an (iso)morphism compatible with ch in the sense of making diagram (1.1) commutative. To achieve our aim, we need the completion $QH(\mathbb{P}^n) = QH_{\text{poly}}(\mathbb{P}^n) \otimes_{\mathbb{Q}[q]} \mathbb{Q}[\![q]\!]$ and introduce the *quantum Todd class* of the tangent bundle $T_{\mathbb{P}^n}$,

$$\mathrm{Td}_q(T_{\mathbb{P}^n}) := \left(\frac{h}{1 - e^{-h}}\right)^{n+1} \in QH(\mathbb{P}^n), \tag{1.2}$$

in whose series expansion each term h^k is read off as the quantum product h^{*k}. In particular, we have $\mathrm{Td}_q(T_{\mathbb{P}^n}) \equiv \mathrm{Td}(T_{\mathbb{P}^n}) \mod \mathbf{q}$. As the warm-up theorem, we have

Theorem 1.2 (**Theorem 3.1**). *The map* $\mathrm{qch} : QK(\mathbb{P}^n) \longrightarrow QH(\mathbb{P}^n)$, *given by*

$$L^{-1} \mapsto e^{-h} = \sum_{m=0}^{+\infty} \frac{(-h)^{*m}}{m!}, \qquad Q \mapsto q\ (\mathrm{Td}_q(T_{\mathbb{P}^n}))^{-1},$$

is a well-defined ring homomorphism, and its classical limit gives ch.

Remark 1.3. *As pointed out in [IMT15, Remark 5.13], Givental-Tonita's work [GT14] implies that the big quantum K-theory is isomorphic to the big quantum cohomology as F-maniolds. From the discussion with Hiroshi Iritani, such isomorphism does not restrict to the small locus in the following sense: when $n \geq 2$, there is no continuous ring homomorphism $\mathrm{ch}^q : QK(\mathbb{P}^n) \to QH(\mathbb{P}^n)$ (with respect to the natural Q-adic/q-adic topology) such that $\mathrm{ch}^q(Q) \in \mathbb{Q}[\![q]\!] \setminus \{0\}$ and (1.1) is a commutative diagram of ring homomorphisms. In Theorem 1.2, the image of Q does lie in $QH(\mathbb{P}^n) \setminus \mathbb{Q}[\![q]\!]$.*

The second simplest case may be the incidence varieties $\mathbb{F}\ell_{1,n-1;n} = \{V_1 \leq V_{n-1} \leq \mathbb{C}^n \mid \dim V_1 = 1, \dim V_{n-1} = n-1\}$. Let $\underline{\mathbb{C}^n}$ denote the trivial $\mathbb{C}^n$-bundle over $\mathbb{F}\ell_{1,n-1;n}$, and $\mathcal{S}_1$ (resp. $\mathcal{S}_{n-1}$) denote the tautological vector subbundle whose fiber at a point $V_\bullet$ is just the vector space V_1 (resp. V_{n-1}). Let L_1 (resp. L_2) denote the class of the dual line bundle $\mathcal{S}_1^\vee$ (resp. quotient line bundle $\underline{\mathbb{C}^n}/\mathcal{S}_{n-1}$) in $K(\mathbb{F}\ell_{1,n-1;n})$. We have the inverse $L_1^{-1} = [\mathcal{S}_1]$ and $L_2^{-1} = [(\underline{\mathbb{C}^n}/\mathcal{S}_{n-1})^\vee]$ in $K(\mathbb{F}\ell_{1,n-1;n})$, and denote $h_i := c_1(L_i) \in H^2(\mathbb{F}\ell_{1,n-1;n})$ for $i = 1,2$. We use the ring presentation for $QH_{\mathrm{poly}}(\mathbb{F}\ell_{1,n-1;n})$ with generators $\{h_1, h_2\}$ in [CP11, Proposition 7.2] (see Proposition 4.1 below). For $QK(\mathbb{F}\ell_{1,n-1;n})$, the ring presentation could be obtained by simplifying the Whitney presentation in [GM$^+$23, Theorem 1.3] (with $2n-1$ generators) or derived from the quantum Littlewood-Richardson rule in [Xu24]. Here we consider the ring presentation with generators $\{L_1^{-1}, L_2^{-1}\}$ read off from Theorem 1.5 as a special case of Milnor hypersurfaces. They are of the following form

$$QK(\mathbb{F}\ell_{1,n-1;n}) = \frac{\mathbb{Q}[L_1^{-1}, L_2^{-1}][\![Q_1, Q_2]\!]}{(F_1^Q, F_2^Q)}, \qquad QH(\mathbb{F}\ell_{1,n-1;n}) = \frac{\mathbb{Q}[h_1, h_2][\![q_1, q_2]\!]}{(f_1^q, f_2^q)}$$

with $F_i^Q = F_i^Q(L_1^{-1}, L_2^{-1}, Q_1, Q_2)$ and $f_i^q = f_i^q(h_1, h_2, q_1, q_2)$ being polynomials. Similar to $\mathbb{P}^n$, we introduce the quantum Todd class

$$(\mathrm{Td}_q(L_i^{\oplus n}))^{-1} * \mathrm{Td}_q(L_1 \otimes L_2) := (\frac{1 - e^{-h_i}}{h_i})^n / (\frac{1 - e^{-h_1 - h_2}}{h_1 + h_2}). \tag{1.3}$$

In particular, $(\mathrm{Td}_q(L_1^{\oplus n})) * \mathrm{Td}_q(L_1 \otimes L_2)^{-1} \mod \mathbf{q}$ is the classical Todd class of the difference between the pullback of $T_{\mathbb{F}\ell_{1;n}}$ and the line bundle $L_1 \otimes L_2$ (see Section 4 for more details). We have the following quantum Chern character homomorphism for $\mathbb{F}\ell_{1,n-1;n}$.

Theorem 1.4 (Theorem 4.4). *The map* $\mathrm{qch} : QK(\mathbb{F}\ell_{1,n-1;n}) \longrightarrow QH(\mathbb{F}\ell_{1,n-1;n})$*, given by*

$$\begin{aligned} L_i^{-1} &\mapsto e^{-c_1(L_i)} := \sum_{m=0}^{+\infty} \frac{(-h_i)^{*m}}{m!}, \\ Q_i &\mapsto q_i (\mathrm{Td}_q(L_i^{\oplus n}))^{-1} * \mathrm{Td}_q(L_1 \otimes L_2), \qquad i = 1, 2. \end{aligned}$$

is a well-defined ring homomorphism, and its classical limit gives ch.

We notice that the inverse of (classical) Todd class appeared in the Grothendieck-Riemann–Roch formula (see e.g. [EH16, Theorem 14.5]). The inverse of Todd function also appeared in the study of BCOV theory [CoLi20, Formula (38)].

The degree-$(1,1)$ hypersurfaces in $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$, known as Milnor hypersurfaces, are an important class of algebraic varieties in topology and algebraic geometry. Amongest them the smooth one is usually denoted as $H_{n-1,m-1}$. In particular, we have $H_{n-1,n-1} =$

$\mathbb{F}\ell_{1,n-1;n}$. We can view $H_{n-1,m-1}$ as a smooth Schubert variety of $\mathbb{F}\ell_{1,n-1;n}$ via a natural embedding ι where $n \geq m \geq 2$ (see Section 5.1 for more details). Then we have the line bundle classes $[\iota^*(\mathcal{S}_1^\vee)]$ and $[\iota^*(\underline{\mathbb{C}^n}/\mathcal{S}_{n-1})]$ in $K(H_{n-1,m-1})$, denoted again as L_1 and L_2 respectively by abuse of notation. By Proposition 5.4, we have $K(H_{n-1,m-1}) = \mathbb{Q}[L_1^{-1}, L_2^{-1}]/(F_1(L_1^{-1}, L_2^{-1}), F_2(L_1^{-1}, L_2^{-1}))$, where

$$F_1(x,y) := (1-y)^m, \qquad F_2(x,y) := (-1)^{n-1}(1-x)^{n-1} + \sum_{t=1}^{m-1} (-1)^{n-1-t} x^{t-1}(1-x)^{n-t-1}(1-y)^t.$$

As we will show in **Theorem 5.7**, we have the following ring presentation.

Theorem 1.5. *Let $n \geq m \geq 3$. The small quantum K-theory of $H_{n-1,m-1}$ is presented by*

$$QK(H_{n-1,m-1}) \cong \mathbb{Q}[L_1^{-1}, L_2^{-1}][\![Q_1, Q_2]\!]/(F_1^Q, F_2^Q),$$

where $F_1^Q(L_1^{-1}, L_2^{-1}, Q_1, Q_2) = F_1(L_1^{-1}, L_2^{-1}) - Q_2 + Q_2 L_1^{-1} L_2^{-1}$ and

$$F_2^Q(L_1^{-1}, L_2^{-1}, Q_1, Q_2) = F_2(L_1^{-1}, L_2^{-1}) - (-1)^{n-m} Q_2 (L_1^{-1})^{m-1}(1-L_1^{-1})^{n-m} - (-1)^{n-1} Q_1 L_2^{-1}.$$

On the one hand, there is an approach, known to experts, to derive a ring presentation of the quantum K-theory of a Fano manifold X. Namely, firstly one may compute the K-theoretic J-function J_X of X; secondly, one may find the symbols of finite difference operators annihilating J_X, which would give relations in $QK(X)$ by [IMT15, Proposition 2.10] and [HK24a, Theorem 4.12]; thirdly, one may show those relations are sufficient by a Nakayama-type result in [GM$^+$23, Appendix A]. We notice that when $K(X)$ is generated by line bundle classes, there has also been a heuristic argument on obtaining the ring presentation of $QK(X)$ by using finite difference equations in [Lee99]. On the other hand, there are very few examples that could really be carried out in this way, including the type A complete flag variety [AHK$^+$25, Appendix A]. Our Theorem 1.5 provides one new class of examples with this approach. We directly write down the J-function of $H_{n-1,m-1}$ by the quantum Lefschetz principle [Gi15, Gi20]. We notice that the K-theoretic J-function has been calculated for type A flag varieties [GL03, Ta13] (see [BrFi11] for the quasi-map version for complete flag varieties).

Remark 1.6. *Theorem 1.4 can be generalized to Milnor hypersurfaces $H_{n-1,m-1}$ as follows. We first derive the following ring presentation*

$$QH(H_{n-1,m-1}) \cong \mathbb{Q}[h_1, h_2][\![q_1, q_2]\!]/(h_2^m - q_2(h_1+h_2), \sum_{k=0}^{m-1} (-1)^k h_1^{n-1-k} h_2^k - q_1 - (-1)^{m-1} q_2 h_1^{n-m})$$

by investigating the quantum differential equations for Givental's (cohomological) J-function of $H_{n-1,m-1}$. Then we can define the map $\mathrm{qch} : QK(H_{n-1,m-1}) \to QH(H_{n-1,m-1})$ *by sending L_i^{-1} to e^{-h_i} and sending Q_1 (resp. Q_2) to $q_1(\mathrm{Td}_q(L_1^{\oplus n}))^{-1} * \mathrm{Td}_q(L_1 \otimes L_2)$ (resp. $q_2(\mathrm{Td}_q(L_2^{\oplus m}))^{-1} * \mathrm{Td}_q(L_1 \otimes L_2)$). The strategy to show that such* qch *gives a ring homomorphism* qch *is similar to the case for $\mathbb{F}\ell_{1,n-1;n} = H_{n-1,n-1}$. The full details will be slightly more involved, and we plan to address the cohomological part in the study of D-module mirror symmetry for $H_{n-1,m-1}$ elsewhere.*

The paper is organized as follows. In Section 2, we review the background of quantum cohomology and quantum K-theory. In Sections 3 and 4, we show the small quantum Chern character homomorphism for the cases $\mathbb{P}^n$ and $\mathbb{F}\ell_{1,n-1;n}$ respectively. In Section 5, we derive a ring presentation of the small quantum K-theory of Milnor hypersurface $H_{n-1,m-1}$. Finally, we prove Lemma 4.5 by studying mirror symmetry for $\mathbb{F}\ell_{1,n-1;n}$.

Acknowledgements. The authors would like to thank Sergey Galkin, Alexander Givental, Tao Huang, Hiroshi Iritani, Bai-Ling Wang and Rui Xiong for helpful discussions. The authors are supported in part by the National Key Research and Development Program of China No. 2023YFA1009801. H. Ke is also supported in part by NSFC Grant 12271532. C. Li is also supported in part by NSFC Grant 12271529.

2. Preliminaries

In the section, we review some basic facts on quantum cohomology and quantum K-theory, mainly following [FP97,Gi00,Le04,IMT15]. For simplicity, we assume the topological K-theory of X coincides with the algebraic K-theory of X, which is sufficient in the rest of this paper. We refer to [Le04] for more details on the set-up of the quantum K-theory in general cases.

2.1. **Cohomological/K-theoretic Gromov-Witten invariants.** Let $\overline{\mathcal{M}}_{0,k}(X,\beta)$ denote the moduli (Degline-Mumford) stack of genus 0, k-pointed stable maps to X of degree $\beta \in H_2(X;\mathbb{Z})$. Let $\mathrm{ev}_i : \overline{\mathcal{M}}_{0,k}(X,\beta) \longrightarrow X$ be the i-th evaluation map, and $\mathcal{L}_i$ denote the universal cotangent line bundle on $\overline{\mathcal{M}}_{0,k}(X,\beta)$, where $1 \le i \le k$. Given nonnegative integers d_i and classes $\xi_i \in H^{\mathrm{ev}}(X)/\gamma_i \in K(X)$, the cohomological/$K$-theoretic Gromov-Witten invariants are respectively defined by

$$\langle \tau_{d_1}\xi_1, \tau_{d_2}\xi_2, \cdots, \tau_{d_k}\xi_k\rangle^H_{0,k,\beta} = \int_{[\overline{\mathcal{M}}_{0,k}(X,\beta)]^{vir}} \prod_{i=1}^k (c_1(\mathcal{L}_i)^{d_i} \cup \mathrm{ev}_i^*(\xi_i)),$$
$$\langle \tau_{d_1}\gamma_1, \tau_{d_2}\gamma_2, \cdots, \tau_{d_k}\gamma_k\rangle_{0,k,\beta} = \chi\Big(\overline{\mathcal{M}}_{0,k}(X,\beta), \mathcal{O}^{vir} \otimes (\otimes_{i=1}^k \mathcal{L}_i^{\otimes d_i} \mathrm{ev}_i^*(\gamma_i))\Big).$$

Here $[\overline{\mathcal{M}}_{0,k}(X,\beta)]^{vir}$ denotes the virtual fundamental class [RT95, BeFa97, LT98, FO99], which is of expected dimension $\dim X + k - 3 + \int_\beta c_1(X)$; $\mathcal{O}^{vir}$ denotes the virtual structure sheaf [Le04, BeFa97], and χ is the pushforward to a point $\mathrm{Spec}(\mathbb{C})$.

Let $\overline{\mathrm{NE}}(X)$ denote the Mori cone of effective curve classes. It follows from the definition that cohomological/K-theoretic Gromov-Witten invariants of degree β are vanishing unless $\beta \in \overline{\mathrm{NE}}(X)$. We may take nef line bundles L_i's, such that $\{h_i := c_1(L_i)\}_{i=1}^r$ form a nef integral basis of $H^2(X;\mathbb{Z})/\text{torison}$. We then introduce the Novikov variables $\{q_i\}_{i=1}^r$ (resp. $\{Q_i\}_{i=1}^r$) for quantum cohomology (resp. K-theory). For $\beta \in \overline{\mathrm{NE}}(X)$, we have $d_i := d_i(\beta) = \int_\beta h_i \ge 0$, and denote

$$q^\beta := \prod_{i=1}^r q_i^{d_i}, \qquad Q^\beta := \prod_{i=1}^r Q_i^{d_i}.$$

2.2. **Quantum cohomology.** Let $\{\xi_i\}_{i=1}^N$ be a basis of $H^{\rm ev}(X)$, and $\{\xi^j\}_j$ be the dual basis of $H^{\rm ev}(X)$, where $\int_{[X]}\xi_b\cup\xi^a=\delta_{a,b}$. Write $\mathbf{s}=\sum_{i=1}^N s_i\xi_i\in H^{\rm ev}(X)$. The (even) big quantum cohomology $QH_{\rm big}(X)=(H^{\rm ev}(X)\otimes\mathbb{Q}[\![\mathbf{q},\mathbf{s}]\!],*_{\mathbf{s}})$ is a $\mathbb{Q}[\![\mathbf{s},\mathbf{q}]\!]$-algebra with the big quantum product defined by

$$\xi_i *_{\mathbf{s}} \xi_j := \sum_{k=0}^{\infty}\sum_{\beta\in\overline{\rm NE}(X)}\sum_{a=1}^{N}\frac{q^\beta}{k!}\langle\xi_i,\xi_j,\xi^a,\mathbf{s},\cdots,\mathbf{s}\rangle^H_{0,k+3,\beta}\xi_a. \tag{2.1}$$

The restriction of $QH_{\rm big}(X)$ to $\mathbf{s}=0$, i.e. $\xi_i*\xi_j=\sum_{\beta,a}\langle\xi_i,\xi_j,\xi^a\rangle^H_{0,3,\beta}\xi_a q^\beta$, gives the small quantum cohomology $QH(X)=(H^{\rm ev}(X)\otimes\mathbb{Q}[\![\mathbf{q}]\!],*)$.

For $\alpha\in H^{\rm ev}(X)$ and $f(x)=x^m$ where $m\geq 0$, we take the notation conventions

$$f_{\rm cl}(\alpha)=\alpha^{\cup m}:=\underbrace{\alpha\cup\cdots\cup\alpha}_{m\text{ copies}},\qquad f(\alpha)=\alpha^{*m}:=\underbrace{\alpha*\cdots*\alpha}_{m\text{ copies}}.$$

We can take similar conventions for formal power series in multi-variables by observing the following. For any $f(x_1,\ldots,x_k)\in\mathbb{Q}[\![x_1,\ldots,x_k]\!]$ and any homogeneous $\alpha_i\in H^{\rm ev}(X)\setminus H^0(X)$, $1\leq i\leq k$, we have

(1) $f_{\rm cl}(\alpha_1,..,\alpha_k)\in H^{\rm ev}(X)$, by the nilpotency of α_i's in $H^{\rm ev}(X)$;
(2) $f(\alpha_1,..,\alpha_k)\in QH(X)$ for the reason: for each q^β, by the dimension axiom, there are only finitely many k-tuples $(m_1,...,m_k)$ such that q^β appears in $\alpha_1^{*m_1}*\cdots*\alpha_k^{*m_k}$.

Moreover, under the natural surjective ring homomorphism $QH(X)\xrightarrow{\bmod q}H^{\rm ev}(X)$, we have $f(\alpha_1,...,\alpha_k)\mapsto f_{\rm cl}(\alpha_1,...,\alpha_k)$.

More generally, for $\alpha\in H^2(X)$ and function $f(x)$ holomorphic at $x=0$, we can define $f_{\rm cl}(\alpha)\in H^{\rm ev}(X)$ and $f(\alpha)\in QH(X)$ by the Taylor expansion of $f(x)$ at $x=0$. In this article, we consider the cases $f(x)=e^x$, $\frac{1-e^{-x}}{x}$ or $\frac{x}{1-e^{-x}}$, giving $f(\alpha)\in QH(X)$.

2.3. **Quantum K-theory.** The definition of quantum K-theory is more involved. Let $\mathbf{t}=\sum_{i=1}^N t_i\gamma_i\in K(X)$. We first introduce the quantum K-potential, which is a generating series of genus-zero K-theoretic Gromov-Witten invariants given by

$$\mathcal{G}(\mathbf{t},\mathbf{Q}):=\frac{1}{2}\sum_{i,j}t_it_jg_{ij}+\sum_{k=0}^{\infty}\sum_{\beta\in\overline{\rm NE}(X)}\frac{Q^\beta}{k!}\langle\mathbf{t},\cdots,\mathbf{t}\rangle_{0,k,\beta}$$

with $g_{ij}=g(\gamma_i,\gamma_j):=\chi(X,\gamma_i\otimes\gamma_j)$ being the classical metric on $K(X)$ given by the Euler characteristic map.

Denote $\partial_i:=\frac{\partial}{\partial t_i}$. The quantum K-metric is defined by

$$G_{ij}=G(\gamma_i,\gamma_j):=\partial_i\partial_j\mathcal{G}.$$

We have $G_{ij}\equiv g_{ij}\mod\mathbf{Q}$, and $\big(G_{ij}\big)\in GL_N(\mathbb{Q}[\![\mathbf{Q},\mathbf{t}]\!])$.

Definition 2.1. *The big quantum K-theory $QK_{big}(X):=(K(X)\otimes[\![\mathbf{Q},\mathbf{t}]\!],\star_{\mathbf{t}})$ is a $[\![\mathbf{Q},\mathbf{t}]\!]$-algebra with the big quantum K-product $\gamma_i\star_{\mathbf{t}}\gamma_j$ defined by using the quantum K-metric,*

$$G(\gamma_i\star_{\mathbf{t}}\gamma_j,\gamma_k):=\partial_i\partial_j\partial_k\mathcal{G}.$$

The big quantum K-theory $QK_{\mathrm{big}}(X)$ is a commutative and associative algebra with identity $\mathbf{1}$ (lying in $K(X)$). Its restriction to $\mathbf{t} = 0$ gives the small quantum K-theory $QK(X) = (K(X) \otimes \mathbb{Q}[\![\mathbf{Q}]\!], \star)$.

2.4. **Quantum connection and K-theoretic J-function.** Let $\hbar$ be a formal variable. The quantum connection is given by the operators

$$\nabla_i^{\hbar} := (1-\hbar)\frac{\partial}{\partial t_i} - \gamma_i \star_{\mathbf{t}}, \ 1 \le i \le N,$$

acting on $K(X)\otimes\mathbb{Q}[\hbar, \hbar^{-1}][\![\mathbf{Q}, \mathbf{t}]\!]$. It could also be seen as a connection on the tangent bundle $T_{K(X)}$ by identifying γ_i with $\frac{\partial}{\partial t_i}$. This quantum connection is flat, with the fundamental solution $\{S_{ab}\}_{1\le a,b\le N}$ to the K-theoretic quantum differential equation $\nabla_i^{\hbar}(\sum_{b=1}^N S_{ab}\gamma_b) = 0$ given by

$$S_{ab}(\mathbf{t}, \mathbf{Q}) = g_{ab} + \sum_{l=0}^{\infty} \sum_{d\in\overline{\mathrm{NE}}(\mathrm{X})} \frac{Q^d}{l!}\langle \gamma_a, \mathbf{t}, \cdots, \mathbf{t}, \frac{\gamma_b}{1-\hbar\mathcal{L}}\rangle_{0,l+2,d}.$$

Here $\frac{\gamma_b}{1-\hbar\mathcal{L}} = \sum_{n=0}^{\infty}\gamma_b\hbar^n\mathcal{L}^{\otimes n}$, where $\mathcal{L}$ is the universal cotangent line bundle at the $(l+2)$th-marked point on $\overline{\mathcal{M}}_{0,l+2}(X, d)$. Following [IMT15], we introduce an endomorphism-valued function $T \in End(K(X)) \otimes \mathbb{Q}(\hbar)[\![Q, t]\!]$ by the formula $g(T\gamma_a, \gamma_b) = S_{ab}$.

Definition 2.2. *The big K-theoretic J-function* [1] *of X is defined by*

$$J_X(\hbar, \mathbf{Q}, \mathbf{t}) := T(\mathbf{1}) = \mathbf{1} + \sum_{l=0}^{\infty} \sum_{d\in\overline{\mathrm{NE}}(\mathrm{X})} \sum_{i,j} \frac{Q^d}{l!}\langle \mathbf{1}, \mathbf{t}, \cdots, \mathbf{t}, \frac{\gamma_i}{1-\hbar\mathcal{L}}\rangle_{0,l+2,d}\ g^{ij}\gamma_j.$$

The restriction $J_X(\hbar, \mathbf{Q}) = J_X(\hbar, \mathbf{Q}, \mathbf{t})|_{\mathbf{t}=0}$ is called the small K-theoretic J-function.

Givental and Tonita [GT14] showed that the space $T(K(X)\otimes\mathbb{Q}[\hbar, \hbar^{-1}][\![\mathbf{Q}, \mathbf{t}]\!])$ is preserved by the operator $L_i^{-1}\hbar^{Q_i\partial_{Q_i}}$ for all $1 \le i \le r$, where L_i^{-1} acts by tensor product, and $\hbar^{Q_i\partial_{Q_i}}$ is the $\hbar$-shift operator acting on functions in $Q_1, Q_2, \cdots, Q_r$ by

$$f(Q_1, Q_2, \cdots, Q_r) \mapsto f(Q_1, \cdots, Q_{i-1}, \hbar Q_i, Q_{i+1}, \cdots, Q_r).$$

Set $A_i := \left(T^{-1} \circ L_i^{-1}\hbar^{Q_i\partial_{Q_i}} \circ T\right)|_{\mathrm{t}=0,\hbar=1}$, which is an element in $\mathrm{End}(K(X)) \otimes \mathbb{Q}[\![\mathbf{Q}]\!]$. Suppose that a $\hbar$-difference operator $D \in \mathbb{Q}[\hbar, \mathbf{Q}]\left\langle L_i^{-1}\hbar^{Q_i\partial_{Q_i}}\right\rangle$ satisfies

$$D(L_1^{-1}\hbar^{Q_1\partial_{Q_1}}, \cdots, L_r^{-1}\hbar^{Q_r\partial_{Q_r}}, \hbar, Q_1, \cdots, Q_r)J_X(\hbar, \mathbf{Q}) = 0. \tag{2.2}$$

Then $D(A_1, \cdots, A_r, 1, Q_1, \cdots, Q_r)(\mathbf{1}) = 0$ by the definition of A_i's. To simplify A_i, we note that if the coefficient of Q^d in $L_i^{-1}\hbar^{Q_i\partial_{Q_i}}(J_X(\hbar, \mathbf{Q}))$ vanish when $\hbar = \infty$ for all $d > 0$, then $A_i = (L_i^{-1}\star)$ by [IMT15, Proposition 2.10] and [HK24a, Theorem 4.12]. In summary, we have

Proposition 2.3. *Suppose that: (i) $D \in \mathbb{Q}[\hbar, \mathbf{Q}]\left\langle L_i^{-1}\hbar^{Q_i\partial_{Q_i}}\right\rangle$ satisfies Equation (2.2); (ii) for $1 \le i \le r$, $\mathrm{Coeff}_{Q^d}\left(L_i^{-1}\hbar^{Q_i\partial_{Q_i}}(J_X(\hbar, \mathbf{Q}))\right)|_{\hbar=\infty} = 0$ for all $d \in \overline{\mathrm{NE}}(X) \setminus \{0\}$. Then*

$$D(L_1^{-1}, \cdots, L_r^{-1}, 1, Q_1, \cdots, Q_r)(\mathbf{1}) = 0 \quad \textit{in } QK(X). \tag{2.3}$$

[1] The K-theoretic J-function in [IMT15, Definition 2.4] is equal to our $(1-\hbar)J$ with $\hbar = q$.

3. Quantum Chern character for $\mathbb{P}^n$

In this section, as a warm-up, we construct a quantum version of Chern character for $\mathbb{P}^n$, i.e. a ring homomorphism $QK(\mathbb{P}^n) \xrightarrow{\mathrm{qch}} QH(\mathbb{P}^n)$ that is a lift of $K(\mathbb{P}^n) \xrightarrow{\mathrm{ch}} H^*(\mathbb{P}^n) = H^{ev}(\mathbb{P}^n)$.

Let us consider line bundle class $L := [\mathcal{O}(1)] \in K(\mathbb{P}^n)$ and hyperplane class $h := c_1(L) \in H^2(\mathbb{P}^n)$. We have the following ring presentations (see e.g. [BM11]):

$$QK(\mathbb{P}^n) \cong \mathbb{Q}[L^{-1}][\![Q]\!]/((1-L^{-1})^{n+1} - Q), \qquad QH(\mathbb{P}^n) \cong \mathbb{Q}[h][\![q]\!]/(h^{n+1} - q),$$

where $L^{-1} = [\mathcal{O}(-1)]$ is the classical inverse of L in $K(\mathbb{P}^n)$.

We define a map $\mathrm{qch} : QK(\mathbb{P}^n) \longrightarrow QH(\mathbb{P}^n)$ as follows. We firstly set

$$\mathrm{qch}(L^{-1}) = e^{-h} \text{ and } \mathrm{qch}(Q) = q\left(\frac{1-e^{-h}}{h}\right)^{n+1},$$

which both lie in $QH(\mathbb{P}^n)$ by our convention in Section 2.2. Secondly, for any $\alpha \in QK(\mathbb{P}^n)$, the above ring presentation implies that there exists $\Phi_\alpha(x, y) \in \mathbb{Q}[x][\![y]\!]$ such that $\alpha = \Phi_\alpha(L^{-1}, Q)$. We then set

$$\mathrm{qch}(\alpha) = \Phi_\alpha\left(\mathrm{qch}(L^{-1}), \mathrm{qch}(Q)\right).$$

Theorem 3.1. *The map* qch *is independent of the choice of* Φ_α*'s and is a ring homomorphism. Moreover, its classical limit gives* ch*.*

Proof. To prove the first statement, by the ring presentation for $QK(\mathbb{P}^n)$, it suffices to show that $\left(1 - \mathrm{qch}(L^{-1})\right)^{n+1} = \mathrm{qch}(Q)$, or equivalently, $\left(1 - e^{-h}\right)^{n+1} = q\left(\frac{1-e^{-h}}{h}\right)^{n+1}$.

The equality $1 - e^{-x} = x \cdot \frac{1-e^{-x}}{x}$ in $\mathbb{Q}[\![x]\!]$ implies that $1 - e^{-h} = h \star \frac{1-e^{-h}}{h}$. Thus we have

$$\left(1-e^{-h}\right)^{n+1} = h^{n+1} * \left(\frac{1-e^{-h}}{h}\right)^{n+1} = q\left(\frac{1-e^{-h}}{h}\right)^{n+1}.$$

In the last equality, we use the relation $h^{n+1} = q$ in $QH(\mathbb{P}^n)$. This proves the first statement.

The second statement follows directly from the definition of qch. This finishes the proof of the proposition. □

Remark 3.2. *The image* $\mathrm{qch}(Q)$ *is actually uniquely determined by* $\mathrm{qch}(L^{-1}) = e^{-h}$ *and the requirement that* qch *is a ring homomorphism. Indeed, that* qch *is a ring homomorphism implies that* $\left(1 - \mathrm{qch}(L^{-1})\right)^{n+1} = \mathrm{qch}(Q)$*. Together with the relation* $h^{n+1} = q$ *in* $QH(\mathbb{P}^n)$*, we obtain*

$$\mathrm{qch}(Q) * h^{n+1} = q\left(1-e^{-h}\right)^{n+1}.$$

This equation for $\mathrm{qch}(Q)$ *has a solution* $\mathrm{qch}(Q) = q\left(\frac{1-e^{-h}}{h}\right)^{n+1}$*. Moreover, noting* $QH(\mathbb{P}^n)$ *is a* $\mathbb{Q}[\![q]\!]$*-algebra, the relation* $h^{n+1} = q$ *implies that* h *is NOT a zero divisor in* $QH(\mathbb{P}^n)$*. As a consequence, the equation for* $\mathrm{qch}(Q)$ *has a unique solution.*

Remark 3.3. *The factor* $\left(\frac{1-e^{-h}}{h}\right)^{n+1}$ *in* $\mathrm{qch}(Q)$ *can be interpreted as a quantum version of the inverse of Todd class of* $T_{\mathbb{P}^n}$*. On the one hand, the factor is equal to* $f(h)$ *with*

$f(x) = \left(\frac{1-e^{-x}}{x}\right)^{n+1}$. *On the other hand, the Euler exact sequence*

$$0 \to \mathcal{O} \to L^{\oplus(n+1)} \to T\mathbb{P}^n \to 0$$

gives

$$\mathrm{Td}(T_{\mathbb{P}^n})^{-1} = \left(\mathrm{Td}(L)^{n+1}\right)^{-1} = f_{\mathrm{cl}}(h).$$

So $\left(\frac{1-e^{-h}}{h}\right)^{n+1}$ *is mapped to* $\mathrm{Td}(T_{\mathbb{P}^n})^{-1}$ *via* $QH(\mathbb{P}^n) \xrightarrow{mod\ q} H^*(\mathbb{P}^n)$.

4. Quantum Chern character for $\mathbb{F}\ell_{1,n-1;n}$

In this section, we construct a quantum version of Chern character for the incidence variety $\mathbb{F}\ell_{1,n-1;n} = \{V_1 \le V_{n-1} \le \mathbb{C}^n | \dim V_1 = 1, \dim V_{n-1} = n-1\}$, i.e. a ring homomorphism $QK(\mathbb{F}\ell_{1,n-1;n}) \xrightarrow{\mathrm{qch}} QH(\mathbb{F}\ell_{1,n-1;n})$ that gives a lift of $K(\mathbb{F}\ell_{1,n-1;n}) \xrightarrow{\mathrm{ch}} H^*(\mathbb{F}\ell_{1,n-1;n}) = H^{\mathrm{ev}}(\mathbb{F}\ell_{1,n-1;n})$.

Let $L_1 := [(\mathcal{S}_1)^\vee], L_2 := [\underline{\mathbb{C}^n}/\mathcal{S}_{n-1}] \in K(\mathbb{F}\ell_{1,n-1;n})$, where $\mathcal{S}_1$ and $\mathcal{S}_{n-1}$ are the tautological vector subbundles of $\mathbb{F}\ell_{1,n-1;n}$. Note that $\{h_a := c_1(L_a)\}_{1\le a\le 2}$ form a nef basis of $H^2(\mathbb{F}\ell_{1,n-1;n}, \mathbb{Z})$. Recall that q_a (resp. Q_a), $1 \le a \le 2$, denote the corresponding Novikov variables in the quantum cohomology (resp. K-theory) of $\mathbb{F}\ell_{1,n-1;n}$. We have

Proposition 4.1 ([CP11, Proposition 7.2])**.** *The small quantum cohomology ring of* $\mathbb{F}\ell_{1,n-1;n}$ *is canonically given by* $QH(\mathbb{F}\ell_{1,n-1;n}) \cong \mathbb{Q}[h_1,h_2][\![q_1,q_2]\!]/(f_1^q, f_2^q)$, *where*

$$f_1^q(h_1,h_2,q_1,q_2) = h_2^n - q_2(h_1+h_2),$$
$$f_2^q(h_1,h_2,q_1,q_2) = \sum_{l=0}^{n-1}(-1)^{n-1-l}h_1^l h_2^{n-1-l} - q_1 - (-1)^{n-1}q_2.$$

The isomorphism in the above proposition is canonical in the sense that the hyperplane class h_a in $QH(\mathbb{F}\ell_{1,n-1;n})$ is maps to the generator h_a on the right, for $a = 1, 2$. Moreover, there is a canonical selfduality of $\mathbb{F}\ell_{1,n-1;n}$, which induces an automorphism of the small quantum cohomology by interchanging (h_1, q_1) and (h_2, q_2). As an immediate consequence, we have

Corollary 4.2. *We have* $h_a^n = q_a(h_1+h_2)$ *in* $QH(\mathbb{F}\ell_{1,n-1;n})$ *for* $a = 1, 2$.

Ring presentations for $QK(\mathbb{F}\ell_{1,n-1;n})$ could be obtained from [GM+23, Xu24]. For our purpose of constructing qch, we treat $\mathbb{F}\ell_{1,n-1;n}$ as the special Milnor hypersurface $H_{n-1,n-1}$, and read off one from the general case $QK(H_{n-1,m-1})$ in Theorem 5.7 directly; namely

Proposition 4.3. *The small quantum K-theory of* $\mathbb{F}\ell_{1,n-1;n}$ *is canonically given by*

$$QK(\mathbb{F}\ell_{1,n-1;n}) \cong \mathbb{Q}[L_1^{-1}, L_2^{-1}][\![Q_1,Q_2]\!]/(F_1^Q, F_2^Q),$$

$$\begin{aligned} \textit{where}\quad & F_1^Q(L_1^{-1},L_2^{-1},Q_1,Q_2) = (1-L_2^{-1})^n - Q_2 + Q_2(L_1^{-1}L_2^{-1}),\\ & F_2^Q(L_1^{-1},L_2^{-1},Q_1,Q_2) = F_2(L_1^{-1},L_2^{-1}) - Q_2(L_1^{-1})^{n-1} - (-1)^{n-1}Q_1L_2^{-1}, \end{aligned}$$

with $F_2(x,y) = (-1)^{n-1}(1-x)^{n-1} + \sum_{t=1}^{n-1}(-1)^{n-1-t}x^{t-1}(1-x)^{n-t-1}(1-y)^t$.

We define a map $QK(\mathbb{F}\ell_{1,n-1;n}) \xrightarrow{\mathrm{qch}} QH(\mathbb{F}\ell_{1,n-1;n})$ as follows. Firstly, for $a=1,2$, we set

$$\mathrm{qch}(L_a^{-1}) = e^{-h_a} \quad \text{and} \quad \mathrm{qch}(Q_a) = q_a \left(\frac{1-e^{-h_a}}{h_a}\right)^n * \left(\frac{(h_1+h_2)}{1-e^{-(h_1+h_2)}}\right).$$

Secondly, for any $\alpha \in QK(\mathbb{F}\ell_{1,n-1;n})$, the above ring presentation implies that there exists $\Phi_\alpha(x_1,x_2,y_1,y_2) \in \mathbb{Q}[x_1,x_2][\![y_1,y_2]\!]$ such that $\alpha = \Phi_\alpha\left(L_1^{-1},L_2^{-1},Q_1,Q_2\right)$, and we set

$$\mathrm{qch}(\alpha) = \Phi_\alpha\left(\mathrm{qch}(L_1^{-1}),\mathrm{qch}(L_2^{-1}),\mathrm{qch}(Q_1),\mathrm{qch}(Q_2)\right).$$

Theorem 4.4. *The map* qch *is independent of the choice of Φ_α's and is a ring homomorphism. Moreover, its classical limit gives* ch.

To prove the theorem, we prepare several lemmas first. Recall that in a commutative ring R, an element $x \in R$ is not a zero-divisor if and only if for any $y \in R$, we have $y=0$ whenever $xy=0$. We first assume the next lemma, and leave its proof at the end of the appendix, which will involve mirror symmetry for $\mathbb{F}\ell_{1,n-1;n}$.

Lemma 4.5. *The element h_1+h_2 is not a zero-divisor in $QH(\mathbb{F}\ell_{1,n-1;n})$.*

Lemma 4.6. *The element $1-e^{-(h_1+h_2)}$ is not a zero-divisor in $QH(\mathbb{F}\ell_{1,n-1;n})$.*

Proof. The equality $(1-e^{-x})\cdot\frac{x}{1-e^{-x}} = x$ in $\mathbb{Q}[\![x]\!]$ implies that $\left(1-e^{-(h_1+h_2)}\right)*\frac{(h_1+h_2)}{1-e^{-(h_1+h_2)}} = h_1+h_2$. So it follows from Lemma 4.5 that $1-e^{-(h_1+h_2)}$ is not a zero-divisor. □

Lemma 4.7. *For $a\in\{1,2\}$, we have $(1-e^{-(h_1+h_2)})*\mathrm{qch}(Q_a) = (1-e^{-h_a})^n$ in $QH(\mathbb{F}\ell_{1,n-1;n})$.*

Proof. By the equalities $x\cdot\frac{1-e^{-x}}{x} = 1-e^{-x}$ and $(1-e^{-x})\cdot\frac{x}{1-e^{-x}} = x$ in $\mathbb{Q}[\![x]\!]$, we have

$$h_a * \frac{1-e^{-h_a}}{h_a} = 1-e^{-h_a} \text{ and } (1-e^{-(h_1+h_2)}) * \frac{h_1+h_2}{1-e^{-(h_1+h_2)}} = h_1+h_2.$$

So we see that

$$\begin{aligned}(1-e^{-(h_1+h_2)}) * \mathrm{qch}(Q_a) &= q_a\left(\frac{1-e^{-h_a}}{h_a}\right)^n * \left((1-e^{-(h_1+h_2)}) * \frac{h_1+h_2}{1-e^{-(h_1+h_2)}}\right)\\ &= q_a\left(\frac{1-e^{-h_a}}{h_a}\right)^n * (h_1+h_2)\\ &= h_a^n * \left(\frac{1-e^{-h_a}}{h_a}\right)^n \qquad \text{(from } \mathit{Corollary}\ 4.2\text{)}\\ &= (1-e^{-h_a})^n.\end{aligned}$$

□

Proof of Theorem 4.4. To prove the first statement, by the ring presentation in Proposition 4.1, it suffices to show that the two elements

$$F_a^Q\left(\mathrm{qch}(L_1^{-1}),\mathrm{qch}(L_2^{-1}),\mathrm{qch}(Q_1),\mathrm{qch}(Q_2)\right) \in QH(\mathbb{F}\ell_{1,n-1;n}), \quad a=1,2,$$

are vanishing.

For F_1^Q, by Lemma 4.6, we only need to prove that

$$(1-e^{-(h_1+h_2)}) * \Big((1-e^{-h_2})^n - \mathrm{qch}(Q_2) + \mathrm{qch}(Q_2) * e^{-h_1} * e^{-h_2}\Big) = 0.$$

It follows from Lemma 4.7 that

$$\begin{aligned}&(1-e^{-(h_1+h_2)}) * \Big((1-e^{-h_2})^n - \mathrm{qch}(Q_2) + \mathrm{qch}(Q_2) * e^{-h_1} * e^{-h_2}\Big)\\ =&\ (1-e^{-h_2})^n * \Big((1-e^{-(h_1+h_2)}) - 1 + e^{-h_1} * e^{-h_2}\Big).\end{aligned}$$

So the required vanishing result follows from $e^{-(h_1+h_2)} = e^{-h_1} * e^{-h_2}$.

For F_2^Q, by Lemma 4.6, we only need to prove that LHS = RHS, where

$$\begin{aligned}\mathrm{LHS} &:= (1-e^{-(h_1+h_2)}) * F_2(e^{-h_1}, e^{-h_2}),\\ \mathrm{RHS} &:= (1-e^{-(h_1+h_2)}) * \Big(\mathrm{qch}(Q_2) * (e^{-h_1})^{*(n-1)} + (-1)^{n-1}\mathrm{qch}(Q_1) * e^{-h_2}\Big).\end{aligned}$$

We use Lemma 4.7 to find that

$$\mathrm{RHS} = (1-e^{-h_2})^n * (e^{-h_1})^{n-1} + (-1)^{n-1}(1-e^{-h_1})^n * e^{-h_2}.$$

For LHS, observe that $F_2(e^{-h_1}, e^{-h_2})$ is equal to

$$(-1)^{n-1}(1-e^{-h_1})^{n-1} + (1-e^{-h_2}) * \sum_{l=0}^{n-2} (e^{-h_1}-1)^{n-2-l} * \Big(e^{-h_1} * (1-e^{-h_2})\Big)^l.$$

Note that $(e^{-h_1}-1) - \big(e^{-h_1} * (1-e^{-h_2})\big) = -(1-e^{-(h_1+h_2)})$, and we find that

$$\begin{aligned}\mathrm{LHS} = &\ (-1)^{n-1}(1-e^{-h_1})^{n-1} * (1-e^{-(h_1+h_2)})\\ &- (1-e^{-h_2}) * \Big((e^{-h_1}-1)^{*(n-1)} - (e^{-h_1} * (1-e^{-h_2}))^{n-1}\Big).\end{aligned}$$

Now one can check that LHS = RHS, and this proves the first statement.

The second statement directly follows from the definition of qch. This finishes the proof. □

Remark 4.8. *If we assume a prior that* qch *is a ring homomorphism, then the images* $\mathrm{qch}(Q_a)$*'s are uniquely determined by* $\mathrm{qch}(L_a^{-1}) = e^{-h_a}$*, similar to Remark 3.2.*

Remark 4.9. *Let* π_1, π_2 *be the natural projection from* $\mathbb{F}\ell_{1,n-1;n}$ *to* $\mathbb{F}\ell_{1,n}, \mathbb{F}\ell_{n-1;n}$*, respectively. We have the exact sequences*

$$0 \to \mathcal{O}_{\mathbb{F}\ell_{1,n-1;n}} \to L_1^{\oplus n} \to \pi_1^* T_{\mathbb{F}\ell_{1,n}} \to 0, \qquad 0 \to \mathcal{O}_{\mathbb{F}\ell_{1,n-1;n}} \to L_2^{\oplus n} \to \pi_2^* T_{\mathbb{F}\ell_{n-1;n}} \to 0.$$

Similar to Remark 3.3, the factor $\left(\frac{1-e^{-h_a}}{h_a}\right)^n$ *in* $\mathrm{qch}(Q_a)$*,* $1 \le a \le 2$*, may be interpreted as a quantum version of the inverse of the Todd classes of* $\pi_1^* T_{\mathbb{F}\ell_{1;n}}$ *and* $\pi_2^* T_{\mathbb{F}\ell_{n-1;n}}$ *respectively. The factor* $\frac{(h_1+h_2)}{1-e^{-(h_1+h_2)}}$ *may be interpreted as a quantum version of the Todd class of* $L_1 \otimes L_2$*.*

5. Quantum K-theory of Milnor hypersurfaces

In the section, we provide a ring presentation of the small quantum K-theory of smooth Milnor hypersurfaces.

5.1. **Milnor hypersurfaces.** Let $n \geq m \geq 3$ [2]. The (smooth) Milnor hypersurface $H_{n-1,m-1}$ is a degree-$(1,1)$ hypersurface in $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$, defined by the equation

$$x_1y_1 + x_2y_2 + \cdots + x_my_m = 0,$$

where $[x_1 : \cdots : x_n]$ and $[y_1 : \cdots : y_m]$ are homogeneous coordinates of $\mathbb{P}^{n-1}, \mathbb{P}^{m-1}$, respectively. Note that the incidence variety $\mathbb{F}\ell_{1,n-1;n}$ is a degree $(1,1)$-hypersurface of $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ defined by $\sum_{j=1}^{n}(-1)^j x_jy_j = 0$ via the Plücker embedding. We can identify $H_{n-1,n-1}$ with $\mathbb{F}\ell_{1,n-1;n}$ by a simple coordinate change ($y_j \mapsto (-1)^j y_j$).

Remark 5.1. *Any smooth Schubert variety in $\mathbb{F}\ell_{1,n-1;n}$ is either a Milnor hypersurface or a product of projective spaces.*

Consider the following commutative diagram:

$$\begin{tikzcd} H_{n-1,m-1} \arrow[r, hook, "\iota"] \arrow[d, "\pi_2^m"] & H_{n-1,n-1} \cong \mathbb{F}\ell_{1,n-1;n} \arrow[d, "\pi_2"] \\ \mathbb{P}^{m-1} \arrow[r, hook, "\iota_m"] & \mathbb{P}^{n-1} \cong \mathbb{F}\ell_{n-1;n} \end{tikzcd} \tag{5.1}$$

Here ι is the natural embedding, π_2 (resp. π_2^m) is the natural projection to the second factor of the product of projective spaces, and ι_m denotes the natural inclusion. Let $\mathcal{U}_{n-1}$ be the tautological bundle of $\mathbb{F}\ell_{n-1;n}$. Then $\mathbb{F}\ell_{1,n-1;n} = \mathbb{P}(\mathcal{U}_{n-1})$ is a $\mathbb{P}^{n-2}$-bundle over $\mathbb{P}^{n-1}$, and $H_{n-1,m-1} = \mathbb{P}^{m-1} \times_{\mathbb{F}\ell_{n-1;n}} \mathbb{F}\ell_{1,n-1;n} = \mathbb{P}(\iota_m^*\mathcal{U}_{n-1})$ is a $\mathbb{P}^{n-2}$-bundle over $\mathbb{P}^{m-1}$.

5.2. **Ring presentation of $K(H_{n-1,m-1})$.** There is a standard procedure to obtain a ring presentation of the classical K-theory of the projective bundle $H_{n-1,m-1} = \mathbb{P}(\iota_m^*\mathcal{U}_{n-1})$. Here we provide a precise presentation of $K(H_{n-1,m-1})$ that is helpful for us to derive a ring presentation of $QK(H_{n-1,m-1})$.

Lemma 5.2. *For any nonnegative integers b, n, t with $0 \leq b \leq n-t-1$, we have*

$$\sum_{c=0}^{b} \binom{n}{n-b+c}\binom{t+c}{c}(-1)^c = \binom{n-t-1}{b}.$$

Proof. It follows from differentiating $\sum_{c=0}^{\infty} z^c = \frac{1}{1-z}$, where ($|z| < 1$), that

$$A(z) := \sum_{c=0}^{\infty} \binom{t+c}{c} z^c = \frac{1}{(1-z)^{t+1}}.$$

The coefficient of z^b-term in $(1+z)^n \cdot A(-z)$ is given by $\sum_{c=0}^{b} \binom{n}{n-b+c}\binom{t+c}{c}(-1)^c$. Hence, it is equal to $\binom{n-t-1}{b}$, by noting $(1+z)^{n-t-1} = (1+z)^n \cdot A(-z)$. $\square$

Recall the following polynomials defined in the introduction.

$$F_1(x,y) := (1-y)^m, \tag{5.2}$$

$$F_2(x,y) := (-1)^{n-1}(1-x)^{n-1} + \sum_{t=1}^{m-1} (-1)^{n-1-t} x^{t-1} (1-x)^{n-t-1} (1-y)^t. \tag{5.3}$$

[2]We exclude the case $m = 2$, for which there is a nontrivial mirror map in the K-theoretic J-function by quantum Lefschetz principle and Proposition 2.3 is not applicable directly.

Lemma 5.3. *There exists $a(x,y) \in \mathbb{Q}[x,y]$ such that*

$$F_2(x,y) - a(x,y)F_1(x,y) = (-1)^n \sum_{l=0}^{n-1} (-x)^l \sum_{s=1}^{l+1} \binom{n}{n-1-l+s} (-y)^s.$$

Proof. Let RHS denote the right-hand side and $a_1(x,y) := -x^{n-1}(1-y)^{n-m}$. By direct calculation, we have RHS $= a_1(x,y)F_1(x,y) + x^{n-1} + (-1)^n \sum_{l=0}^{n-2}(-x)^l \sum_{s=1}^{l+1} \binom{n}{n-1-l+s}(-y)^s$. Note $(-y)^s = \sum_{t=1}^{s}(1-y)^t\binom{s}{t}(-1)^{s-t} + (-1)^s$. By changing the order of summation over t and s,l, we have RHS $= a_1(x,y)F_1(x,y) + g_0(x) + \sum_{t=1}^{n-1} g_t(x)(1-y)^t$, where

$$g_0(x) = x^{n-1} + (-1)^n \sum_{l=0}^{n-2}(-x)^l \sum_{s=1}^{l+1} \binom{n}{n-1-l+s}(-1)^s,$$

$$g_t(x) = (-1)^n \sum_{l=t-1}^{n-2} (-x)^l \sum_{s=t}^{l+1} \binom{n}{n-1-l+s}\binom{s}{t}(-1)^{s-t}.$$

By Lemma 5.2 with $t=0$, we have

$$g_0(x) = x^{n-1} + (-1)^n \sum_{l=0}^{n-2}(-x)^l(-1)\sum_{c=0}^{l}\binom{n}{n-l+c}(-1)^c = x^{n-1} + (-1)^n \sum_{l=0}^{n-2}(-x)^l(-1)\binom{n-1}{l}.$$

Hence, $g_0(x) = (-1)^{n-1}(1-x)^{n-1}$. For $1 \le t \le m-1$, by substituting l with $b = l+1-t$ and then applying Lemma 5.2 with $c = s-t$, we have

$$g_t(x) = (-1)^n \sum_{b=0}^{n-t-1} (-x)^{b+t-1} \sum_{s=t}^{b+t} \binom{n}{n-b-t+s}\binom{s}{t}(-1)^{s-t} = (-1)^n \sum_{b=0}^{n-t-1} \binom{n-t-1}{b}(-x)^{b+t-1}.$$

Hence, $g_t(x) = (-1)^{n-1-t}x^{t-1}(1-x)^{n-t-1}$.

Set $a(x,y) = a_1(x,y) + \sum_{t=m}^{n-1} g_t(x)(1-y)^{t-m}$. We are done. □

Recall that $\mathcal{S}_1$ (resp. $\mathcal{S}_{n-1}$) denotes the tautological vector subbundle of $\mathbb{F}\ell_{1,n-1;n}$ whose fiber at $V_\bullet$ is given by the vector space V_1 (resp. V_{n-1}). Let P_1 (resp. P_2) be the class of line bundle $\iota^*\mathcal{S}_1$ (resp. $\iota^*(\underline{\mathbb{C}^n}/\mathcal{S}_{n-1})^\vee$) in $K(H_{n-1,m-1})$, namely $P_i = L_i^{-1}$ in Theorem 1.4.

Proposition 5.4. *The classical K-theory of $H_{n-1,m-1}$ is generated by P_1, P_2 with*

$$K(H_{n-1,m-1}) = \mathbb{Q}[P_1,P_2]/(F_1(P_1,P_2), F_2(P_1,P_2)).$$

Proof. Note $H_{n-1,m-1} \cong \mathbb{P}(\iota_m^*\mathcal{U}_{n-1})$. By the classical K-theory for projective bundles (see e.g. [Ka78, Theorem 2.16]), we have

$$K(H_{n-1,m-1}) = K(\mathbb{P}^{m-1})[P_1]/(P_1^{n-1} - [\lambda^1(\iota_m^*\mathcal{U}_{n-1})]P_1^{n-2} + \cdots + (-1)^{n-1}[\lambda^{n-1}(\iota_m^*\mathcal{U}_{n-1})]).$$

Here $\lambda^k(\iota_m^*\mathcal{U}_{n-1})$ is the k-th exterior power of $\iota_m^*\mathcal{U}_{n-1}$, and $K(H_{n-1,m-1})$ is a free $K(\mathbb{P}^{m-1})$-module via the ring homomorphism $(\pi_2^m)^* : K(\mathbb{P}^{m-1}) \longrightarrow K(H_{n-1,m-1})$.

Let $\underline{\mathbb{C}^n}$ also denote the trivial bundle over $\mathbb{F}\ell_{n-1;n}$ by abuse of notation, and denote $\mathcal{Q} := \underline{\mathbb{C}^n}/\mathcal{U}_{n-1}$, which is a line bundle. So $(1-(\iota_m^*\mathcal{Q})^\vee)^m = 0$ in $K(\mathbb{P}^{m-1})$. Notice $P_2 = [\iota^*(\underline{\mathbb{C}^n}/\mathcal{S}_{n-1})^\vee] = [(\pi_2^m)^*(\iota_m^*\mathcal{Q})^\vee] \in K(H_{n-1,m-1})$. Hence, $(1-P_2)^m = 0$.

Let $\mathbf{1} \in K(\mathbb{P}^{m-1})$ be the class of the trivial line bundle. For $1 \leq k \leq n$, it follows from the exact sequence $0 \to \mathcal{U}_{n-1} \to \underline{\mathbb{C}^n} \to \mathcal{Q} \to 0$, together with $\mathcal{Q}$ being a line bundle, that

$$\binom{n}{k}\mathbf{1} = \sum_{t+b=k} [\iota_m^*(\lambda^t(\mathcal{U}_{n-1}))] \cdot [\iota_m^*(\lambda^b \mathcal{Q})] = [\lambda^{k-1}(\iota_m^*\mathcal{U}_{n-1}) \otimes \iota_m^* \mathcal{Q})] + [\lambda^k(\iota_m^*\mathcal{U}_{n-1})]. \tag{5.4}$$

In particular, we have $[\lambda^{n-1}(\iota_m^*\mathcal{U}_{n-1}) \otimes \iota_m^*\mathcal{Q})] = \mathbf{1}$, implying $[\lambda^{n-1}(\iota_m^*\mathcal{U}_{n-1})] = [\iota_m^*\mathcal{Q})^\vee]$. By (reverse) induction on k, we have $[\lambda^k(\iota_m^*\mathcal{U}_{n-1})] = \sum_{s=1}^{n-k}(-1)^{s-1}\binom{n}{k+s}[(\iota_m^*\mathcal{Q})^\vee]^s$ in $K(\mathbb{P}^{m-1})$ for $1 \leq k \leq n-1$. Hence, $K(H_{n-1,m-1})$ is the quotient of $\mathbb{Q}[P_1, P_2]$ by the ideal generated by $(1-P_2)^m$ and

$$\sum_{l=0}^{n-1}(-1)^{n-1-l}(P_1)^l\left(\sum_{s=1}^{l+1}(-1)^{s-1}\binom{n}{n-1-l+s}P_2^s\right). \tag{5.5}$$

Then we are done by Lemma 5.3. $\square$

5.3. **Ring presentation of $QK(H_{n-1,m-1})$.** Recall $L_i = P_i^{-1} \in K(H_{n-1,m-1})$, and $\{h_i = c_1(L_i)\}_{1\leq i\leq 2}$ form a nef basis of $H^2(H_{n-1,m-1}, \mathbb{Z})$. We have the Mori cone $\overline{\mathrm{NE}}(H_{n-1,m-1}) = \mathbb{Z}_{\geq 0}\beta_1 + \mathbb{Z}_{\geq 0}\beta_2 \subset H_2(H_{n-1,m-1}, \mathbb{Z})$ with with $\int_{\beta_i} h_j = \delta_{i,j}$. By [Ta13, Proposition 1] and [Lee99, Theorem 10], the small K-theoretic J-function of $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ is given by

$$J_{\mathbb{P}^{n-1}\times\mathbb{P}^{m-1}}(\hbar, \mathbf{Q}) = \sum_{d_1\geq 0,\, d_2\geq 0}^{\infty} \frac{Q_1^{d_1}Q_2^{d_2}}{\prod_{l=1}^{d_1}(1-\mathcal{L}_1^{-1}\hbar^l)^n \prod_{l=1}^{d_2}(1-\mathcal{L}_2^{-1}\hbar^l)^m}. \tag{5.6}$$

Here $\mathcal{L}_1$ (resp. $\mathcal{L}_2$) is the class of the line bundle obtained by the pullback of $\mathcal{O}_{\mathbb{P}^n}(1)$ (resp. $\mathcal{O}_{\mathbb{P}^m}(1)$), and we still denote by Q_1, Q_2 the corresponding Novikov variables for $QK(\mathbb{P}^{n-1} \times \mathbb{P}^{m-1})$ by abuse of notation. As a degree-$(1,1)$ hypersurface of $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$, by the quantum Lefschetz principle [Gi15, the last Theorem] (see also [Gi20, Theorem 3] for general cases), we obtain the small K-theoretic J-function of $H_{n-1,m-1}$, [3]

$$J_{H_{n-1,m-1}}(\hbar, \mathbf{Q}) = \sum_{d_1\geq 0, d_2\geq 0}^{\infty} \frac{Q_1^{d_1}Q_2^{d_2}\prod_{l=1}^{d_1+d_2}(1-L_1^{-1}\cdot L_2^{-1}\hbar^l)}{\prod_{l=1}^{d_1}(1-L_1^{-1}\hbar^l)^n \prod_{l=1}^{d_2}(1-L_2^{-1}\hbar^l)^m}. \tag{5.7}$$

Theorem 5.5. *Denote $\vartheta_k := 1 - L_k^{-1}\hbar^{Q_k\partial_{Q_k}}, k = 1, 2$. For the $\hbar$-difference operators*

$$D_1 := \vartheta_2^m, \qquad D_2 := (-1)^{n-1}(\vartheta_1)^{n-1} + \sum_{t=1}^{m-1}(-1)^{n-1-t}(1-\vartheta_1)^{t-1}\vartheta_1^{n-t-1}\vartheta_2^t,$$

we have

$$\big(D_1 - Q_2 + Q_2\hbar(1-\vartheta_1)(1-\vartheta_2)\big)J_{H_{n-1,m-1}} = 0, \tag{5.8}$$

$$\big(D_2 - (-1)^{n-m}Q_2(1-\vartheta_1)^{m-1}\vartheta_1^{n-m} - (-1)^{n-1}Q_1(1-\vartheta_2)\big)J_{H_{n-1,m-1}} = 0. \tag{5.9}$$

Proof. We show the statement by direct computations as follows.

[3] The J-function of $H_{2,2} = F\ell_{1,2;3}$ was computed in [GL03, Example 2.5] up to a scalar by $(1-\hbar)$.

$$\begin{aligned}
D_1(J_{H_{n-1,m-1}}) &= \Bigg(\sum_{d_2=0} \frac{(1-L_2^{-1})^m Q_1^{d_1} \prod_{l=1}^{d_1}(1-L_1^{-1}\cdot L_2^{-1}\hbar^l)}{\prod_{l=1}^{d_1}(1-L_1^{-1}\hbar^l)^n} \\
&\quad + \sum_{d_2\geq 1}^{\infty} \frac{(1-L_2^{-1}\hbar^{d_2})^m Q_1^{d_1} Q_2^{d_2} \prod_{l=1}^{d_1+d_2}(1-L_1^{-1}\cdot L_2^{-1}\hbar^l)}{\prod_{l=1}^{d_1}(1-L_1^{-1}\hbar^l)^n \prod_{l=1}^{d_2}(1-L_2^{-1}\hbar^l)^m} \Bigg) \\
&= 0 + \sum_{d_2\geq 1}^{\infty} \frac{Q_1^{d_1} Q_2^{d_2} \prod_{l=1}^{d_1+d_2-1}(1-L_1^{-1}\cdot L_2^{-1}\hbar^l)\cdot 1}{\prod_{l=1}^{d_1}(1-L_1^{-1}\hbar^l)^n \prod_{l=1}^{d_2-1}(1-L_2^{-1}\hbar^l)^m} \\
&\quad + Q_2\hbar \sum_{d_2\geq 1}^{\infty} \frac{Q_1^{d_1} Q_2^{d_2-1} \prod_{l=1}^{d_1+d_2-1}(1-L_1^{-1}\cdot L_2^{-1}\hbar^l)(-L_1^{-1}\hbar^{d_1}\cdot L_2^{-1}\hbar^{d_2-1})}{\prod_{l=1}^{d_1}(1-L_1^{-1}\hbar^l)^n \prod_{l=1}^{d_2-1}(1-L_2^{-1}\hbar^l)^m} \\
&= Q_2 J_{H_{n-1,m-1}} - Q_2\hbar(L_1^{-1}\hbar^{Q_1\partial_{Q_1}})(L_2^{-1}\hbar^{Q_2\partial_{Q_2}})(J_{H_{n-1,m-1}}).
\end{aligned}$$

Here the term 0 in the second equality follows from the relation $(1-L_2^{-1})^m = 0$ in $K(H_{n-1,m-1})$. The third equality holds because $d_2 \geq 1$ is replaced by $d_2 - 1 \geq 0$, so that the J-function $J_{H_{n-1,m-1}}$ appears again.

To simplify the expressions, we denote $S_i := (1 - L_i^{-1}\hbar^{d_i}), i = 1, 2$. We have

$$\begin{aligned}
&D_2(J_{H_{n-1,m-1}}) \\
&= \Bigg(F_2(L_1^{-1}, L_2^{-1}) + \sum_{0\neq(d_1,d_2)}^{\infty} Q_1^{d_1}Q_2^{d_2} \frac{F_2(L_1^{-1}\hbar^{d_1}, L_2^{-1}\hbar^{d_2}) \prod_{l=1}^{d_1+d_2}(1-L_1^{-1}\cdot L_2^{-1}\hbar^l)}{\prod_{l=1}^{d_1}(1-L_1^{-1}\hbar^l)^n \prod_{l=1}^{d_2}(1-L_2^{-1}\hbar^l)^m} \Bigg) \\
&= 0 + \sum_{0\neq(d_1,d_2)}^{\infty} Q_1^{d_1}Q_2^{d_2} \frac{F_2(1-S_1, 1-S_2)(-S_1S_2+S_1+S_2) \prod_{l=1}^{d_1+d_2-1}(1-L_1^{-1}\cdot L_2^{-1}\hbar^l)}{\prod_{l=1}^{d_1}(1-L_1^{-1}\hbar^l)^n \prod_{l=1}^{d_2}(1-L_2^{-1}\hbar^l)^m} \\
&= \Bigg(\sum_{d_1=0,d_2\geq 1}^{\infty} Q_2^{d_2} \frac{((-1)^{n-m}(1-S_1)^{m-1}S_1^{n-m}S_2^m) \prod_{l=1}^{d_2-1}(1-L_1^{-1}\cdot L_2^{-1}\hbar^l)}{\prod_{l=1}^{d_2}(1-L_2^{-1}\hbar^l)^m} \\
&\quad + \sum_{d_1\geq 1,d_2=0}^{\infty} Q_1^{d_1} \frac{((-1)^{n-1}S_1^n(1-S_2)) \prod_{l=1}^{d_1-1}(1-L_1^{-1}\cdot L_2^{-1}\hbar^l)}{\prod_{l=1}^{d_1}(1-L_1^{-1}\hbar^l)^n} \\
&\quad + \sum_{d_1\geq 1,d_2\geq 1}^{\infty} Q_1^{d_1}Q_2^{d_2} \frac{((-1)^{n-1}S_1^n(1-S_2) + (-1)^{n-m}(1-S_1)^{m-1}S_1^{n-m}S_2^m) \prod_{l=1}^{d_1+d_2-1}(1-L_1^{-1}\cdot L_2^{-1}\hbar^l)}{\prod_{l=1}^{d_1}(1-L_1^{-1}\hbar^l)^n \prod_{l=1}^{d_2}(1-L_2^{-1}\hbar^l)^m} \Bigg) \\
&= \Bigg(Q_2 \sum_{d_2\geq 1}^{\infty} Q_1^{d_1}Q_2^{d_2-1} \frac{((-1)^{n-m}(1-S_1)^{m-1}S_1^{n-m}S_2^m) \prod_{l=1}^{d_1+d_2-1}(1-L_1^{-1}\cdot L_2^{-1}\hbar^l)}{\prod_{l=1}^{d_1}(1-L_1^{-1}\hbar^l)^n \prod_{l=1}^{d_2-1}(1-L_2^{-1}\hbar^l)^m S_2^m} \\
&\quad + Q_1 \sum_{d_1\geq 1}^{\infty} Q_1^{d_1-1}Q_2^{d_2} \frac{((-1)^{n-1}S_1^n(1-S_2)) \prod_{l=1}^{d_1-1+d_2}(1-L_1^{-1}\cdot L_2^{-1}\hbar^l)}{S_1^n \prod_{l=1}^{d_1-1}(1-L_1^{-1}\hbar^l)^n \prod_{l=1}^{d_2}(1-L_2^{-1}\hbar^l)^m} \Bigg) \\
&= (-1)^{n-m}Q_2(L_1^{-1}\hbar^{Q_1\partial_{Q_1}})^{m-1}(1-L_1^{-1}\hbar^{Q_1\partial_{Q_1}})^{n-m}(J_{H_{n-1,m-1}}) + (-1)^{n-1}Q_1(L_2^{-1}\hbar^{Q_2\partial_{Q_2}})(J_{H_{n-1,m-1}}).
\end{aligned}$$

Here $F_2(x,y)$ is the polynomial defined in Equation (5.3). The term 0 in the second equality follows from the relation $F_2(L_1^{-1},L_2^{-1})=0$ in $K(H_{n-1,m-1})$ by Proposition 5.4. The term $(-S_1S_2+S_1+S_2)$ in the second equality comes from $1-L_1^{-1}L_2^{-1}\hbar^{d_1+d_2}$. The last equality holds because $d_2\geq 1$ (resp. $d_1\geq 1$) could be replaced by $d_2-1\geq 0$ (resp. $d_1-1\geq 0$) after canceling out the same term S_2^m (resp. S_1^n), so that the J-function appears again. □

To achieve a ring presentation of $QK(H_{n-1,m-1})$, it remains to apply the following Nakayama-type result proved in [GMSZ22, Proposition A.3].

Proposition 5.6 (Gu-Mihalcea-Sharpe-Zou)**.** *Let A be a Noetherian integral domain, and let $\mathfrak{a}\subset A$ be an ideal. Assume that A is complete in the $\mathfrak{a}$-adic topology. Let M,N be finitely generated A-modules. Assume that the A-module N, and the $A/\mathfrak{a}$-module $N/\mathfrak{a}N$, are both free modules of the same rank $r\leq\infty$, and that we are given an A-module homomorphism $f:M\to N$ such that the induced $A/\mathfrak{a}$-module map $\overline{f}:M/\mathfrak{a}M\to N/\mathfrak{a}N$ is an isomorphism of $A/\mathfrak{a}$-modules. Then f is an isomorphism.*

Now we restate Theorem 1.5 as follows.

Theorem 5.7. *Let $n\geq m\geq 3$. The small quantum K-theory of $H_{n-1,m-1}$ is presented by*

$$QK(H_{n-1,m-1})\cong\mathbb{Q}[L_1^{-1},L_2^{-1}][\![Q_1,Q_2]\!]/(F_1^Q,F_2^Q),$$

where $F_1^Q(L_1^{-1},L_2^{-1},Q_1,Q_2)=F_1(L_1^{-1},L_2^{-1})-Q_2+Q_2L_1^{-1}L_2^{-1}$ *and*

$F_2^Q(L_1^{-1},L_2^{-1},Q_1,Q_2)=F_2(L_1^{-1},L_2^{-1})-(-1)^{n-m}Q_2(L_1^{-1})^{m-1}(1-L_1^{-1})^{n-m}-(-1)^{n-1}Q_1L_2^{-1}.$

Proof. Let M be the right hand side of the expected isomorphism, set $N:=QK(H_{n-1,m-1})$, and take the ideal $\mathfrak{a}=(Q_1,Q_2)$ of $A:=\mathbb{Q}[\![Q_1,Q_2]\!]$. Clearly, we have $A/\mathfrak{a}=\mathbb{Q}$. Moreover, N (resp. $N/\mathfrak{a}N$) is a free A-module (resp. $A/\mathfrak{a}$-module) of rank $\dim_{\mathbb{Q}}K(H_{n-1,m-1})=m(n-1)<\infty$. Note $n\geq m\geq 3$. It follows directly from the expression of $J_{H_{n-1,m-1}}$ that each (d_1,d_2)-term of $L_i^{-1}\hbar^{Q_i\partial_{Q_i}}(J_{H_{n-1,m-1}})$ vanishes at $\hbar=\infty$ for $1\leq i\leq 2$, whenever $d_1>0$ or $d_2>0$. By Proposition 2.3 and Theorem 5.5, the relations $F_1^Q(L_1^{-1},L_2^{-1},Q_1,Q_2)=0$ and $F_2^Q(L_1^{-1},L_2^{-1},Q_1,Q_2)=0$ hold in $QK(H_{n-1,m-1})$ (with respect to the quantum K-product). Therefore we have a canonical ring homomorphism $f:M\to N$ defined by $L_i^{-1}\mapsto L_i^{-1}$ and $Q_i\mapsto Q_i, i=1,2$. Since $f(\mathfrak{a}M)\subset\mathfrak{a}N$, it induces an $A/\mathfrak{a}$-module homomorphism $\overline{f}:M/\mathfrak{a}M\to N/\mathfrak{a}N$. Note $M/\mathfrak{a}M\cong(A/\mathfrak{a})\otimes_A M\cong\mathbb{Q}[L_1^{-1},L_2^{-1}]/(F_1(L_1^{-1},L_2^{-1}),F_2(L_1^{-1},L_2^{-1})$ and $N/\mathfrak{a}N\cong K(H_{n-1,m-1})$. Thus $\overline{f}$ is an isomorphism of $A/\mathfrak{a}$-modules by Proposition 5.4. Therefore f is an isomorphism of A-modules by Proposition 5.6. □

Remark 5.8. *The presentation for the special case $QK(H_{n-1,n-1})$ can also be derived from the quantum Littlewood-Richardson rule for $QK(\mathbb{F}\ell_{1,n-1;n})$ [Xu24, Section 5]. Therein, the Schubert class $\mathcal{O}^{[1,n-1]}$ (resp. $\mathcal{O}^{[2,n]}$) is given by $1-L_1^{-1}$, (resp. $1-L_2^{-1}$).*

6. Appendix: proof of Lemma 4.5

By [BCFKS00], the toric superpotential mirror to $\mathbb{F}\ell_{1,n-1;n}$ is the Laurent polynomial

$$f_{\mathrm{tor}}:=\sum_{k=1}^{2n-3}\frac{x_k}{x_{k-1}}+\frac{q_2}{x_{n-2}}+\frac{x_n}{q_2}+\frac{q_1q_2}{x_{2n-3}}\in\mathbb{Q}[x_1^{\pm1},\cdots,x_{2n-3}^{\pm1},q_1^{\pm1},q_2^{\pm1}],$$

where $x_0 := 1$. The Jacobi ring of f_{tor} is defined by

$$\mathrm{Jac}(f_{\mathrm{tor}}) := \mathbb{Q}[x_1^{\pm 1}, \cdots, x_{2n-3}^{\pm 1}, q_1^{\pm 1}, q_2^{\pm 1}] \Big/ \Big(x_1 \frac{\partial f_{\mathrm{tor}}}{\partial x_1}, \cdots, x_{2n-3} \frac{\partial f_{\mathrm{tor}}}{\partial x_{2n-3}} \Big). \tag{6.1}$$

Proposition 6.1. *There is an injective homomorphism of* $\mathbb{Q}[\![q_1, q_2]\!]$*-algebras*

$$\Phi : QH(\mathbb{F}\ell_{1,n-1;n}) \hookrightarrow \mathrm{Jac}(f_{\mathrm{tor}}) \otimes_{\mathbb{Q}[q_1,q_2]} \mathbb{Q}[\![q_1, q_2]\!],$$

such that $\Phi(h_1) = [\frac{q_1 q_2}{x_{2n-3}}], \quad \Phi(h_2) = [x_1]$.

Proof. Assume $n \geq 4$ first. Let R_a be the relation given by $x_a \frac{\partial f_{\mathrm{tor}}}{\partial x_a}$. From $R_1, ..., R_{n-3}$ and $R_{n+1}, \ldots, R_{2n-3}$, we get

$$x_k = \begin{cases} x_1^k, & \text{if } 2 \leq k \leq n-2, \\ (\frac{x_{2n-3}}{q_1 q_2})^{2n-3-k} x_{2n-3}, & \text{if } n \leq k \leq 2n-4. \end{cases} \tag{6.2}$$

It follows immediately that

$$\mathrm{Jac}(f_{\mathrm{tor}}) \cong \mathbb{Q}[x_1^{\pm 1}, x_{2n-3}^{\pm 1}, x_{n-1}^{\pm 1}, q_1^{\pm 1}, q_2^{\pm 1}] \big/ \big(R_{n-2}, R_{n-1}, R_n \big). \tag{6.3}$$

It follows from R_{n-2} that

$$x_{n-1} = x_1^{n-1} - q_2. \tag{6.4}$$

It follows from R_{n-1} and (6.2) that in $\mathrm{Jac}(f_{\mathrm{tor}})$,

$$x_n = \frac{(x_1^{n-1} - q_2)^2}{x_1^{n-2}} = (\frac{x_{2n-3}}{q_1 q_2})^{n-3} x_{2n-3}. \tag{6.5}$$

By R_n we have $\frac{x_n}{x_{n-1}} = \frac{x_{n+1}}{x_n} - \frac{x_n}{q_2}$. By (6.4) and (6.5), we have $\frac{x_n}{x_{n-1}} = \frac{x_1^{n-1} - q_2}{x_1^{n-2}}$. By (6.2), we have $\frac{x_{n+1}}{x_n} = \frac{q_1 q_2}{x_{2n-3}}$. By (6.5), we have $\frac{x_n}{q_2} = \frac{1}{q_2} \frac{(x_1^{n-1} - q_2)^2}{x_1^{n-2}}$. All these relations together give the relation in $\mathrm{Jac}(f_{\mathrm{tor}})$

$$\frac{x_1^n}{q_2} - x_1 - \frac{q_1 q_2}{x_{2n-3}} = 0. \tag{6.6}$$

One can check that the ideal (R_{n-2}, R_{n-1}, R_n) in (6.3) is generated by the three relations (6.4), (6.5), (6.6). Recall $QH(\mathbb{F}\ell_{1,n-1;n}) = \mathbb{Q}[h_1, h_2][\![q_1, q_2]\!]/(f_1^q(h_1, h_2, q_1, q_2), f_2^q(h_1, h_2, q_1, q_2))$ by Proposition 4.1. It remains to show the relations (6.5) and (6.6) are equivalent to $f_1^q(\frac{q_1 q_2}{x_{2n-3}}, x_1, q_1, q_2)$ and $f_2^q(\frac{q_1 q_2}{x_{2n-3}}, x_1, q_1, q_2)$ in (6.3).

Since q_2 is a unit in (6.3), (6.6) is equivalent to $x_1^n - q_2(x_1 + \frac{q_1 q_2}{x_{2n-3}}) = f_1^q(\frac{q_1 q_2}{x_{2n-3}}, x_1, q_1, q_2)$. By direct calculations using (6.5) and (6.6), we have

$$q_1 q_2 - (\frac{q_1 q_2}{x_{2n-3}})^{n-2} (-\frac{x_1^{n-1} - q_2}{x_1^{n-2}} + \frac{q_1 q_2}{x_{2n-3}}) q_2 = 0 \tag{6.7}$$

Therefore, in (6.3), we note q_2 is a unit again, and then by (6.7) we have

$$\begin{aligned}0 =& q_1 - (\frac{q_1q_2}{x_{2n-3}})^{n-2}(-x_1 + \frac{q_2}{x_1^{n-2}} + \frac{q_1q_2}{x_{2n-3}})\\ =& q_1 - \big((\frac{q_1q_2}{x_{2n-3}})^{n-1} - (\frac{q_1q_2}{x_{2n-3}})^{n-2}x_1 + \frac{q_1q_2}{x_{2n-3}}\frac{q_2}{x_1^{n-2}}(\frac{q_1q_2}{x_{2n-3}})^{n-3}\big)\\ =& q_1 - \big((\frac{q_1q_2}{x_{2n-3}})^{n-1} - (\frac{q_1q_2}{x_{2n-3}})^{n-2}x_1 + (\frac{q_1q_2}{x_{2n-3}})^{n-3}x_1^2 - \frac{q_2}{x_1^{n-3}}(\frac{q_1q_2}{x_{2n-3}})^{n-3}\big)\\ =& q_1 - \big((\frac{q_1q_2}{x_{2n-3}})^{n-1} - (\frac{q_1q_2}{x_{2n-3}})^{n-2}x_1 + (\frac{q_1q_2}{x_{2n-3}})^{n-3}x_1^2\\ &\qquad - (\frac{q_1q_2}{x_{2n-3}})^{n-4}x_1^3 + \cdots + (-1)^{n-1}x_1^{n-1} - (-1)^{n-1}q_2\big).\end{aligned}$$

Here the third equality follows from (6.6), and the last equality follows from (6.6) together with the induction on the power of x_1. Therefore we obtain the following relation

$$0 = \sum_{k=0}^{n-1}(-1)^k(\frac{q_1q_2}{x_{2n-3}})^{n-1-k}x_1^k - (-1)^{n-1}q_2 - q_1 = f_2^q(\frac{q_1q_2}{x_{2n-3}}, x_1, q_1, q_2). \tag{6.8}$$

It follows directly from the process of obtaining f_1^q, f_2^q that

$$\big(R_{n-2}, R_{n-1}, R_n\big) = \big(x_{n-1} - (x_1^{n-1} - q_2), f_1^q(\frac{q_1q_2}{x_{2n-3}}, x_1, q_1, q_2), f_2^q(\frac{q_1q_2}{x_{2n-3}}, x_1, q_1, q_2)\big)$$

as ideals in $\mathbb{Q}[x_1^{\pm1}, x_{2n-3}^{\pm1}, x_{n-1}^{\pm1}, q_1^{\pm1}, q_2^{\pm1}]$. Hence, $h_1 \mapsto \frac{q_1q_2}{x_{2n-3}}, h_2 \mapsto x_1$, $q_1 \mapsto q_1$ and $q_2 \mapsto q_2$ cannonically defines an injective ring homomorphism Φ.

The case $n = 3$ can be verified by similar but easier calculations (in which case $\mathbb{F}\ell_{1,2;3}$ is a complete flag variety and the expected statement is known earlier to the experts). $\square$

Remark 6.2. *The injective homomorphism Φ can naturally be extended to an isomorphism from a suitable localization of $QH(\mathbb{F}\ell_{1,n-1;n})$ to $\mathrm{Jac}(f_{\mathrm{tor}}) \otimes_{\mathbb{Q}[q_1,q_2]} \mathbb{Q}[\![q_1, q_2]\!]$.*

Proof of Lemma 4.5. By Proposition 6.1, $QH(\mathbb{F}\ell_{1,n-1;n})$ is a subring of $\mathrm{Jac}(f_{\mathrm{tor}}) \otimes_{\mathbb{Q}[q_1,q_2]} \mathbb{Q}[\![q_1, q_2]\!]$ via the embeding Φ. Note that both $\Phi(h_1) = \frac{q_1q_2}{x_{2n-3}}$ and q_1 are invertible in the Jacobi ring. Since $h_1^n = q_1(h_1 + h_2)$ holds in $QH(\mathbb{F}\ell_{1,n-1;n})$, $\Phi(h_1 + h_2) = q_1^{-1}\Phi(h_1)^n$ is invertible in the Jacobi ring. Hence, $h_1 + h_2$ is not a zero-divisor in (the subring) $QH(\mathbb{F}\ell_{1,n-1;n})$. $\square$

References

[AHK+25] K. Amini, I. Huq-Kuruvilla, L. C. Mihalcea, D. Orr and W. Xu, *Toda-type presentations for the quantum K theory of partial flag varieties*, preprint at arXiv: math.AG/2504.07412 (2025).

[ACTI22] D Anderson, L Chen and H. H. Tseng, *On the finiteness of quantum K-theory of a homogeneous space*, Int Math Res Not, 2022, 1313–1349. With Appendix B by H. Iritani.

[AH61] M.F. Atiyah, F. Hirzebruch, *Vector bundles and homogeneous spaces*, Proc. Sympos. Pure Math., Vol. III, pp. 7–38, American Mathematical Society, Providence, RI, 1961.

[BeFa97] K. Behrend and B. Fantechi, *The intrinsic normal cone*, Invent. Math. 128 (1997), 45–88.

[BCFKS00] V. Batyrev, I. Ciocan-Fontanine, B. Kim and D. van Straten, *Mirror symmetry and toric degenerations of partial flag manifolds*, Acta Math. 184 (2000), no. 1, 1-39.

[BrFi11] A. Braverman and M. Finkelberg, *Semi-infinite Schubert varieties and quantum K-theory of flag manifolds*, J. Amer. Math. Soc. 27 (2014), no. 4, 1147–1168.

[BCMP13] A. S. Buch, P.-E. Chaput, L. C. Mihalcea and N. Perrin, *Finiteness of cominuscule quantum K-theory*, Ann. Sci. Éc. Norm. Supér. (4) 46 (2013), no. 3, 477–494.

[BCMP16] A. S. Buch, P.-E. Chaput, L. C. Mihalcea and N. Perrin, *Rational connectedness implies finiteness of quantum K-theory*, Asian J. Math. 20 (2016), no. 1, 117–122.

[BCMP18] A. S. Buch, P.-E. Chaput, L. C. Mihalcea and N. Perrin, *A Chevalley formula for the equivariant quantum K-theory of cominuscule varieties*, Algebr. Geom. 5 (2018), no. 5, 568–595.

[BCP23] A. S. Buch, P.-E. Chaput and N. Perrin, *Seidel and Pieri products in cominuscule quantum K-theory*, preprint at arXiv: math.AG/2308.05307 (2023).

[BM11] A. S. Buch and L. C. Mihalcea, *Quantum K-theory of Grassmannians*, Duke Math. J. 156 (2011), no. 3, 501–538.

[CP11] P.E. Chaput and N. Perrin, *On the quantum cohomology of adjoint varieties*, Proc. Lond. Math. Soc. (3) 103 (2011), no. 2, 294–330.

[Co03] T. Coates, *Riemann-Roch theorems in Gromov-Witten theory*, ProQuest LLC, Ann Arbor, MI, 2003, 147 pp.

[CG06] T. Coates and A. Givental, *Quantum cobordisms and formal group laws*, Progr. Math., 244, Birkhäuser Boston, Inc., Boston, MA, 2006, 155–171.

[ChLe20] C. H. Chow and N. C. Leung, *Quantum K-theory of G/P and K-homology of affine Grassmannian*, preprint at arXiv: math.AG/2201.12951 (2022).

[CoLi20] K. Costello and S. Li, *Anomaly cancellation in the topological string*, Adv. Theor. Math. Phys. 24 (2020), no. 7, 1723–1771.

[EH16] D. Eisenbud and J. Harris, *3264 and all that—a second course in algebraic geometry*, Cambridge University Press, Cambridge, 2016.

[FP97] W. Fulton, R. Pandharipande, *Notes on stable maps and quantum cohomology. Algebraic geometry-Santa Cruz*, Proc. Sympos. Pure Math., 62, Part 2, Amer. Math. Soc., Providence, RI, 1997.

[FO99] Kenji Fukaya, Kaoru Ono. Arnold conjecture and Gromov-Witten invariant. Topology 38 (1999), no. 5, 933–1048.

[Gi00] A. Givental, *On the WDVV equation in quantum K-theory*, Michigan Math. J. 48 (2000), 295–304.

[Gi04] A. Givental, *Symplectic geometry of Frobenius structures*, Aspects Math., E36, Friedr. Vieweg & Sohn, Wiesbaden, 2004, 91–112.

[Gi15] A. Givental, *Permutation-equivariant quantum K-theory V. Toric q-hypergeometric functions*, preprint at arXiv: math.AG/1509.03903 (2015).

[Gi20] A. Givental, *Permutation-equivariant quantum K-theory X. Quantum Hirzebruch-Riemann-Roch in genus 0*, SIGMA Symmetry Integrability Geom. Methods Appl. 16 (2020), Paper No. 031, 16 pp.

[GL03] A. Givental and Y.-P. Lee, *Quantum K-theory on flag manifolds, finite-difference Toda lattices and quantum groups*, Invent. Math. 151 (2003), no. 1, 193–219.

[GT14] A. Givental and V. Tonita, *The Hirzebruch-Riemann-Roch theorem in true genus-0 quantum K-theory*, Math. Sci. Res. Inst. Publ., 62, Cambridge University Press, New York, 2014, 43–91.

[Gr58] A. Grothendieck, *La théorie des classes de Chern*, Bull. Soc. Math. France 86 (1958), 137–154.

[GM$^+$23] W. Gu, L. C. Mihalcea, E. Sharpe, W. Xu, H. Zhang and H. Zou, *Quantum K Whitney relations for partial flag varieties*, preprint at arXiv: math.AG/2310.03826 (2023).

[GM$^+$24] W. Gu, L. C. Mihalcea, E. Sharpe, W. Xu, H. Zhang and H. Zou, *Quantum K theory rings of partial flag manifolds*, J. Geom. Phys. 198 (2024), Paper No. 105127, 30pp.

[GMSZ22] W. Gu, L. C. Mihalcea, E. Sharpe and H. Zou, *Quantum K theory of Grassmannians, Wilson line operators, and Schur bundles*, preprint at arXiv: math.AG/2208.01091 (2022).

[HK24a] I. Huq-Kuruvilla, *Relations in Twisted Quantum K-Rings*, preprint at arXiv: math.AG/2406.00916 (2024).

[HK24b] I. Huq-Kuruvilla, *Quantum K-rings of partial flag varieties, Coulomb branches, and the Bethe ansatz*, preprint at arXiv: math.AG/2409.15575 (2024).

[IIM20] T. Ikeda, S. Iwao and T. Maeno, *Peterson isomorphism in K-theory and relativistic Toda lattice*, Int. Math. Res. Not. IMRN 19 (2020), 6421–6462.

[IMT15] H. Iritani, T. Milanov and V. Tonita, *Reconstruction and convergence in quantum K-theory via difference equations*, Int. Math. Res. Not. IMRN 2015, no. 11, 2887–2937.

[Ka78] M. Karoubi, *K-theory. An introduction*, Grundlehren der Mathematischen Wissenschaften, Band 226, Springer-Verlag, Berlin-New York, 1978.

[Ka18] S. Kato, *Loop structure on equivariant K-theory of semi-infinite flag manifolds*, to appear in Ann. Math.; preprint at arXiv: math.AG/1805.01718.

[Ka19] S. Kato, *On quantum K-groups of partial flag manifolds*, preprint at arXiv: math.AG/1906.09343 (2019).

[KPSZ21] P. Koroteev, P. P. Pushkar, A. V. Smirnov and A. M. Zeitlin, *Quantum K-theory of quiver varieties and many-body systems*, Selecta Math. (N.S.) 27 (2021), no. 5, Paper No. 87, 40pp.

[KN24] T. Kouno and S. Naito, *Borel-type presentation of the torus-equivariant quantum K ring of flag manifolds of type C*, preprint at arXiv: math.AG/2410.10575 (2024).

[KLNS24] T. Kouno, C. Lenart, S. Naito and D. Sagaki, *Quantum K-theory Chevalley formulas in the parabolic case*, J. Algebra 645 (2024), 1–53. With Appendix B joint with W. Xu.

[LLMS18] T. Lam, C. Li, L. C. Mihalcea and M. Shimozono, *A conjectural Peterson isomorphism in K-theory*, J. Algebra 513 (2018), 326–343.

[LNS24] C. Lenart, S. Naito and D. Sagaki, *A general Chevalley formula for semi-infinite flag manifolds and quantum K-theory*, Selecta Math. (N.S.) 30 (2024), no. 3, Paper No. 39, 44 pp.

[Lee99] Y.-P. Lee, *Quantum K-theory*, ProQuest LLC, Ann Arbor, MI, 1999, 58 pp.

[Le04] Y.-P. Lee, *Quantum K-theory. I. Foundations*, Duke Math. J. 121 (2004), no. 3, 389–424.

[LL17] N.C. Leung and C. Li, *An update of quantum cohomology of homogeneous varieties*, Proceedings of the Sixth International Congress of Chinese Mathematicians. Vol. II, 211–235, Adv. Lect. Math. (ALM), 37, Int. Press, Somerville, MA, 2017.

[LLSY25] C. Li, Z. Liu, J. Song and M. Yang, *On Seidel representation in quantum K-theory of Grassmannians*, Sci. China Math. 68 (2025), no. 7, 1523–1548.

[LT98] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties*, J. Amer. Math. Soc. 11 (1998), no. 1, 119–174.

[MNS25a] T. Maeno, S. Naito, and D. Sagaki, *A presentation of the torus-equivariant quantum K-theory ring of flag manifolds of type A, Part I: The defining ideal*, J. Lond. Math. Soc. (2) 111 (2025), no. 3, Paper No. e70095, 43pp.

[MNS25b] T. Maeno, S. Naito, and D. Sagaki, *A presentation of the torus-equivariant quantum K-theory ring of flag manifolds of type A, Part II: quantum double Grothendieck polynomials*, Forum Math. Sigma 13 (2025), Paper No. e19, 26pp.

[RT95] Y. Ruan and G. Tian, *A mathematical theory of quantum cohomology*, J. Differential Geom. 42 (1995), no. 2, 259–367.

[Ta13] K. Taipale, *K-theoretic J-functions of type A flag varieties*, Int. Math. Res. Not. IMRN 2013, no. 16, 3647–3677.

[Xu24] W. Xu, *Quantum K-theory of incidence varieties*, Eur. J. Math. 10 (2024), no. 2, Paper No. 22, 47 pp.

School of Mathematics, Sun Yat-sen University, Guangzhou 510275, P.R. China
Email address: `kehuazh@mail.sysu.edu.cn`

School of Mathematics, Sun Yat-sen University, Guangzhou 510275, P.R. China
Email address: `lichangzh@mail.sysu.edu.cn`

School of Mathematics, Sun Yat-sen University, Guangzhou 510275, P.R. China
Email address: `songjy29@mail2.sysu.edu.cn`