A CONJECTURAL PETERSON ISOMORPHISM IN K-THEORY

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Abstract. We state a precise conjectural isomorphism between localizations of the equivariant quantum K-theory ring of a flag variety and the equivariant K-homology ring of the affine Grassmannian, in particular relating their Schubert bases and structure constants. This generalizes Peterson's isomorphism in (co)homology. We prove a formula for the Pontryagin structure constants in the K-homology ring, and we use it to check our conjecture in few situations.

1. The K-Peterson conjecture

The goal of this manuscript is to present a precise conjecture which asserts the coincidence of the Schubert structure constants for the Pontryagin product in K-homology of the affine Grassmannian, with those for the quantum K-theory of the flag manifold. This is a K-theoretic analogue of the celebrated Peterson isomorphism between the homology of the affine Grassmannian and the quantum cohomology of the flag manifold [Pet, LS, LL].

Let G be a simple and simply-connected complex Lie group with chosen Borel subgroup B and maximal torus T, Weyl group W and affine Weyl group $W_{\mathrm{af}} = W \ltimes Q^\vee$ where $Q^\vee$ denotes the coroot lattice. Let Λ denote the weight lattice of G, so that the representation ring $R(T)$ of T is given by $R(T) \simeq K_T(\mathrm{pt}) \simeq \mathbb{Z}[\Lambda]$. The torus-equivariant quantum K-theory $\mathrm{QK}_T(G/B)$ of the flag variety G/B has as basis over $\mathbb{Z}[[q]] \otimes R(T)$ the Schubert classes $\mathcal{O}^w$, for $w \in W$, of structure sheaves of Schubert varieties in G/B. The torus-equivariant K-homology $K_0^T(\mathrm{Gr})$ of the affine Grassmannian $\mathrm{Gr} = \mathrm{Gr}_G$ of G has as basis over $\mathbb{Z}[\Lambda]$ the Schubert classes $\mathcal{O}_x$ of structure sheaves of Schubert varieties in Gr, where x varies over the set $W_{\mathrm{af}}^- \subset W_{\mathrm{af}}$ of affine Grassmannian elements.

Conjecture 1. *Let $ut_\lambda, vt_\mu, wt_\nu \in W_{\mathrm{af}}^-$, and let $\eta \in Q^\vee$. Assume that $\nu = \lambda + \mu$. Then*

$$c_{ut_\lambda, vt_\mu}^{wt_{\nu+\eta}} = N_{u,v}^{w,\eta} \qquad \text{in } K_T^*(\mathrm{pt})$$

where c's are the structure constants in $K_0^T(\mathrm{Gr})$ with respect to $\mathcal{O}_x$ and N's are the structure constants in $\mathrm{QK}_T(G/B)$ with respect to $\mathcal{O}^w$.

Conjecture 1 implies that the multiplication in the ring $\mathrm{QK}_T(G/B)$ is finite, and thus it is possible to define it over $\mathbb{Z}[q]$ instead of $\mathbb{Z}[[q]]$. On the affine side, it implies that we have the formula $\mathcal{O}_{wt_\lambda} \cdot \mathcal{O}_{t_\nu} = \mathcal{O}_{wt_{\lambda+\nu}}$ in the K-homology ring $K_0^T(\mathrm{Gr})$ endowed with the Pontryagin product. Conjecture 1 can then be alternatively formulated as follows.

Conjecture 2. *The $R(T)$-module homomorphism*

$$\Psi : K_0^T(\mathrm{Gr})[\mathcal{O}_{t_\lambda}^{-1}] \longrightarrow \mathrm{QK}_T(G/B)[q_i^{-1}]$$
$$\mathcal{O}_{wt_\lambda}\mathcal{O}_{t_\nu}^{-1} \longmapsto q_{\lambda-\nu}\mathcal{O}^w$$

is an isomorphism of $R(T)$-algebras.

A direct verification of Conjecture 1 requires computations of the structure constants on both sides. In this paper, we will mainly focus on the affine side. There has been an algebraic model $\mathbb{L}$ for the Hopf algebra structures of the (topological) K-homology $K_0^T(\mathrm{Gr})$ studied in [LSS]. Therein the Hopf algebra $\mathbb{L}$ is the centralizer of $R(T)$ in the so-called K-theoretic nilHecke ring $\mathbb{K}$. Moreover, a K-theoretic Peterson's j-isomorphism $k : K_0^T(\mathrm{Gr}) \to \mathbb{L}$ was given, similar to that for the (co)homology case in [Pet, LS]. We will give the geometric meaning of the Peterson's j-isomorphism $k : K_0^T(\mathrm{Gr}) \to \mathbb{L}$ in Theorem 1, by describing the image $k(\mathcal{O}_w)$ of Schubert classes $\mathcal{O}_w$ in $K_0^T(\mathrm{Gr})$. The extension of $R(T)$-module $\mathbb{L}$ to a module over the fraction field $Q(T)$ of $R(T)$ has a $Q(T)$-basis $\{t_\lambda\}_\lambda$, which correspond to the torus-fixed points in the affine Grassmannian Gr. The Pontryagin product among t_λ is quite simple. The basis change between t_λ and $k(\mathcal{O}_w)$ are given in terms of rational functions $b_{u,v}$ and $e_{u,v}$, precise expressions of which will be provided in Proposition 1. As a consequence, we obtain an explicit formula of the structure constants $c_{x,y}^z$ in Theorem 2. Combining with calculations on the quantum side in few cases, this leads to computational evidence for the aforementioned conjectures.

The equivariant K-theory was introduced for a flag variety X by Givental [Giv], and was defined for a general projective manifold by Lee [Lee]. Even in the nice case of a complete flag variety G/B, the structure constants $N_{u,v}^{w,d}$ for $QK_T(G/B)$ are not a single but a combination of K-theoretic Gromov-Witten invariants, and are difficult to compute in general. While there is no algorithm to calculate $N_{u,v}^{w,d}$ for $X = G/B$ or for arbitrary flag varieties G/P, there are several particular instances where algorithms are available. In the case of a cominuscule Grassmannian, a "quantum = classical" statement, calculating KGW invariants in terms of certain K-theoretic intersection numbers on two-step flag manifolds was obtained in [BM11]; in Lie types different from type A, this uses rationality results from [CP11]. As a result, a *Chevalley formula*, which calculates the multiplication by a divisor class, was obtained in [BM11] for the type A Grassmannians, and it was recently extended in [BCMP16+] to all cominuscule Grassmannians. In the equivariant context, this formula determines an algorithm to calculate any product of Schubert classes, generalizing the result from quantum cohomology [Mih]. Formulas to calculate $N_{u,v}^{w,d}$ for $X = G/B$ and d a "line class", i.e. $d = [X(s_i)]$, were obtained in [LM] by making use of the geometry of lines on flag manifolds. There are also algorithms based on reconstruction formula [LP, IMT] which in principle can be used to calculate KGW invariants. In practice, however, these lead to quantities which quickly become unfeasible for explicit calculations. We remark that it is shown in [HL, Corollary 5.10] that the K-homology Schubert structure constants determine the 3-point K-theoretic Gromov-Witten invariants of a cominuscule flag variety G/P. However, a direct formula relating the two sets of invariants is not given.

There are other evidences to support our conjectures. As mentioned earlier, one consequence of Conjecture 1 is the finiteness of $N_{u,v}^{w,d}$ for $QK_T(G/B)$. The finiteness property was conjectured to be true for any flag variety G/P by Buch, Chaput, Mihalcea and Perrin, and was proved in the case of Grassmannians, namely when P is a maximal parabolic group [BCMP13, BCMP16] by studying curve neighborhoods of Schubert varieties. It is now proved for any G/P by Anderson, Chen and Tseng by a different technique [ACT, ACT18]. There is a similar statement to Conjecture 2. Ikeda, Iwao, and Maeno [IIM] have recently shown that the K-homology ring $K_0(\mathrm{Gr}_{\mathrm{SL}_n})$ is isomorphic to Kirillov-Maeno's conjectural presentation [LeMa] of the quantum K-theory $QK(\mathrm{Fl}_n)$ of complete flag manifold Fl_n after localization. Their approach is via the relativistic Toda lattice, and the behavior of Schubert classes under their isomorphism is also studied. We remark that a similar presentation of $QK(\mathrm{Fl}_n)$ was given in [KPSZ] with respect to a different definition of quantum K-theory. The quantum K-theory introduced by Givental and Lee was in the formularism of stable maps. There is another one in the formularism of quasimaps. Braverman and Finkelberg [BF] showed that the coefficients of Givental's K-theoretic J-function [Giv] for a flag variety are the equivariant characters of the polynomial functions on a Zastava space, which consists of based quasimaps to the flag variety. Moreover, in each homogeneous degree, the functions on a Zastava space are isomorphic to the functions on a transverse slice of a G-stable stratum inside another G-stable stratum in the affine Grassmannian. Together with the K-theoretic reconstruction theorems [LP, IMT], this provides a conceptual connection between quantum K-theory of flag varieties and K-homology of affine Grassmannians. We refer to section 1.4 of [ACT18] for some discussions on quantum K-theory with respect to different compactifications of the space of degree d maps from $\mathbb{P}^1$ to G/P.

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2. Quantum K-theory of flag varieties

Let G be a complex, simple, simply-connected Lie group and $B, B^- \subset G$ is a pair of opposite Borel subgroups containing the fixed torus $T := B \cap B^-$. For each element $w \in W$ in the (finite) Weyl group there are the Schubert cells $X(w)^\circ := BwB/B$, $Y(w) := B^-wB/B$ and the Schubert varieties $X(w) := \overline{BwB/B}$ and $Y(w) := \overline{B^-wB/B}$ in the flag manifold $X := G/B$. Then $\dim_{\mathbb{C}} X(w) = \operatorname{codim}_{\mathbb{C}} Y(w) = \ell(w)$ (the length of w). The *boundary* of the Schubert varieties is defined by $\partial X(w) = X(w) \setminus X(w)^\circ$ and $\partial Y(w) := Y(w) \setminus Y(w)^\circ$. The boundary is generally a reduced, Cohen-Macaulay, codimension 1 subscheme of the corresponding Schubert variety.

We briefly recall the relevant definitions regarding the equivariant K-theory ring, following e.g. [CG]. For any (complex) projective variety Z with an algebraic action of a torus T, one can define the equivariant K-theory ring $K^T(Z)$. This is the ring generated by symbols $[E]_T$ of T-equivariant vector bundles $E \to Z$ subject to the relations $[F]_T + [H]_T = [E]_T$ whenever there is a short exact sequence of T-equivariant vector bundles $0 \to F \to E \to H \to 0$ on Z. The two ring operations on $K^T(Z)$ are defined by

$$[E]_T + [F]_T := [E \oplus F]_T; \quad [E]_T \cdot [F]_T := [E \otimes F]_T.$$

There is a pairing $\langle \cdot, \cdot \rangle : K^T(Z) \otimes K^T(Z) \to K^T(pt) = R(T)$, where $R(T)$ is the representation ring of T, given by

$$\langle [E]_T, [F]_T \rangle = \chi_T(Z; E \otimes F) = \sum_{i=0}^{\dim Z} (-1)^i \mathrm{ch}_T\big(H^i(Z; E \otimes F)\big);$$

here χ_T denotes the equivariant Euler characteristic and $\mathrm{ch}_T \in R(T)$ denotes the character of a T-module. If in addition Z is smooth then any T-equivariant coherent sheaf $\mathcal{F}$ on Z has a *finite* resolution by equivariant vector bundles, and thus there is a well defined class $[\mathcal{F}]_T \in K^T(Z)$. This identifies the Grothendieck *group* $K_T(Z)$ of equivariant coherent sheaves with $K^T(Z)$. For any T-equivariant map of projective varieties $f : Z_1 \to Z_2$, there is a well defined push-forward $f_* : K_T(Z_1) \to K_T(Z_2)$ given by $f_*[\mathcal{F}]_T = \sum_{i \geq 0} (-1)^i [R^i f_* \mathcal{F}]_T$; in this language the pairing above is given by $\langle [E]_T, [F]_T \rangle = \pi_*([E]_T \cdot [F]_T)$ where $\pi : Z \to pt$ is the structure map.

The maximal torus T acts on $X = G/B$ by left multiplication and the Schubert varieties $X(w), Y(w)$ are T-stable. Then the structure sheaves of the Schubert varieties determine the Grothendieck classes $\mathcal{O}_w := [\mathcal{O}_{X(w)}]_T$ and $\mathcal{O}^w := [\mathcal{O}_{Y(w)}]_T$ in the T-equivariant K-theory ring $K_T(X)$. We will also need the *ideal sheaf* classes $\xi_w := [\mathcal{O}_{X(w)}(-\partial X(w))]_T$ and $\xi^w := [\mathcal{O}_{Y(w)}(-\partial Y(w))]_T$ determined by the boundaries of the corresponding Schubert varieties. The ideal sheaf classes are duals of the Schubert classes:

$$\langle \mathcal{O}_u, \xi^v \rangle = \langle \mathcal{O}^u, \xi_v \rangle = \delta_{u,v},$$

where δ is the Kronecker delta symbol. We refer to [Bri05, §3.3] or [AGM] for the proofs of this.

Motivated by the relation between quantum cohomology and Toda lattice discovered by Givental and Kim [GK, Kim], Givental and Lee [Giv, Lee] defined a ring which deforms both the (equivariant) K-theory and the quantum cohomology rings for the flag manifold X. This is the equivariant *quantum K-theory* ring $\mathrm{QK}_T(X)$. Additively, $\mathrm{QK}_T(X)$ is the free module over the power series ring $K_T(pt)[[q]] = R(T)[[q_1, \ldots, q_r]]$ which has a $R(T)[[q]]$-basis given by Schubert classes $\mathcal{O}^w$ (or $\mathcal{O}_w$) as w varies in W. Here r denotes the rank of $H_2(X)$, which for $X = G/B$ is the same as the number of simple reflections $s_i \in W$. The multiplication is determined by Laurent polynomials $N_{u,v}^{w,d} \in R(T)$ such that

$$\mathcal{O}^u \star \mathcal{O}^v = \sum_{w \in W, d \in H_2(X,\mathbb{Z})} N_{u,v}^{w,d} q^d \mathcal{O}^w \tag{1}$$

where the structure coefficient $N_{u,v}^{w,d}$ is nonzero only if $d = \sum_{i=1}^r m_i [X(s_i)] \in H_2(X)$ is effective (i.e. each $m_i \geq 0$), and we mean $\prod_{i=1}^r q_i^{m_i}$ by q^d. The precise definition of $N_{u,v}^{w,d}$ requires taking Euler characteristic of certain K-theory classes on the Kontsevich moduli space of stable maps $\overline{\mathcal{M}}_{0,3}(X, d)$ and over some of its boundary components $\overline{\mathcal{M}}_{\mathbf{d}}(X)$:

$$N_{u,v}^{w,d} = \sum (-1)^j \chi_{\overline{\mathcal{M}}_{\mathbf{d}}(X)}(\mathrm{ev}_1^*(\mathcal{O}^u) \cdot \mathrm{ev}_2^*(\mathcal{O}^v) \cdot \mathrm{ev}_3^*(\xi_w)).$$

In other words, each $N_{u,v}^{w,d}$ is a combination of K-theoretic Gromov-Witten invariants. This is unlike the case of quantum cohomology, where boundary components do not contribute to the structure constants.

3. K-homology of the affine Grassmannian

3.1. ***K*-groups for thick and thin affine Grassmannians.** The foundational reference for the thick affine Grassmannian is [Kas] and for the thin affine Grassmannian we use [Kum02] and [Kum15].

We use notation from Section 1. The (thin) affine Grassmannian Gr is an ind-finite scheme: it is the union of finite-dimensional projective Schubert varieties X_w, for $w \in W_{\mathrm{af}}^-$ (in analogy with the Schubert varieties $X(w)$ for G/B). The dimension of the complex projective variety X_w is equal to the length $\ell(w)$. Let $K_0^T(\mathrm{Gr})$ be the Grothendieck group of the category of T-equivariant finitely-supported (that is, supported on some X_w) coherent sheaves on Gr. We have

$$K_0^T(\mathrm{Gr}) \simeq \bigoplus_{w \in W_{\mathrm{af}}^-} R(T) \cdot \mathcal{O}_w$$

where $\mathcal{O}_w = [\mathcal{O}_{X_w}]_T$ denotes the class of the structure sheaf of X_w (cf. [Kum15, Section 3]). We call the $R(T)$-module $K_0^T(\mathrm{Gr})$ the *(equivariant) K-homology* of Gr. We notice that $\xi_w^{\mathrm{Gr}} := [\mathcal{O}_{X_w}(-\partial X_w)]_T$, $w \in W_{\mathrm{af}}^-$, form another $R(T)$-basis of $K_0^T(\mathrm{Gr})$, which we simply denote as ξ_w whenever it is clear from the context.

The thick affine Grassmannian $\overline{\mathrm{Gr}}$ is an infinite-dimensional non quasicompact scheme: it is a union of finite-codimensional Schubert varieties X^w, for $w \in W_{\mathrm{af}}^-$, of codimension $\ell(w)$. Let $K_T^0(\overline{\mathrm{Gr}})$ be the Grothendieck group of the category of T-equivariant coherent sheaves on $\overline{\mathrm{Gr}}$, defined for example in [LSS, Section 3.2]. We have

$$K_T^0(\overline{\mathrm{Gr}}) \simeq \prod_{w \in W_{\mathrm{af}}^-} R(T) \cdot \mathcal{O}^w$$

where $\mathcal{O}^w = [\mathcal{O}_{X^w}]_T$ denotes the class of the coherent structure sheaf of X^w.

Let Fl denote the (ind-)affine flag manifold, and $\overline{\mathrm{Fl}}$ denote the thick version. As for the affine Grassmannian, one defines Schubert varieties $X_w \subset \mathrm{Fl}$ and $X^w \subset \overline{\mathrm{Fl}}$ such that $\dim X(w) = \operatorname{codim} X^w = \ell(w)$. In this case w varies in the affine Weyl group W_{af}. Let $\mathcal{O}_w = [\mathcal{O}_{X_w}] \in K_0^T(\mathrm{Fl})$ and $\mathcal{O}^w := [\mathcal{O}_{X^w}] \in K_T^0(\overline{Fl})$; we refer to [KaSh] or [Kum15] for the (rather delicate) details. There are T-equivariant projection maps $\pi : \mathrm{Fl} \to \mathrm{Gr}$ and (abusing notation) $\pi : \overline{\mathrm{Fl}} \to \overline{\mathrm{Gr}}$ which are locally trivial G/B-bundles. In particular they are flat, and

$$\pi^* \mathcal{O}_{X^w_{\overline{\mathrm{Gr}}}} = \mathcal{O}_{X^w_{\overline{\mathrm{Fl}}}} \tag{2}$$

for any $w \in W_{\mathrm{af}}^-$. Further, similar arguments to those in the finite case show that for any $w \in W_{\mathrm{af}}$,

$$\pi_* \mathcal{O}_{X_w^{\mathrm{Fl}}} = \mathcal{O}_{X_{\pi(w)}^{\mathrm{Gr}}}, \tag{3}$$

where $\pi(w)$ denotes the image of w in W_{af}^- under the projection map. (See e.g. [BK05, Thm. 3.3.4] for a proof based on Frobenius splitting; or [BCMP13, Prop. 3.2] for an argument based on a theorem of Kollár.) There is a pairing $\langle \cdot, \cdot \rangle_{\mathrm{Fl}} : K_T^0(\overline{\mathrm{Fl}}) \otimes K_0^T(\mathrm{Fl}) \to$

$R(T)$ defined in [Kum15, §3] by

$$\begin{aligned} \langle \cdot, \cdot \rangle_{\mathrm{Fl}} :& K^0_T(\overline{\mathrm{Fl}}) \otimes K_0^T(\mathrm{Fl}) \longrightarrow R(T) \\ & \langle [\mathcal{F}]\,,\,[\mathcal{G}] \rangle_{\mathrm{Fl}} := \sum_i (-1)^i \chi_T((\mathrm{Fl})_n, \mathscr{T}or_i^{\mathcal{O}_{\overline{\mathrm{Fl}}}}(\mathcal{F},\mathcal{G})), \end{aligned} \tag{4}$$

for any classes $[\mathcal{F}], [\mathcal{G}]$ such $\mathcal{F}$ is a T-equivariant sheaf on $\overline{\mathrm{Fl}}$ and $\mathcal{G}$ is a T-equivariant sheaf supported on a finite dimensional stratum $(\mathrm{Fl})_n$ of the ind-variety Fl. By [Kum15, Lemma 3.4] this pairing is well defined. In fact, the definition of this pairing extends in an obvious way to any partial flag variety, in particular to the affine Grassmannian Gr. It was proved in [BK, Prop. 3.9] that the pairing satisfies the property $\langle \mathcal{O}^u_{\overline{\mathrm{Fl}}}, \xi_v^{\mathrm{Fl}} \rangle = \delta_{u,v}$. We will need the following additional properties of this pairing.

Lemma 1. *Consider the pairing $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ and take any $u, v \in W_{\mathrm{af}}$ in the case when $\mathcal{X} = \mathrm{Fl}$ and $u, v \in W_{\mathrm{af}}^-$ for $\mathcal{X} = \mathrm{Gr}$. Then*

$$\langle \mathcal{O}^u, \mathcal{O}_v \rangle_{\mathcal{X}} = \begin{cases} 1 & \textit{if } u \le v; \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. Consider first $\mathcal{X} = \mathrm{Fl}$. By definition we have

$$\langle \mathcal{O}^u, \mathcal{O}_v \rangle_{\mathcal{X}} = \sum (-1)^i \chi_T(X_v, \mathcal{T}or_i^{\mathcal{O}_{\overline{\mathcal{X}}}}(\mathcal{O}_{X^u}, \mathcal{O}_{X_v})).$$

According to [Kum15, Lemma 5.5] all Tor sheaves are 0 for $i > 0$, and by definition $\mathcal{T}or_0^{\mathcal{O}_{\overline{\mathcal{X}}}}(\mathcal{O}_{X^u}, \mathcal{O}_{X_v}) = \mathcal{O}_{X_u^v}$ where $X_v^u := X^u \cap X_v$ is the Richardson variety (cf. e.g. §2 of *loc.cit*). According to [KuSc, Cor. 3.3], the higher cohomology groups $H^i(X_v^u, \mathcal{O}_{X_v^u}) = 0$ for $i > 0$ and since X_v^u is irreducible $H^0(X_v^u, \mathcal{O}_{X_v^u}) = \mathbb{C}$. It follows that the sheaf Euler characteristic $\chi_T(X_v^u, \mathcal{O}_{X_v^u}) = 1$.

We now turn to the situation when $\mathcal{X} = \mathrm{Gr}$. Let $u, v \in W_{\mathrm{af}}^-$. The same argument as before reduces the statement to the calculation of $\chi_T(X_v^u, \mathcal{O}_{X_v^u})$ where $X_v^u \subset \mathrm{Gr}$ is the Richardson variety. By definition of Schubert varieties, $\pi^{-1}(X_v) = X_{vw_0} \subset \mathrm{Fl}$ where $w_0 \in W$ is the longest element in the finite Weyl group, and $\pi^{-1}(X^u) = X^u$. It follows that the preimage of the grassmannian Richardson variety is $\pi^{-1}(X_v^u) = X^u_{vw_0}$. Then a standard argument based on the Leray spectral sequence (taking into account that the fiber of $\pi : \pi^{-1}(X_v^u) \to X_v^u$ is the finite flag manifold G/B, and that $H^i(G/B, \mathcal{O}_{G/B}) = 0$ for $i > 0$) gives that $H^i(X^u_{vw_0}, \mathcal{O}_{X^u_{vw_0}}) = H^i(X_v^u, \mathcal{O}_{X_v^u})$ for all i, thus the required Euler characteristic equals 1, as needed. □

Lemma 2. *For any $u, v \in W_{\mathrm{af}}^-$, we have*

$$\text{(i)} \quad \langle \mathcal{O}^u, \xi_v \rangle_{\mathrm{Gr}} = \delta_{u,v}; \qquad \text{(ii)} \quad \mathcal{O}_v = \sum\nolimits_{w \le v; w \in W_{\mathrm{af}}^-} \xi_w.$$

Proof. (i) The statement follows from the same arguments as for Fl in [BK, Prop. 3.9]. To prove (ii), we write $\mathcal{O}_v = \sum_w a_{w,v} \xi_w$, which is a finite sum because the class $\mathcal{O}_v$ is supported on a finite-dimensional variety. By statement (i) and Lemma 1,

$$a_{w,v} = \langle \mathcal{O}^w, \mathcal{O}_v \rangle_{\mathrm{Gr}} = \begin{cases} 1 & \text{if } w \le v, \\ 0 & \text{otherwise.} \end{cases}$$

□

3.2. K**-Peterson algebra.** The K-groups $K_0^T(\mathrm{Gr})$ and $K^0_T(\overline{\mathrm{Gr}})$ acquire dual Hopf algebra structures from the homotopy equivalence $\mathrm{Gr} \simeq \Omega K$, where $K \subset G$ is a maximal compact subgroup and ΩK is the group of based loops into K. An algebraic model for these Hopf algebras is constructed in [LSS]. Only the product structure of $K_0^T(\mathrm{Gr})$, arising from the Pontryagin product $\Omega K \times \Omega K \to \Omega K$ will be of concern to us. Here we are using the topological K-theory for the Pontryagin product in K-homology classes.

We consider a variation of Kostant and Kumar's K-nilHecke ring, the "small torus" affine K-nilHecke ring of [LSS], which was inspired by the homological analogue [Pet].

The affine Weyl group W_{af} acts on the weight lattice Λ of T by the level-zero action (that is, we take the null root $\delta = 0$)

$$wt_\mu \cdot \lambda = w \cdot \lambda \qquad \text{for } w \in W,\ \mu \in Q^\vee \text{ and } \lambda \in \Lambda.$$

Let I_{af} denote the vertex set of the affine Dynkin diagram. Abusing notation, we denote by $\{\alpha_i \mid i \in I_{\mathrm{af}}\}$ the images of the simple affine roots in Λ. In particular, $\alpha_0 = -\theta \in \Lambda$ where θ is the highest root of G. Let $Q(T) = \mathrm{Frac}(R(T))$ and equip $\mathbb{K}_Q = Q(T) \otimes_{R(T)} \mathbb{Q}[W_{\mathrm{af}}]$ with product $(p \otimes v)(q \otimes w) = p(v \cdot q) \otimes vw$ for $p, q \in Q(T)$ and $v, w \in W_{\mathrm{af}}$. Define

$$T_i = (1 - e^{\alpha_i})^{-1}(s_i - 1) \qquad \text{for } i \in I_{\mathrm{af}}. \tag{5}$$

The T_i satisfy $T_i^2 = -T_i$ and the same braid relations as the s_i do. Therefore for a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_\ell} \in W$ there are well defined elements $T_w = T_{i_1} T_{i_2} \cdots T_{i_\ell}$. Let $\mathbb{K}$ be the subring generated by T_i for $i \in I_{\mathrm{af}}$ and $R(T)$. We call it the *small-torus affine K-nilHecke ring*.

Let $\mathbb{L} \subset \mathbb{K}$ be the centralizer of $R(T)$ in $\mathbb{K}$; this is called the K-Peterson subalgebra. The following theorem clarifies the geometric meaning of [LSS, Theorems 5.3 and 5.4]. Recall that the ideal sheaf basis $\{\xi_w \mid w \in W_{\mathrm{af}}^-\} \in K_0^T(\mathrm{Gr})$ are the unique elements characterized by $\langle \mathcal{O}^v , \xi_w \rangle_{\mathrm{Gr}} = \delta_{vw}$.

Theorem 1. *There is an isomorphism of $R(T)$-Hopf algebras $k : K_0^T(\mathrm{Gr}) \cong \mathbb{L}$ such that for every $w \in W_{\mathrm{af}}^-$*

(a) *the element $k_w := k(\xi_w)$ is the unique element in $\mathbb{L}$ of the form*

$$k_w = T_w + \sum_{x \in W_{\mathrm{af}} \setminus W_{\mathrm{af}}^-} k_w^x T_x \tag{6}$$

where $k_w^x \in R(T)$, and

(b) *the element $l_w := k(\mathcal{O}_w)$ is given by*

$$l_w = \sum_{v \le w} k_w. \tag{7}$$

Proof. In [LSS], a K-homology Hopf algebra $K_0^T(\mathrm{Gr})$ was constructed as a Hopf dual to $K^0_T(\overline{\mathrm{Gr}})$. In [LSS, Theorem 5.3], an isomorphism $K_0^T(\mathrm{Gr}) \simeq \mathbb{L}$ is constructed, and the $R(T)$-bilinear pairing $\langle \cdot\, , \cdot \rangle_{\mathbb{L}} : K^0_T(\overline{\mathrm{Gr}}) \times \mathbb{L}$ is given by

$$\langle \mathcal{O}^w , a \rangle_{\mathbb{L}} = a_w, \tag{8}$$

where $w \in W_{\mathrm{af}}^-$ and $a = \sum_{v \in W_{\mathrm{af}}} a_v T_v \in \mathbb{L} \subset \mathbb{K}$ with $a_v \in R(T)$; see [LSS, §2.4], especially equation (2.10). The uniqueness of the elements k_w given by (6) is [LSS, Theorem 5.4].

We now identify the $\mathbb{L}$ with $K_0^T(\mathrm{Gr})$ via (4) and (8). It follows from [LSS, Theorem 5.4] that under the resulting isomorphism $k : K_0^T(\mathrm{Gr}) \cong \mathbb{L}$, we have $k(\xi_w) = k_w$. Statement (b) follows immediately from Lemma 2. □

3.3. **Closed formula for structure constants.** For $x, y, z \in W_{\mathrm{af}}$, define the structure constants $c_{x,y}^z$ by

$$\mathcal{O}_x \cdot \mathcal{O}_y = \sum_{z \in W_{\mathrm{af}}} c_{x,y}^z \mathcal{O}_z$$

with the product structure given by the isomorphism of Theorem 1. We now give a closed formula for $c_{x,y}^z$ in terms of equivariant localizations.

Define the elements $y_i = 1 + T_i$ for $i \in I_{\mathrm{af}}$. Then $y_i^2 = y_i$ and the y_i satisfy the braid relations so that for $w \in W_{\mathrm{af}}$ we can define $y_w \in \mathbb{K}$. The $\{y_w \mid w \in W_{\mathrm{af}}\}$ form a $R(T)$-basis of $\mathbb{K}$. For any $q \in Q(T)$, we have $q y_{s_i} = y_{s_i}(s_i q) + \frac{q - s_i q}{1 - e^{-\alpha_i}} y_{\mathrm{id}}$. Define $b_{w,u} \in Q(T)$ and $e_{w,u} \in Q(T)$ respectively by[1]

$$y_w = \sum_{u \in W_{\mathrm{af}}} b_{w,u} u, \qquad w = \sum_{u \in W_{\mathrm{af}}} e_{w,u} y_u. \tag{9}$$

In particular, we notice that $b_{\mathrm{id},u} = \delta_{\mathrm{id},u} = e_{\mathrm{id},u}$ for any u. The matrix $\big(b_{w,u}\big)$ is invertible, and its inverse is given by $\big(e_{w,u}\big)$.

Proposition 1. *Let $u, v \in W_{\mathrm{af}}$ with $u \neq \mathrm{id}$. Let $u = s_{\beta_1} \cdots s_{\beta_m}$ be a reduced expression of u. We have*

$$b_{u,v} = \sum \prod_{k=1}^m s_{\beta_1}^{\varepsilon_1} \cdots s_{\beta_{k-1}}^{\varepsilon_{k-1}} \Big(\frac{(-e^{-\beta_k})^{\varepsilon_k}}{1 - e^{-\beta_k}} \Big) \Big|_{\alpha_0 = -\theta}, \tag{10}$$

the summation over all $(\varepsilon_1, \cdots, \varepsilon_m) \in \{0,1\}^m$ satisfying $s_{\beta_1}^{\varepsilon_1} \cdots s_{\beta_m}^{\varepsilon_m} = v$.

Denote $\gamma_j = s_{\beta_1} \cdots s_{\beta_{j-1}}(\beta_j)$ for each j. Then we have

$$e_{u,v} = \sum \prod_{k=1}^m \big((1 - \varepsilon_k) e^{\gamma_k} + \varepsilon_k (1 - e^{\gamma_k}) \big) \Big|_{\alpha_0 = -\theta}, \tag{11}$$

the summation over all $(\varepsilon_1, \cdots, \varepsilon_m) \in \{0,1\}^m$ satisfying $y_{s_{\beta_1}}^{\varepsilon_1} \cdots y_{s_{\beta_m}}^{\varepsilon_m} = y_v$.

In the present work, we work in $Q(T)$, where $T \subset G$ is the finite torus. Our proof below holds in $Q(T_{\mathrm{af}})$ where T_{af} denotes the affine torus.

Proof. The formula for $b_{u,v}$ follows immediately from the expression $y_u = y_{s_{\beta_1}} \cdots y_{s_{\beta_k}}$ and the following equality.

$$y_{\beta_i} = 1 + (1 - e^{\beta_i})^{-1}(s_{\beta_i} - 1) = \frac{-e^{\beta_i}}{1 - e^{\beta_i}} + \frac{1}{1 - e^{\beta_i}} s_{\beta_i} = \frac{1}{1 - e^{-\beta_i}} + \frac{-e^{-\beta_i}}{1 - e^{-\beta_i}} s_{\beta_i}.$$

The formula for $e_{u,v}$ holds by showing that both sides satisfy the same recursive formulas. Precisely, let $\tilde{e}_{u,v}$ denote the RHS of (11). We shall show that $e_{u,v}$ and $\tilde{e}_{u,v}$

[1] In the notation of [KK90], they are denoted as $b_{u^{-1},v^{-1}}$ and $e^{v^{-1},u^{-1}}$ respectively.

satisfy the same recursions. We have

$$\begin{aligned}
s_iu &= (e^{\alpha_i}y_{\mathrm{id}} + (1-e^{\alpha_i})y_{s_i})\sum_v e_{u,v}y_v\\
&= \sum_v e^{\alpha_i}e_{u,v}y_v + \sum_v (1-e^{\alpha_i})y_{s_i}e_{u,v}y_v\\
&= \sum_v e^{\alpha_i}e_{u,v}y_v + \sum_v (1-e^{\alpha_i})(s_i(e_{u,v})y_{s_i} - \frac{s_i(e_{u,v})-e_{u,v}}{1-e^{-\alpha_i}}y_{\mathrm{id}})y_v\\
&= \sum_v e^{\alpha_i}e_{u,v}y_v + \sum_v (1-e^{\alpha_i})s_i(e_{u,v})y_{s_i}y_v + \sum_v e^{\alpha_i}(s_i(e_{u,v})-e_{u,v})y_v\\
&= \sum_{v:s_iv<v}\big(e^{\alpha_i}e_{u,v} + (1-e^{\alpha_i})s_i(e_{u,v}) + (1-e^{\alpha_i})s_i(e_{u,s_iv}) + e^{\alpha_i}(s_i(e_{u,v})-e_{u,v})\big)y_v\\
&\quad + \sum_{v:s_iv>v}\big(e^{\alpha_i}e_{u,v} + e^{\alpha_i}(s_i(e_{u,v})-e_{u,v})\big)y_v\\
&= \sum_{v:s_iv<v}\big(s_i(e_{u,v}) + (1-e^{\alpha_i})s_i(e_{u,s_iv})\big)y_v + \sum_{v:s_iv>v} e^{\alpha_i}(s_i(e_{u,v})y_v).
\end{aligned}$$

That is, we have

$$e_{s_iu,v} = \begin{cases} s_i(e_{u,v}) + (1-e^{\alpha_i})s_i(e_{u,s_iv}), & \text{if } s_iv<v,\\ e^{\alpha_i}s_i(e_{u,v}), & \text{if } s_iv>v. \end{cases} \tag{12}$$

It remains to show that $\tilde{e}_{s_iu,v}$ satisfies the same recursive rule and using induction on the length of u.

Indeed for any v, we notice $e_{\mathrm{id},v} = \delta_{\mathrm{id},v}$ by definition and set $\tilde{e}_{\mathrm{id},v} = \delta_{\mathrm{id},v}$ for convention. Assume that for any $u \in W_{\mathrm{af}}$ of length m, $e_{u,v} = \tilde{e}_{u,v}$ holds for any $v \in W_{\mathrm{af}}$. Then any element in W_{af} of length $m+1$ must be of the form s_iu with $u = s_{\beta_1}\cdots s_{\beta_m}$ being of length m and $s_iu > u$. In order to compute $\tilde{e}_{s_iu,v}$, we need to analyze those $\{\epsilon, \varepsilon_1, \cdots, \varepsilon_m\} \subset \{0,1\}$ satisfying $y_{s_i}^{\epsilon}y_{s_{\beta_1}}^{\varepsilon_1}\cdots y_{s_{\beta_m}}^{\varepsilon_m} = y_v$. If $s_iv > v$, then we have $\epsilon = 0$ and hence $\tilde{e}_{s_iu,v} = e^{\alpha_i}\sum\prod_{k=1}^m\big((1-\varepsilon_k)e^{\tilde{\gamma}_k} + \varepsilon_k(1-e^{\tilde{\gamma}_k})\big) = e^{\alpha_i}s_i(\tilde{e}_{u,v})$, where the second equality holds because $\tilde{\gamma}_k = s_is_{\beta_1}\cdots s_{\beta_{k-1}}(\beta_k) = s_i(\gamma_k)$. If $s_iv < v$, then $y_v = y_{s_i}y_v = y_{s_i}y_{s_iv}$. The summation for $\tilde{e}_{s_iu,v}$ is taken over the following three parts:

i) $\epsilon = 1, y_{s_{\beta_1}}^{\varepsilon_1}\cdots y_{s_{\beta_m}}^{\varepsilon_m} = y_v$; ii) $\epsilon = 1, y_{s_{\beta_1}}^{\varepsilon_1}\cdots y_{s_{\beta_m}}^{\varepsilon_m} = y_{s_iv}$; iii) $\epsilon = 0, y_{s_{\beta_1}}^{\varepsilon_1}\cdots y_{s_{\beta_m}}^{\varepsilon_m} = y_v$.

The corresponding contributions are equal to $(1-e^{\alpha_i})s_i(\tilde{e}_{u,v})$, $(1-e^{\alpha_i})s_i(\tilde{e}_{u,s_iv})$ and $e^{\alpha_i}s_i(\tilde{e}_{u,v})$ respectively. Hence, $\tilde{e}_{s_iu,v} = s_i(\tilde{e}_{u,v}) + (1-e^{\alpha_i})s_i(\tilde{e}_{u,s_iv})$ if $s_iv < v$. By the induction hypothesis, $e_{u,v} = \tilde{e}_{u,v}$. It follows that $e_{s_iu,v} = \tilde{e}_{s_iu,v}$ in either cases. □

Remark 1. *The above expressions of $b_{u,v}$ and $e_{u,v}$ depend on reduced expressions of u. Nevertheless, it follows from the definition of $b_{u,v}$ and $e_{u,v}$ that the summations depend only on u and v.*

Consider the left $Q(T)$-module homomorphism $\kappa : Q(T)\otimes_{R(T)}\mathbb{K} \to Q(T)\otimes_{R(T)}\mathbb{L}$ defined by

$$\kappa(t_\lambda w) = t_\lambda \qquad \text{for } w \in W \text{ and } \lambda \in Q^\vee.$$

Proposition 2. *The map κ restricts to a $R(T)$-module map $\kappa : \mathbb{K} \to \mathbb{L}$, and*

$$\begin{aligned} \kappa(T_u) &= 0 && \text{if } u \in W_{\mathrm{af}} \backslash W_{\mathrm{af}}^-. \\ \kappa(T_u) &= k_u && \text{if } u \in W_{\mathrm{af}}^-. \\ \kappa(y_u) &= l_u && \text{if } u \in W_{\mathrm{af}}^-. \end{aligned}$$

Proof. The first claim follows from the three formulas. From the definition, $\kappa(T_i) = 0$ for $i \neq 0$. It follows easily that $\kappa(T_u) = 0$ if $u \notin W_{\mathrm{af}}^-$. By [LSS, (5.1)], the element $k_u \in \mathbb{L}$ can be characterized as follows. Let $T_u = \sum_{x \in W_{\mathrm{af}}} a_x\, x$ for $a_x \in Q(T)$ and $k_u = \sum_{\lambda \in Q^\vee} a'_{t_\lambda}\, t_\lambda$ for $a'_{t_\lambda} \in Q(T)$. Then for any function $f : W_{\mathrm{af}} \to R(T)$ satisfying $f(x) = f(xv)$ for $v \in W$, we have

$$\sum_{x \in W_{\mathrm{af}}} a_x f(x) = \sum_{\lambda \in Q^\vee} a'_{t_\lambda} f(t_\lambda).$$

It follows that $\kappa(T_u) = k_u$. The last claim follows from the first two formulas, the equality $y_w = \sum_{v \in W_{\mathrm{af}},\, v \le w} T_v$ and (7). □

Denote

$$b_{x,[y]} := \sum_{z \in yW} b_{x,z}, \qquad e_{x,[y]} := \sum_{z \in yW} e_{x,z}.$$

Theorem 2. *For any $x, y, z \in W_{\mathrm{af}}^-$, the coefficient $c_{x,y}^z$ is given by*

$$c_{x,y}^z = \sum_{t_1, t_2 \in Q^\vee} b_{x,[t_1]} b_{y,[t_2]} e_{t_1 t_2,[z]}. \tag{13}$$

Proof. By Theorem 1, we have $l_x l_y = \sum_{z \in W_{\mathrm{af}}^-} c_{x,y}^z l_z$. By Proposition 2, we have

$$\begin{aligned} l_x l_y &= \sum_{u,v \in W_{\mathrm{af}}} \kappa(b_{x,u} u) \kappa(b_{y,v} v) \\ &= \sum_{t_1,t_2 \in Q^\vee} \sum_{u,v \in W} b_{x,t_1 u} b_{y,t_2 v} \kappa(t_1 u) \kappa(t_2 v) \\ &= \sum_{t_1,t_2 \in Q^\vee} \sum_{u,v \in W} b_{x,t_1 u} b_{y,t_2 v} t_1 t_2 \\ &= \sum_{z \in W_{\mathrm{af}}^-} \sum_{t_1,t_2 \in Q^\vee} b_{x,[t_1]} b_{y,[t_2]} e_{t_1 t_2,[z]} l_z. \end{aligned}$$

□

3.4. **Geometric remarks.** We will provide a brief geometric interpretation of the previous approach. There is an $R(T)$-module identification $\mathbb{K} = K_0^T(\mathrm{Fl})$ and an $R(T)$-Hopf algebra identification $\mathbb{L} = K_0^T(\mathrm{Gr})$. The classes $y_w \in \mathbb{K}$ play two roles: on one side $y_w = \mathcal{O}_w$ are the structure (finite dimension) Schubert structure sheaves on the affine flag manifold Fl; on the other side they *act* on $K_T^0(\mathrm{Fl})$ as the K-theoretic BGG operators ∂_w - see [LSS, Lemma 2.2]. Similarly, the elements $T_w \in \mathbb{K}$ correspond to the ideal sheaves ξ_w on $K_0(\mathrm{Fl})$, or to the BGG-type operators $\partial_w - id$. The map $\kappa : \mathbb{K} \to \mathbb{L}$ is the K-theoretic projection map $\pi_* : K_0^T(\mathrm{Fl}) \to K_0^T(\mathrm{Gr})$, and the classes k_w and l_w (for

$w \in W_{\mathrm{af}}^-$) correspond respectively to the ideal sheaves and Schubert structure sheaves in the affine Grassmannian. In particular, Proposition 2 states that

$$\pi_*(\xi_w^{\mathrm{Fl}}) = \begin{cases} \xi_w & \text{if } w \in W_{\mathrm{af}}^- \\ 0 & \text{otherwise} \end{cases}; \qquad \pi_*(\mathcal{O}_u^{\mathrm{Fl}}) = \mathcal{O}_u \ \text{ for } \ u \in W_{\mathrm{af}}^-.$$

It is not difficult to prove these identities directly, using Lemma 2 and identities (2), (3).

For each of $K_0^T(\mathrm{Fl})$ and $K_0^T(\mathrm{Gr})$, there is a third basis $\{\iota_w\}$, indexed respectively by W_{af} and by W_{af}/W, called the *localization basis*. If $w \in W_{\mathrm{af}}$ then $\iota_w \in K_0^T(\mathrm{Fl})$ is the map $\iota_w : K_T^0(\mathrm{Fl}) \to R(T)$ defined by sending the K-cohomology class $\mathcal{O}^u$ to its localization to the fixed point w. Then equation (9) above corresponds to expanding the structure sheaf basis into localization basis and viceversa. A key observation from [LL] and [LSS], which is used in the proof of Theorem 2, is that the Pontryagin multiplication on $K_0^T(\mathrm{Gr})$ is easy to write in the localization basis: if $\lambda, \mu \in Q^\vee$ and $\iota_{t_\lambda}, \iota_{t_\mu} \in K_0^T(\mathrm{Gr})$ are the corrsponding localization elements, then $\iota_{t_\lambda} \cdot \iota_{t_\mu} = \iota_{t_{\lambda+\mu}}$; see [LSS, Lemma 5.1].

4. Data and Evidence

As we observed above, the cohomological versions of Conjectures 1 and 2 were proved in [LS]. In the K-theoretic version, we can verify Conjecture 1 when the degree d in $N_{u,v}^{w,d}$ is $d = 0$ or $d = \alpha_i^\vee$ is a simple coroot. Our arguments are similar to those in [LL], but are quite involved, even in these situations. It would be desirable to find more conceptual explanations. Next we provide two computational examples.

4.1. **Conjecture is true for $G = SL_2$.** The complete flag manifold SL_2/B is the complex projective line $\mathbb{P}^1$. The Weyl group $W = \mathbb{Z}_2$ is generated by the simple reflection $s_1 = s_\alpha$ of the unique simple root $\alpha = \alpha_1$. The equivariant quantum K-theory $QK_T(\mathbb{P}^1)$ has an $R(T)[q]$-basis $\{\mathcal{O}^{\mathrm{id}}, \mathcal{O}^{s_1}\}$. As shown in [BM11], the only nontrivial quantum product is given by[2]

$$\mathcal{O}^{s_1} \star \mathcal{O}^{s_1} = (1 - e^{-\alpha})\mathcal{O}^{s_1} + e^{-\alpha} q. \tag{14}$$

On the affine side, we notice that $s_0 = s_1 t_{-\alpha^\vee}$ and that

$$W_{\mathrm{af}}^- = \{\mathrm{id}\} \cup \{w t_{n\alpha^\vee} \mid n \in \mathbb{Z}_{<0}, w = \mathrm{id} \text{ or } s_1\}.$$

Let g_m be the unique element of W_{af}^- of length m for $m \geq 0$. Let h_m be the unique element of $W_{\mathrm{af}} \setminus W_{\mathrm{af}}^-$ of length m for $m \geq 1$. For example, $g_3 = s_0 s_1 s_0$ and $h_4 = s_0 s_1 s_0 s_1$. Notice that $T_i f = s_i(f) T_i + T_i(f)$ and $T_i^2 = -T_i$ for any $f \in R(T)$ and $i \in \{0, 1\}$.

Lemma 3. *We have $k_{\mathrm{id}} = 1$. For $r \geq 1$, we have*

$$k_{g_{2r-1}} = T_{g_{2r-1}} + T_{h_{2r-1}} + (1 - e^{-\alpha}) T_{h_{2r}} \qquad \textit{and} \qquad k_{g_{2r}} = T_{g_{2r}} + e^{-\alpha} T_{h_{2r}}.$$

Proof. Denote by $\tilde{k}_{g_m}$ the expected formula. By Theorem 1(a), it suffices to show $\tilde{k}_{g_m} \in \mathbb{L}$, or equivalently, $\tilde{k}_{g_m} e^{-\alpha} = e^{-\alpha} \tilde{k}_{g_m}$. Clearly, this holds when $m = 0$. It also holds for $m \in \{1, 2\}$ by direct calculations. We simply denote $T_{01} := T_0 T_1$ and $T_{10} := T_1 T_0$. Then

$$(T_0 + T_1 + (1 - e^{-\alpha}) T_{01}) e^{\pm\alpha} = e^{\pm\alpha} (T_0 + T_1 + (1 - e^{-\alpha}) T_{01});$$
$$(T_{10} + e^{-\alpha} T_{01}) e^{\pm\alpha} = e^{\pm\alpha} (T_{10} + e^{-\alpha} T_{01}).$$

[2] We use the opposite identification $e^{\varepsilon_i} = -[\mathbb{C}_{\varepsilon_i}] \in R(T)$ compared with [BM11, Section 5.5].

Assume that it holds for $m \leq 2r$ where $r \geq 1$. Then we have

$$\begin{aligned}
\tilde{k}_{g_{2r+1}} e^{-\alpha} &= \big(T_{g_{2r+1}} + T_1 T_{h_{2r}} + (1-e^{-\alpha})T_{01}T_{h_{2r}}\big)e^{-\alpha} \\
&= T_{g_{2r+1}} e^{-\alpha} + (T_1 + (1-e^{-\alpha})T_{01})e^{\alpha}e^{-\alpha}T_{h_{2r}}e^{-\alpha} \\
&= T_{g_{2r+1}} e^{-\alpha} + \big(e^{\alpha}(T_0 + T_1 + (1-e^{-\alpha})T_{01}) - T_0 e^{\alpha}\big)\big(e^{-\alpha}(T_{g_{2r}} + e^{-\alpha}T_{h_{2r}}) - T_{g_{2r}}e^{-\alpha}\big) \\
&= T_{g_{2r+1}} e^{-\alpha} + \big(T_0 + T_1 + (1-e^{-\alpha})T_{01}\big)T_{g_{2r}} + e^{-\alpha}\big(T_0 + T_1 + (1-e^{-\alpha})T_{01}\big)T_{h_{2r}} \\
&\qquad - T_0(T_{g_{2r}} + e^{-\alpha}T_{h_{2r}}) + T_0 e^{\alpha}T_{g_{2r}}e^{-\alpha} - e^{\alpha}\big(T_0 + T_1 + (1-e^{-\alpha})T_{01}\big)T_{g_{2r}}e^{-\alpha} \\
&= e^{-\alpha}\tilde{k}_{g_{2r+1}} + T_1 T_{g_{2r}} + e^{-\alpha}T_0 T_{h_{2r}} - T_0 T_{h_{2r}} e^{-\alpha} - e^{\alpha}T_1 T_{g_{2r}} e^{-\alpha} \\
&= e^{-\alpha}\tilde{k}_{g_{2r+1}} - (T_{g_{2r}} + e^{-\alpha}T_{h_{2r}})e^{\alpha}e^{-\alpha} + T_{h_{2r}}e^{-\alpha} + e^{\alpha}T_{g_{2r}}e^{-\alpha} \\
&= e^{-\alpha}\tilde{k}_{g_{2r+1}}
\end{aligned}$$

Similarly, we can show $\tilde{k}_{g_{2r+2}} e^{-\alpha} = e^{-\alpha}\tilde{k}_{g_{2r+2}}$. Thus the statement follows. □

The following result follows from Lemma 3 and Theorem 1(b).

Lemma 4. *We have $l_{\mathrm{id}} = 1$. For $r \geq 1$, we have*

$$\ell_{g_{2r-1}} = (1-e^{-\alpha})T_{h_{2r}} + \sum_{\substack{v \in W_{\mathrm{af}} \\ \ell(v) \leq 2r-1}} T_v \qquad \text{and} \qquad \ell_{g_{2r}} = \sum_{\substack{v \in W_{\mathrm{af}} \\ \ell(v) \leq 2r}} T_v.$$

Proposition 3. *For $x \in W_{\mathrm{af}}^-$ and $n \in \mathbb{Z}_{<0}$, we have in $K_0^T(\mathrm{Gr}_{\mathrm{SL}_2})$*

$$\mathcal{O}_x \cdot \mathcal{O}_{t_{n\alpha^\vee}} = \mathcal{O}_{x t_{n\alpha^\vee}}.$$

Proof. It suffices to prove the statement for $n = -1$. Notice that $t_{-\alpha^\vee} = s_1 s_0 = g_2$ and $x = g_m$ for some $m \in \mathbb{Z}_{\geq 0}$. By Theorem 1, we just need to show $l_{g_m} l_{g_2} = l_{g_{m+2}}$. This follows from Lemma 4. □

Thanks to the above formula, it remains to compute $\mathcal{O}_{s_1 t_{-\alpha^\vee}} \cdot \mathcal{O}_{s_1 t_{-\alpha^\vee}}$. For $x = s_1 t_{-\alpha^\vee} = s_0 = g_1$, by direct calculations we have $l_{g_1}^2 = e^{-\alpha} l_{g_2} + (1-e^{-\alpha}) l_{g_3}$. Therefore

$$\mathcal{O}_{s_1 t_{-\alpha^\vee}} \cdot \mathcal{O}_{s_1 t_{-\alpha^\vee}} = (1-e^{-\alpha})\mathcal{O}_{s_1 t_{-2\alpha^\vee}} + e^{-\alpha}\mathcal{O}_{t_{-\alpha^\vee}}. \tag{15}$$

Remark 2. *We can also calculate the above product by using Theorem 2. For instance, for $z = s_1 s_0 = t_{-\alpha^\vee}$, all the terms in the formula* (13) *for $c_{x,x}^z$ vanish unless $t_1 = t_2 = t_{\alpha^\vee}$. Therefore*

$$c_{x,x}^z = b_{s_0,[t_{\alpha^\vee}]}^2 e_{t_{2\alpha^\vee},[s_1 s_0]} = \Big(\frac{-e^{\alpha}}{1-e^{\alpha}}\Big)^2 e^{-\alpha}(1-e^{-\alpha})^2 = e^{-\alpha}.$$

Formulas (14) and (15), together with Proposition 3, implies that Conjectures 1 and 2 hold when $G = \mathrm{SL}_2$.

4.2. **Multiplication for** Gr_{SL_3}**.** The complete flag manifold $\mathrm{SL}_3/B = \mathrm{Fl}_3 = \{V_1 \subset V_2 \subset \mathbb{C}^3 \mid \dim V_1 = 1, \dim V_2 = 2\}$ parameterizes complete flags in $\mathbb{C}^3$. The Weyl group W is the permutation group S_3 of three objects generated by simple reflections s_1, s_2. We have the highest root $\theta = \alpha_1 + \alpha_2$ and coroot $\theta^\vee = \alpha_1^\vee + \alpha_2^\vee$. By calculations using Theorem 2, we obtain $\mathcal{O}_{wt_{-\theta^\vee}} \mathcal{O}_{t_{-\theta^\vee}} = \mathcal{O}_{wt_{-2\theta^\vee}}$ in $K_0^T(\mathrm{Gr}_{\mathrm{SL}_3})$ for any $w \in W$, in addition to the following multiplication table.

$$
\begin{aligned}
\mathcal{O}_{s_1t_{-\theta^\vee}}\mathcal{O}_{s_1t_{-\theta^\vee}} &= (1-e^{-\alpha_1})\mathcal{O}_{s_1t_{-2\theta^\vee}} + e^{-\alpha_1}\mathcal{O}_{s_2s_1t_{-2\theta^\vee}} + e^{-\alpha_1}\mathcal{O}_{t_{-2\theta^\vee+\alpha_1^\vee}} - e^{-\alpha_1}\mathcal{O}_{s_2t_{-2\theta^\vee+\alpha_1^\vee}}\\
\mathcal{O}_{s_1t_{-\theta^\vee}}\mathcal{O}_{s_2t_{-\theta^\vee}} &= \mathcal{O}_{s_1s_2t_{-2\theta^\vee}} + \mathcal{O}_{s_2s_1t_{-2\theta^\vee}} - \mathcal{O}_{s_1s_2s_1t_{-2\theta^\vee}}\\
\mathcal{O}_{s_1t_{-\theta^\vee}}\mathcal{O}_{s_1s_2t_{-\theta^\vee}} &= (1-e^{-\alpha_1})\mathcal{O}_{s_1s_2t_{-2\theta^\vee}} + e^{-\alpha_1}\mathcal{O}_{s_1s_2s_1t_{-2\theta^\vee}}\\
\mathcal{O}_{s_1t_{-\theta^\vee}}\mathcal{O}_{s_2s_1t_{-\theta^\vee}} &= (1-e^{-\alpha_1-\alpha_2})\mathcal{O}_{s_2s_1t_{-2\theta^\vee}} + e^{-\alpha_1-\alpha_2}\mathcal{O}_{s_2t_{-2\theta^\vee+\alpha_1^\vee}}\\
\mathcal{O}_{s_1t_{-\theta^\vee}}\mathcal{O}_{s_1s_2s_1t_{-\theta^\vee}} &= (1-e^{-\alpha_1-\alpha_2})\mathcal{O}_{s_1s_2s_1t_{-2\theta^\vee}} + e^{-\alpha_1-\alpha_2}\mathcal{O}_{s_1s_2t_{-2\theta^\vee+\alpha_1^\vee}}\\
&\quad + e^{-\alpha_1-\alpha_2}\mathcal{O}_{t_{-2\theta^\vee+\alpha_1^\vee+\alpha_2^\vee}} - e^{-\alpha_1-\alpha_2}\mathcal{O}_{s_1t_{-2\theta^\vee+\alpha_1^\vee+\alpha_2^\vee}}\\
\mathcal{O}_{s_1s_2t_{-\theta^\vee}}\mathcal{O}_{s_1s_2t_{-\theta^\vee}} &= (1-e^{-\alpha_1})(1-e^{-\alpha_1-\alpha_2})\mathcal{O}_{s_1s_2t_{-2\theta^\vee}} + e^{-\alpha_1}\mathcal{O}_{s_2s_1t_{-2\theta^\vee+\alpha_2^\vee}}\\
&\quad + (1-e^{-\alpha_1})e^{-\alpha_1-\alpha_2}\mathcal{O}_{s_1t_{-2\theta^\vee+\alpha_2^\vee}}\\
\mathcal{O}_{s_1s_2t_{-\theta^\vee}}\mathcal{O}_{s_2s_1t_{-\theta^\vee}} &= (1-e^{-\alpha_1-\alpha_2})\mathcal{O}_{s_1s_2s_1t_{-2\theta^\vee}} + e^{-\alpha_1-\alpha_2}\mathcal{O}_{t_{-2\theta^\vee+\alpha_1^\vee+\alpha_2^\vee}}\\
\mathcal{O}_{s_1s_2t_{-\theta^\vee}}\mathcal{O}_{s_1s_2s_1t_{-\theta^\vee}} &= (1-e^{-\alpha_1})(1-e^{-\alpha_1-\alpha_2})\mathcal{O}_{s_1s_2s_1t_{-2\theta^\vee}} + (1-e^{-\alpha_1-\alpha_2})e^{-\alpha_1}\mathcal{O}_{s_2s_1t_{-2\theta^\vee+\alpha_2^\vee}}\\
&\quad + (1-e^{-\alpha_1})e^{-\alpha_1-\alpha_2}\mathcal{O}_{t_{-2\theta^\vee+\alpha_1^\vee+\alpha_2^\vee}} + e^{-2\alpha_1-\alpha_2}\mathcal{O}_{s_2t_{-2\theta^\vee+\alpha_1^\vee+\alpha_2^\vee}}\\
\mathcal{O}_{s_1s_2s_1t_{-\theta^\vee}}\mathcal{O}_{s_1s_2s_1t_{-\theta^\vee}} &= (1-e^{-\alpha_1})(1-e^{-\alpha_2})(1-e^{-\alpha_1-\alpha_2})\mathcal{O}_{s_1s_2s_1t_{-2\theta^\vee}} + e^{-\alpha_1-\alpha_2}\mathcal{O}_{s_1s_2t_{-2\theta^\vee+\alpha_1^\vee+\alpha_2^\vee}}\\
&\quad + (1-e^{-\alpha_1})(1-e^{-\alpha_1-\alpha_2})e^{-\alpha_2}\mathcal{O}_{s_1s_2t_{-2\theta^\vee+\alpha_1^\vee}} + e^{-\alpha_1-\alpha_2}\mathcal{O}_{s_2s_1t_{-2\theta^\vee+\alpha_1^\vee+\alpha_2^\vee}}\\
&\quad + (1-e^{-\alpha_2})(1-e^{-\alpha_1-\alpha_2})\mathcal{O}_{s_2s_1t_{-2\theta^\vee+\alpha_2^\vee}} - e^{-\alpha_1-\alpha_2}\mathcal{O}_{s_1s_2s_1t_{-2\theta^\vee+\alpha_1^\vee+\alpha_2^\vee}}\\
&\quad + e^{-\alpha_1-2\alpha_2}(1-e^{-\alpha_1^\vee})\mathcal{O}_{s_1t_{-2\theta^\vee+\alpha_1^\vee+\alpha_2^\vee}} + e^{-2\alpha_1-\alpha_2}(1-e^{-\alpha_2^\vee})\mathcal{O}_{s_2t_{-2\theta^\vee+\alpha_1^\vee+\alpha_2^\vee}}\\
&\quad + e^{-\alpha_1-\alpha_2}(1-e^{-\alpha_1^\vee})(1-e^{-\alpha_2^\vee})\mathcal{O}_{t_{-2\theta^\vee+\alpha_1^\vee+\alpha_2^\vee}}
\end{aligned}
$$

The remaining products are read off immediately from the above table by the symmetry of the Dynkin diagram of Lie type A_2.

Comparing the above table with the appendix in [LM], we conclude that Conjecture 1 holds whenever the degree d in $N_{u,v}^{w,d}$ is given by $(0,0),(1,0)$ or $(0,1)$.

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