AN ANTICANONICAL PERSPECTIVE ON G/P SCHUBERT VARIETIES

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Abstract. We describe a natural basis of the Cartier class group of an arbitrary Schubert variety $\mathbb{X}_{w,P}$ in a flag variety G/P of general Lie type. We then characterise when the Schubert variety is factorial/Fano, along with an explicit formula for the anticanonical line bundle in these cases. We also prove that, for Schubert varieties in simply-laced types (only), being factorial is equivalent to being $\mathbb{Q}$-factorial, and is equivalent to the equality of the Betti numbers $b_2(\mathbb{X}_{w,P}) = b_{2\ell(w)-2}(\mathbb{X}_{w,P})$. Finally, we give a convenient characterisation of when a simply-laced Schubert variety is Gorenstein and when it is Gorenstein Fano.

Contents

1. Introduction

Let G be a simply-connected simple complex algebraic group together with a choice of upper-triangular Borel subsgroup B and parabolic P with $B \subseteq P$. Our main object of study is the Schubert variety

$$\mathbb{X}_{w,P} := \overline{BwP/P}$$

associated to a Weyl group element w, where we choose w to be a minimal coset representative, $w \in W^P$, for the associated parabolic subgroup of the Weyl group W. The variety $\mathbb{X}_{w,P}$ is an irreducible $\ell(w)$-dimensional subvariety of G/P. Moreover, it is well-known that $\mathbb{X}_{w,P}$ is normal, Cohen-Macaulay and has rational singularities [Ra85, An85, LSe86]. More recently, Woo and Yong characterised when type A Schubert varieties $\mathbb{X}_{w,B}$ satisfy local properties that measure the singularities in [WY08]. They gave a combinatorial characterisation of when a type A Schubert variety is Gorenstein or Fano in [WY06], and posed a problem there and again in [WY23, Problem 8.24]. Namely

Problem A. *Determine which Schubert varieties $\mathbb{X}_{w,P}$ in G/P are ($\mathbb{Q}$-)Gorenstein/($\mathbb{Q}$-)factorial.*

The Schubert variety $\mathbb{X}_{w,P}$ comes with a natural anticanonical divisor. In this paper we describe when this divisor is Cartier, and in that case express its associated line bundle in the Picard group in a natural way. Along the way we provide an answer to Problem A. The answer is particularly nice in the simply-laced case, where we show that $\mathbb{Q}$-Gorenstein is equivalent to

Gorenstein and $\mathbb{Q}$-factorial to factorial. As a result we prove a conjecture of Enomoto [En21, Conjecture A.8] in simply-laced types, but we disprove the conjecture in general.

1.1. We have two natural sets of divisors in $\mathbb{X}_{w,P}$. On the one hand, we have the Schubert divisors $\mathbb{X}_{x,P} \subset \mathbb{X}_{w,P}$, which are given by elements $x \prec w$, namely $x \in W^P$ with $x < w$ and $\ell(x) = \ell(w) - 1$. We recall that the Schubert divisors determine a basis of $H_{2\ell(w)-2}(\mathbb{X}_{w,P}, \mathbb{Z})$. Or equivalently, they form a basis of the (Weil) divisor class group $A_{\ell(w)-1}(\mathbb{X}_{w,P})$. The Chow groups of a Schubert variety agree with the integral homology groups [FMSS95].

On the other hand we have the projected Richardson varieties. Namely, in G/B we have the 'Richardson varieties' of $\mathbb{X}_{w,B}$ associated to elements $v \leq w$ of W by

$$\mathbb{X}_{w,B}^{v} = \overline{B_{-}vB \cap BwB/B}.$$

These are reduced irreducible subvarieties of dimension $\ell(w) - \ell(v)$ by [KL79]. If $v \not\leq w$ the intersection above is empty. Amongst these we have the codimension one Richardson varieties corresponding to simple reflections s_i appearing in w. The G/P analogue is the 'projected Richardson variety', and we obtain divisors in $\mathbb{X}_{w,P}$ of the form

$$\mathbb{X}_{w,P}^{s_i} := \pi_P(\mathbb{X}_{w,B}^{s_i}),$$

where $\pi_P : G/B \to G/P$ is the natural projection. Let us write $I_w^B := \{i \in I \mid s_i < w\}$ for the set indexing the Richardson divisors in $\mathbb{X}_{w,B}$. This set simultaneously indexes also the projected Richardson varieties in $\mathbb{X}_{w,P}$. We have a natural subset $I_w^P := \{k \in I_w^B \mid s_k \in W^P\}$. The role of the divisors $\mathbb{X}_{w,P}^{s_k}$ is elucidated by the following result, which forms the starting point for our paper.

Proposition A. *The projected Richardson divisor $\mathbb{X}_{w,P}^{s_k}$ in $\mathbb{X}_{w,P}$ associated to $k \in I_w^P$ is Cartier, and the Picard group $\mathrm{Pic}(\mathbb{X}_{w,P})$ is freely generated by the line bundles $\mathcal{O}(\mathbb{X}_{w,P}^{s_k})$ with $k \in I_w^P$.*

By [Ma88] the line bundles on $\mathbb{X}_{w,P}$ all arise as restrictions of line bundles on G/P. From this perspective the line bundle $\mathcal{O}(\mathbb{X}_{w,P}^{s_k})$ from the proposition is just isomorphic to the restriction of the line bundle $\mathcal{L}_{\omega_k}$ associated to the fundamental weight ω_k via the Borel-Weil construction. See also [WY06, RW24] for such statements in the G/B and the Grassmannian Schubert variety case, respectively.

Our perspective is now that the first set of divisors, the set of Schubert divisors $\mathbb{X}_{x,P}$, gives a basis of the Weil divisor class group, while the second set of divisors, $\mathbb{X}_{w,P}^{s_k}$, gives a basis of the Cartier divisor class group. Moreover, the former group is naturally isomorphic to $H_{2\ell(w)-2}(\mathbb{X}_{w,P}, \mathbb{Z})$ and the latter group can be identified with $H^2(\mathbb{X}_{w,P}, \mathbb{Z})$. Our approach will be to use the relationship between these two sets of divisors to analyse properties of $\mathbb{X}_{w,P}$ such as the property of factoriality or of being Gorenstein.

1.2. Our first application involving Proposition A is about characterising factorial Schubert varieties. Let b_i denote the i^{th} Betti number, so $b_i(\mathbb{X}_{w,P}) = \dim H_i(\mathbb{X}_{w,P}, \mathbb{Z})$. We recall the work of Carrell, Kuttler and Peterson [Ca94, CK03] which proves that in simply-laced type $\mathbb{X}_{w,B}$ is smooth if and only if its Poincaré polynomial is palindromic, namely $b_i(\mathbb{X}_{w,P}) = b_{2\ell(w)-i}(\mathbb{X}_{w,P})$ for all i. Below we provide a similar characterization for simply-laced Schubert varieties with smoothness replaced by factoriality.

Theorem A. *In simply-laced types, a Schubert variety $\mathbb{X}_{w,P}$ is factorial if and only if $b_2(\mathbb{X}_{w,P}) = b_{2\ell(w)-2}(\mathbb{X}_{w,P})$. Moreover, any $\mathbb{Q}$-Cartier divisor in $\mathbb{X}_{w,P}$ is automatically Cartier.*

Remark 1.1. For arbitrary Schubert varieties $\mathbb{X}_{w,P}$, the condition $b_2(\mathbb{X}_{w,P}) = b_{2\ell(w)-2}(\mathbb{X}_{w,P})$ is equivalent to being $\mathbb{Q}$-factorial. This follows from the fact that Schubert varieties are normal varieties with rational singularities using a general result of Park and Popa [PP24, Theorem A]. From this perspective, our result says that $\mathbb{Q}$-factoriality is equivalent to factoriality for simply-laced Schubert varieties. The second statement of Theorem A also implies that being $\mathbb{Q}$-Gorenstein is equivalent to being Gorenstein for simply-laced Schubert varieties.

Remark 1.2. Our results have further parallels to the results of Carrell, Kuttler and Peterson. Namely, in [Ca94] it is proved that $\mathbb{X}_{w,B}$ having a palindromic Poincaré polynomial is equivalent to it being rationally smooth, which can also be interpreted in terms of the Kazhdan-Lusztig polynomial [KL79] as saying $P_{e,w} = 1$. Then [CK03] shows that in simply-laced type being rationally smooth is equivalent to being smooth. Here in this work we consider $\mathbb{Q}$-factorialty of $\mathbb{X}_{w,P}$, which is equivalent $b_2 = b_{2\ell(w)-2}$ by [PP24], which can further be restated as vanishing of the degree 1 term of $P_{e,w}$ by [BE09, Theorem D]. And we prove that for $\mathbb{X}_{w,P}$ of simply-laced type, being $\mathbb{Q}$-factorial is equivalent to being factorial.

Remark 1.3. For the complete flag variety G/B of type A, the characterisation of factoriality given in Theorem A also follows from an earlier combinatorial characterisation for Schubert varieties to be factorial that is found in work of [BMB07, En21]. Thus in type A it was already known that factoriality is equivalent to $b_2(\mathbb{X}_{w,P}) = b_{2\ell(w)-2}(\mathbb{X}_{w,P})$. Our Theorem A extends this result to all simply-laced types with a new and uniform proof. This verifies for simply-laced Schubert varieties a conjecture made by Enomoto in [En21, Conjecture A.8]. However, we give counterexamples to the conjecture of Enomoto in the non-simply-laced case.

Underlying Theorem A is another result that characterises factoriality for general Schubert varieties. In Definition 3.1 we associate a set of coroots $\check{R}^+_{w,P}$ to $\mathbb{X}_{w,P}$, consisting of positive coroots $\check{\eta}$ such that $ws_{\check{\eta}} \in W^P$ has length $\ell(w) - 1$. We characterise this set $\check{R}^+_{w,P}$ entirely in terms of root system combinatorics. For example, $\check{R}^+_{w,B}$ is the set of positive coroots sent to negative that are *indecomposable*, see Definition 3.2 and Proposition 3.4. For the additional condition to lie in the parabolic subset $\check{R}^+_{w,P}$, see Proposition 3.7. The general criterion for factoriality in these terms is as follows.

Theorem B. *A Schubert variety $\mathbb{X}_{w,P}$ of arbitrary Lie type is factorial if and only if the integer matrix $M = \left(\langle\omega_k, \check{\eta}\rangle\right)_{\check{\eta}\in\check{R}^+_{w,P},\, k\in I^P_w}$ is square and has* $\det(M) = \pm 1$.

Note that the row indexing set $\check{R}^+_{w,P}$ of the matrix M from the theorem is in bijection with the set of Schubert divisors in $\mathbb{X}_{w,P}$, and the column indexing set with the Cartier divisors $\mathbb{X}^{s_k}_{w,P}$, so that the matrix M has dimensions $b_{2\ell(w)-2} \times b_2$. Therefore $\mathbb{Q}$-factoriality is equivalent to the matrix being square, see Remark 1.1. From this perspective, Theorem A says that in simply-laced types, as soon as the matrix is square it will automatically be invertible over $\mathbb{Z}$.

One ingredient in the proof of these theorems is showing that the matrix M from Theorem B always has maximal rank, for any P and $w \in W^P$. This is another application of Proposition A, and we note that as a purely combinatorial statement about root systems it is rather non-trivial. We also note that using Theorem B, it becomes straightforward to determine whether a given Schubert variety is factorial via root system combinatorics.

1.3. The next main object of interest is the anticanonical class, and determining whether individual Schubert varieties are Fano. For this we consider the total sum of *all* the projected Richardson divisors $\mathbb{X}^{s_i}_{w,P}$ and all the Schubert divisors $\mathbb{X}_{x,P}$ in a given Schubert variety. This sum is an anticanonical divisor in $\mathbb{X}_{w,P}$, which was implicit in [Ra87, KLS14] and we prove in

Proposition 6.1 following [LSZ23, Proposition 2.2 and Lemma 3.8]. We write

$$-K_{\mathbb{X}_{w,P}} \quad = \quad \sum_{i\in I_w^B} \mathbb{X}_{w,P}^{s_i} \;+\; \sum_{x\prec w} \mathbb{X}_{x,P}.$$

Let us now consider the setting of *factorial* Schubert varieties $\mathbb{X}_{w,P}$ and characterise which of these have ample anticanonical divisor. Associate to $\mathbb{X}_{w,P}$ the (in this case, square) matrix $M_{w,P} = (\langle \omega_k, \check\eta\rangle)_{\check\eta,k}$ Let $b = b_2(\mathbb{X}_{w,P}) = |I_w^P|$. By ordering rows and columns appropriately we may assume that $M_{w,P} \in SL_b(\mathbb{Z})$, see Theorem B. We have the following result.

Theorem C. *Suppose $\mathbb{X}_{w,P}$ is a factorial Schubert variety of arbitrary Lie type. Let $N_{w,P}$ be the inverse of the matrix $M_{w,P}$ associated to $\mathbb{X}_{w,P}$ above and let $\mathbf{1}$ denote the constant vector $(1)_{\check\eta\in\check R^+_{w,P}}$ and $\mathbf{h} = (h_{\check\eta})_{\check\eta\in\check R^+_{w,P}}$ the vector of heights $h_{\check\eta} = \mathrm{ht}(\check\eta)$. Let $N_{w,P}(\mathbf{h}+\mathbf{1}) = (\hat n_k)_{k\in I_w^P}$. Then the anticanonical class of $\mathbb{X}_{w,P}$ is given by*

$$[-K_{\mathbb{X}_{w,P}}] = \sum_{k\in I_w^P} \hat n_k [\mathbb{X}_{w,P}^{s_k}],$$

in terms of the projected Richardson divisor classes from Proposition A. Moreover, the factorial Schubert variety $\mathbb{X}_{w,P}$ is Fano if and only if the vector $N_{w,P}(\mathbf{h}+\mathbf{1})$ is positive, that is, all $\hat n_k > 0$.

In terms of the Borel-Weil description of line bundles, we have that the anticanonical bundle of $\mathbb{X}_{w,P}$ is the restriction of the line bundle $\mathcal{L}_\lambda$ on G/P where $\lambda = \sum_{k\in I_w^P} \hat n_k\omega_k$.

1.4. Let us now assume $\mathbb{X}_{w,P}$ is a Schubert variety of simply-laced type (with no factoriality assumption). Recall that elements of $H_2(\mathbb{X}_{w,P},\mathbb{Z})$ can be represented by coroots. If $\cap[\mathbb{X}_{w,P}] : H^2(\mathbb{X}_{w,P},\mathbb{Z}) \to H_{2\ell(w)-2}(\mathbb{X}_{w,P},\mathbb{Z})$ is the cap-product map, which here can also be interpreted as the map sending the Chern class of a line bundle to its divisor class, we consider the dual map

$$(\cap[\mathbb{X}_{w,P}])^* : H^{2\ell(w)-2}(\mathbb{X}_{w,P},\mathbb{Z}) \longrightarrow H_2(\mathbb{X}_{w,P},\mathbb{Z}).$$

In simply-laced types we prove that one can choose a subset of Schubert cohomology classes whose image under $(\cap[\mathbb{X}_{w,P}])^*$ forms a (nonstandard) basis of $H_2(\mathbb{X}_{w,P},\mathbb{Z})$. This basis is encoded in a subset $\mathcal{B}_{w,P}$ of coroots inside $\check R^+_{w,P}$ indexed by I_w^P which we construct non-canonically.

In the case of a G/B Schubert variety (still simply-laced), the element $\check\mu_w(i) \in \mathcal{B}_{w,B}$ associated to $i \in I_w^B$ is chosen from amongst positive coroots $\check\mu$ satisfying $w(\check\mu) < 0$ and $\langle\omega_i,\check\mu\rangle > 0$, as indecomposable summand of the *minimal* element in terms of a suitable 'reflection ordering', see Section 5.1. The basis $\mathcal{B}_{w,P}$ for a general parabolic is constructed starting from the subset of $\mathcal{B}_{w,B}$ indexed by I_w^P via a sequence of lemmas that culminate in Definition 5.16. We do not repeat the construction in the introduction. However, the set $\mathcal{B}_{w,P}$ has the remarkable property of agreeing with the subset $\{\check\mu_w(k) \mid k \in I_w^P\}$ of $\mathcal{B}_{w,B}$ modulo $\langle\check\alpha_j \mid j \in I_P\rangle_{\mathbb{Z}}$. This property, in particular, is highly reliant on the simply-laced condition.

Let now $\mathbb{X}_{w,P}$ be a Schubert variety of simply-laced type, and $M_{w,P}$ the $I_w^P \times I_w^P$-matrix

$$M_{w,P} := (\langle\omega_i, \check\mu_w(k)\rangle)_{k,i\in I_w^P},$$

where $\mathcal{B}_{w,P} = \{\check\mu_w(k) \mid k \in I_w^P\}$ is the subset of $\check R^+_{w,P}$ described above. As part of Theorem D below, $M_{w,P}$ constructed in this way will be invertible over $\mathbb{Z}$. We let $N_{w,P}$ denote the inverse matrix to $M_{w,P}$, and consider the vector $N_{w,P}(\mathbf{h}+\mathbf{1}) = (\hat n_k)_{k\in I_w^P}$ in $\mathbb{Z}^{I_w^P}$, where $\mathbf{1} = (1,\ldots,1)$, and $\mathbf{h} = (h_k)_{k\in I_w^P}$ is the vector of heights of the $\check\mu_w(k)$.

Theorem D. *Let $\mathbb{X}_{w,P}$ be a simply-laced Schubert variety. Then the following statements hold.*

(1) *The matrix* $M_{w,P}$ *is invertible over* $\mathbb{Z}$.

(2) *The Schubert variety* $\mathbb{X}_{w,P}$ *is Gorenstein if and only if for every* $\check\eta \in \check R^+_{w,P} \setminus \mathcal{B}_{w,P}$ *we have* $\left\langle \sum_{k\in I^P_w} \hat n_k \omega_k\, ,\, \check\eta \right\rangle - \mathrm{ht}(\check\eta) = 1$.

(3) *Assuming* $\mathbb{X}_{w,P}$ *is Gorenstein, we have*

$$[-K_{\mathbb{X}_{w,P}}] = \sum_{k\in I^P_w} \hat n_k [\mathbb{X}^{s_k}_{w,P}],$$

in terms of the basis of Cartier divisor classes from Proposition A. Moreover, $\mathbb{X}_{w,P}$ *is Gorenstein Fano if and only if* $\hat n_k > 0$ *for all* $k \in I^P_w$.

For the first part of this theorem, that $M_{w,P}$ is invertible over $\mathbb{Z}$, we in fact show that this matrix is a unipotent triangular matrix for some ordering of I^P_w. This result is also a key ingredient in the proof of Theorem A. It is a special feature of the simply-laced case that cannot be extended to the non-simply-laced setting.

Note that the columns of the matrix $N_{w,P}$ may be naturally interpreted as elements of a dual basis to $\mathcal{B}_{w,P}$. Namely, $\check\mu^*_w(k) = \sum_{k'\in I^P_w} n_{k'k}\omega_{k'}$ is the dual basis element to $\check\mu_w(k)$. Furthermore, $\mathcal{B}^*_{w,P} = \{\check\mu^*_w(k) \mid k \in I^P_w\}$ can be interpreted as a basis of $H^2(\mathbb{X}_{w,P}, \mathbb{Z})$. In these terms we have whenever $\mathbb{X}_{w,P}$ is Gorenstein that

$$c_1(\mathbb{X}_{w,P}) = \sum_{k\in I^P_w} (\mathrm{ht}(\check\mu_w(k)) + 1)\, \check\mu^*_w(k), \tag{1.1}$$

and if additionally $P = B$, then this simplifies to

$$c_1(\mathbb{X}_{w,B}) = \sum_{i\in I^B_w} \omega_i + \sum_{i\in I^B_w} \check\mu^*_w(i), \tag{1.2}$$

which, if $I^B_w = I$, is just the ρ-shift of the sum of the $\check\mu^*_w(i)$. Moreover, in the simply-laced G/B Schubert variety case, the condition to be Gorenstein from Theorem D can be reformulated as

$$\mathbb{X}_{w,B} \text{ is Gorenstein} \quad\iff\quad \langle \sum_{i\in I^B_w} \check\mu^*_w(i), \check\eta\rangle = 1 \text{ for all } \check\eta \in \check R^+_{w,B} \setminus \mathcal{B}_{w,B}. \tag{1.3}$$

We finally remark that it was shown in [KWY13] that local properties of Richardson varieties $\mathbb{X}^{u,P}_{w,P}$ can be reduced to that of the Schubert varieties $\mathbb{X}_{w,P}$ and $\mathbb{X}^{u,P}$. Our theorems above may have further applications in the study of the singularities of Richardson varieties.

1.5. The paper is organised as follows. After introducing notations we define the important set $\check R^+_{w,P}$ of positive coroots associated to a Schubert variety $\mathbb{X}_{w,P}$. In Section 3 we first characterise the set $\check R^+_{w,B}$ in terms of root system combinatorics as set of 'indecomposable' elements in $\check R^+(w)$, with special consideration of the simply-laced case. See Proposition 3.4 and Corollary 3.5. These results are extended in Proposition 3.7 to a characterisation of $\check R^+_{w,P}$ for general P. Next, in Section 4, we describe the Picard group $\mathrm{Pic}(\mathbb{X}_{w,P})$, identifying it with a subgroup of the Weil divisor class group. This is where Proposition A is proved. We obtain a characterisation of factorial and $\mathbb{Q}$-factorial Schubert varieties in terms of $\check R^+_{w,P}$ in Section 5. The main results in this section are Proposition 5.2 and Theorem 5.3, which imply both Theorem A and Theorem B. The proof of Theorem 5.3 involves the construction of a subset $\mathcal{B}_{w,P}$ in $\check R^+_{w,P}$ for any simply-laced Schubert variety $\mathbb{X}_{w,P}$. The construction of this subset is carried out in Section 5.1 (Borel case) and Section 5.2 (extension to arbitrary parabolic P). In Section 6 we make use of this set $\mathcal{B}_{w,P}$ to determine when a simply-laced Schubert variety is Gorenstein and in that case compute its first Chern class. In that section we also compute the first Chern class for any factorial Schubert variety in general type, see Proposition 6.7 and Proposition 6.10. We then characterise which of these Schubert varieties are Fano in Corollary 6.12. These propositions prove Theorem D and Theorem C. We provide a wealth of examples in Section 7. We also formulate three conjectures related to Weyl group combinatorics that are stated at the end of Section 3 and Section 5.1.

Acknowledgements. The authors thank Hongsheng Hu, Haidong Liu, Nick Shepherd-Barron, Lauren Williams, David Sheard, Rui Xiong and Alexander Yong for helpful conversations. C. Li is supported in part by the National Key Research and Development Program of China No. 2023YFA1009801 and NSFC Grant 12271529.

2. Notation

In this section, we review some relevant background from Lie theory and algebraic geometry, and refer to [BL00, Bo81, Sp98, BB05, Br05, Ko13, Ku02] for more details.

2.1. Notations in Lie theory. Let G be a simply-connected simple complex algebraic group, B be a Borel subgroup of G, and $P \supseteq B$ be a parabolic subgroup of G. Let B_- denote the Borel subgroup opposite to B, and then $T := B \cap B_-$ is a maximal torus of G with Lie algebra $\mathfrak{t} = \mathrm{Lie}(T)$. Let $\Pi = \{\alpha_i \mid i \in I\} \subset \mathfrak{t}^*$ (resp. $\check{\Pi} = \{\check{\alpha}_i \mid i \in I\} \subset \mathfrak{t}$) denote the set of simple roots (resp. coroots) inside the positive roots R^+ (resp. coroots $\check{R}^+$). Let $\{\omega_i \mid i \in I\} \subset \mathfrak{t}^*$ denote the fundamental weights. Denote $\rho := \sum_{i \in I} \omega_i$.

We express elements of the Weyl group W in terms of the simple reflection generators $W = \langle s_i \mid i \in I \rangle$. We also take a lifting $\dot{s}_i$ in the normalizer $N_G(T)$ of T, so that $s_i \mapsto \dot{s}_i T$ defines a (canonical) isomorphism $W \cong N_G(T)/T$. Associate to P, we have a subset $I_P := \{i \in I \mid \dot{s}_i T \subset P\}$, and denote $I^P := I \setminus I_P$. Let $W_P = \langle s_i \mid i \in I_P \rangle$ denote the Weyl subgroup of W associated to P, and $\ell : W \to \mathbb{Z}_{\geq 0}$ denote the standard length function. Then we have the subset $W^P \subset W$, consisting of minimal-length coset representatives for W/W_P. We note that $w \in W^P$ if and only if $w(\alpha_j) > 0$ for all $j \in I_P$. Let $(W, <)$ denote the Bruhat order on W. Namely for $u, w \in W$, we have $u < w$ if some substring of a reduced expression for w gives a reduced expression for u. We further write $u \prec w$ if w covers u in the Bruhat order, that is, if $u < w$ and $\ell(u) = \ell(w) - 1$.

For $w \in W$, we denote by $\check{R}^+(w)$ the inversion set of coroots, and write out its elements using a reduced expression $w = s_{i_1} \dots s_{i_r}$,

$$\check{R}^+(w) = \{\check{\alpha} \in \check{R}^+ \mid w(\check{\alpha}) < 0\} = \{\check{\alpha}_{i_r}, s_{i_r}(\check{\alpha}_{i_{r-1}}), \dots, s_{i_r} s_{i_{r-1}} \dots s_{i_2}(\check{\alpha}_1)\}.$$

There is a total ordering $<_{\mathbf{i}}$ of $\check{R}^+(w)$ determined by the chosen reduced expression of w, according to which the elements in $\check{R}^+(w)$ as listed above are in order. See [Pa94] for a full characterisation of these orderings. We only note that the ordering $<_{\mathbf{i}}$ has the property that if $\check{\eta}' <_{\mathbf{i}} \check{\eta}'' \in \check{R}^+(w)$ and $r', r'' \in \mathbb{R}_{>0}$ are such that $\check{\eta} = r'\check{\eta}' + r''\check{\eta}''$ lies in $\check{R}^+$, then $\check{\eta} \in \check{R}^+(w)$ and $\check{\eta}' <_{\mathbf{i}} \check{\eta} <_{\mathbf{i}} \check{\eta}''$. An ordering with this property is called a *reflection ordering*, see [Dy93]. We may also write $\check{\eta}^{\mathbf{i}}_{(k)}$ for the k-th element of this list, with $\mathbf{i} = (i_1, \dots, i_r)$ encoding the reduced expression.

2.2. Geometry of flag varieties. The flag variety G/P is a smooth projective variety. Abuse the notation for w with its image $\dot{w}T$ in $N_G(T)/T$. Inside G/P, there are Schubert varieties

$$\mathbb{X}_{w,P} := \overline{BwP/P}, \qquad \mathbb{X}^{v,P} := \overline{B_- vP/P}$$

of dimension $\ell(w)$ and codimension $\ell(v)$ respectively, where $w, v \in W^P$. Their intersections $\mathbb{X}^{v,P}_{w,P} = \mathbb{X}_{w,P} \cap \mathbb{X}^{v,P}$ are called Richardson varieties in G/P. Note that $\mathbb{X}^{v,P}_{w,P}$ is a reduced irreducible subvariety of dimension $\ell(w) - \ell(v)$ if $v \leq w$, or an empty set otherwise. Let $\pi_P : G/B \to G/P$ denote the natural projection. We simply denote

$$\mathbb{X}^v_{w,B} := \mathbb{X}^{v,B}_{w,B}, \qquad \mathbb{X}^v_{w,P} := \pi_P(\mathbb{X}^v_{w,B})$$

the latter of which is called a projected Richardson variety in G/P, distinct from the Richardson variety $\mathbb{X}^{v,P}_{w,P}$ in G/P in general.

Remark 2.1. In $SL_3/P = \mathbb{P}^2$ where $I^P = \{1\}$ the projected Richardson variety $\mathbb{X}^{s_2}_{s_2s_1,P}$ is already not a Richardson variety in $\mathbb{P}^2$, see [Sp23, Example 1.23]. Though it is still a divisor. However, when $k \in I^P_w$, we have

$$\mathbb{X}^{s_k}_{w,P} = \pi(\mathbb{X}^{s_i,B} \cap \mathbb{X}_{w,B}) \subset \pi(\mathbb{X}^{s_k,B}) \cap \pi(\mathbb{X}_{w,B}) = \mathbb{X}^{s_k,P} \cap \mathbb{X}_{w,P}.$$

Both are irreducible and of the same dimension $\ell(w)-1$. Hence, they coincide with each other.

The Chow ring $A^*(G/P)$ is isomorphic to the integral cohomology $H^*(G/P, \mathbb{Z})$, which has a canonical basis of Schubert classes $P.D.[\mathbb{X}^{v,P}] \in H^{2\ell(v)}(G/P, \mathbb{Z})$. In particular, the Picard group $\mathrm{Pic}(G/P) = A^1(G/P)$ is given by $H^2(G/P, \mathbb{Z})$.

2.3. Local properties. In birational geometry, it is fundamental to study the local properties of a complex variety, such as whether it is factorial, Gorenstein, or Cohen-Macaulay among other properties. Schubert varieties are known to be normal and Cohen-Macaulay with rational singularities in complete generality. But they may or may not be Gorenstein or factorial.

Definition 2.2. A normal variety X is called Gorenstein (resp. $\mathbb{Q}$-Gorenstein) if its canonical divisor K_X is Cartier (resp. $\mathbb{Q}$-Cartier).

Definition 2.3. A variety X is called factorial (resp. $\mathbb{Q}$-factorial) if every Weil (resp. $\mathbb{Q}$-Weil) divisor of X is Cartier (resp. $\mathbb{Q}$-Cartier).

We note that there are notions of Gorenstein/factorial defined by properties on the local ring $\mathcal{O}_{X,x}$ (see e.g. [BH93]). To define Gorenstein, X is assumed to be Cohen-Macaulay instead of normal. The two kinds of definitions are equivalent when X is normal and Cohen-Macaulay, in particular when $X = \mathbb{X}_{w,P}$ is a Schubert variety. These properties measure the singularities of X with the following relationships.

smooth $\subset$ factorial $\subset$ Gorenstein $\subset$ normal & Cohen-Macaulay $\subset$ arbitrary singularities

3. The set of coroots $\check{R}^+_{w,P}$

In this section, we define a special set of coroots $\check{R}^+_{w,P}$ associated to $\mathbb{X}_{w,P}$. The heart of this section is Proposition 3.4, which provides a characterisation of the special case $\check{R}^+_{w,B}$, and Proposition 3.7 which adapts the characterisation of $\check{R}^+_{w,B}$ to general $\check{R}^+_{w,P}$. We also prove a key lemma, Lemma 3.6, which will play an important role again in Section 5.2.

Definition 3.1. For $w \in W^P$, we define a subset of $\check{R}^+(w)$ by

$$\check{R}^+_{w,P} \quad := \quad \{\check{\eta} \in \check{R}^+(w) \mid \ell(ws_{\check{\eta}}) = \ell(w) - 1, ws_{\check{\eta}} \in W^P\}. \tag{3.1}$$

Note $W^B = W$. The special case,

$$\check{R}^+_{w,B} \quad = \quad \{\check{\eta} \in \check{R}^+(w) \mid \ell(ws_{\check{\eta}}) = \ell(w) - 1\}, \tag{3.2}$$

is known as the *cover inversion set* of coroots for $w \in W$.

Definition 3.2. We call an element $\check{\eta} \in \check{R}^+(w)$

- *simply decomposable* in $\check{R}^+(w)$, if $\check{\eta} = \check{\mu} + \check{\mu}'$ for some $\check{\mu}, \check{\mu}' \in \check{R}^+(w)$,

- *decomposable* in $\check{R}^+(w)$, if $\check{\eta} = \frac{1}{c}(\check{\mu} + \check{\mu}')$ for two distinct $\check{\mu}, \check{\mu}' \in \check{R}^+(w)$ and $c \in \mathbb{Z}_{>0}$,
- *weakly decomposable* in $\check{R}^+(w)$, if $\check{\eta} = r'\check{\mu} + r''\check{\mu}'$ for two distinct $\check{\mu}, \check{\mu}' \in \check{R}^+(w)$ and $r, r' \in \mathbb{R}_{>0}$.

We say that $\check{\eta} \in \check{R}^+(w)$ is *indecomposable*, if it is not decomposable in $\check{R}^+(w)$.

Lemma 3.3. *Suppose G is simply-laced and let $w \in W$ and $\check{\eta} \in \check{R}^+(w)$. If $\check{\eta} = r\check{\mu} + r'\check{\mu}'$ for some $r, r' \in \mathbb{R}_{>0}$ and distinct $\check{\mu}, \check{\mu}' \in \check{R}^+(w)$, then we have the following.*

(1) *$r = r' = 1$, that is, $\check{\eta} = \check{\mu} + \check{\mu}'$.*
(2) *$\langle \eta, \check{\mu} \rangle = \langle \eta, \check{\mu}' \rangle = 1$ and $\langle \mu, \check{\mu}' \rangle = -1$.*

In particular, the three notions of 'decomposable' from Definition 3.2 are equivalent in simply-laced types.

Proof. We may assume $\check{\mu}' <_{\mathbf{i}} \check{\eta} <_{\mathbf{i}} \check{\mu}$ for the reflection ordering associated to some reduced expression $\mathbf{i}$ of w. We may moreover assume that $\check{\mu}' = \check{\alpha}_i$, a simple coroot. Namely, if not then write $\check{\mu}' = s_{i_r} \dots s_{i_{\ell+1}}(\check{\alpha}_{i_\ell})$, factorise w as $w = w_1 w_2 = (s_{i_1} \dots s_{i_\ell})(s_{i_{\ell+1}} \dots s_{i_r})$, and apply $s_{i_{\ell+1}} \dots s_{i_r}$ to $\check{\eta} = r\check{\mu} + r'\check{\mu}'$. We then obtain an identity of the same type, but now in $\check{R}^+(w_1)$ and with $\check{\mu}'$ replaced by $\check{\alpha}_{i_\ell}$.

We now assume $\check{\mu}' = \check{\alpha}_i$ and $\check{\eta} = r\check{\mu} + r'\check{\alpha}_i$, with $r, r' \in \mathbb{R}_{>0}$. For any positive coroot expanded in terms of the simple coroots $\check{\alpha}_j$ we make the following straightforward observation that holds in the simply-laced case.

($\star$) If a positive coroot is not simple, then at least *two* of its coefficients will be equal to 1.

We make use of this fact while comparing the two sides of the equation,

$$\check{\eta} - r'\check{\alpha}_i = r\check{\mu}. \tag{3.3}$$

The left-hand side of (3.3) has the following properties.

(i) All coefficients are integral except for at most one: the coefficient of $\check{\alpha}_i$.
(ii) At least one coefficient is equal to 1, by ($\star$).

For the right-hand side, observe that if r is not an integer, then at least *two* coefficients of $r\check{\mu}$ are not integers by ($\star$), contradicting (i). Therefore r must be an integer. Now if $r \neq 1$, then since r is positive and an integer we have $r \geq 2$. This implies that every nonzero coefficient of $r\check{\mu}$ is ≥ 2, in contradiction with (ii). Therefore it follows that $r = 1$.

Next consider $r' \in \mathbb{R}_{>0}$. Since $\check{\eta} = \check{\mu} + r'\check{\alpha}_i$ we have that r' is a positive integer and

$$\langle \alpha_i, \check{\eta} \rangle = \langle \alpha_i, \check{\mu} \rangle + 2r'.$$

Since in the simply-laced type the pairings take values in $\{\pm 1, 0, 2\}$, the only solution is $\langle \alpha_i, \check{\eta} \rangle = 1, \langle \alpha_i, \check{\mu} \rangle = -1$ and $r' = 1$. Both statements of the lemma now follow. □

Proposition 3.4. *Fix $w \in W$ and suppose $\check{\eta} \in \check{R}^+(w)$. The following statements are equivalent.*

(1) *$\ell(ws_{\check{\eta}}) = \ell(w) - 1$.*
(2) *$\check{\eta}$ is not decomposable in $\check{R}^+(w)$.*

The set $\check{R}^+_{w,B}$ is the subset of indecomposable elements in $\check{R}^+(w)$.

The following corollary is simply the combination of Proposition 3.4 and Lemma 3.3.

Corollary 3.5. *Suppose G is simply-laced. Then $\check{\eta} \in R^+(w)$ lies in the subset $R^+_{w,B}$ if and only if there is no decomposition $\check{\eta} = \check{\mu} + \check{\mu}'$ with $\check{\mu}, \check{\mu}' \in R^+(w)$.* □

Proof of Proposition 3.4. Recall that $\check{\eta} \in \check{R}^+(w)$ implies $\ell(ws_{\check{\eta}}) < \ell(w)$. We first prove the following statement.

($\star\star$) If $\ell(ws_{\check\eta}) < \ell(w)-1$, then $\check\eta$ must be decomposable in $\check R^+(w)$.

This will prove that (2) $\implies$ (1).

We use induction on $\ell(w)$ to prove ($\star\star$). If $\ell(w)=1$ then the statement is empty. Now we may assume that the statement is proved whenever w has length $\leq r-1$. Let us suppose $\ell(w)=r$. We consider an element $\check\eta \in \check R^+(w)$ for which $\ell(ws_{\check\eta}) \leq \ell(w)-2$. Write $\check\eta = s_{i_r}\dots s_{i_{\ell+1}}(\check\alpha_{i_\ell})$ using a reduced expression $\mathbf{i}=(i_1,\dots,i_r)$ of w. Then $ws_{\check\eta}$ is obtained by removing the s_{i_ℓ} factor from w where $1<\ell<r$, and we may write

$$\hat w := ws_{\check\eta} = (s_{i_1}\dots s_{i_{\ell-1}})(s_{i_{\ell+1}}\dots s_{i_r}), \tag{3.4}$$

keeping in mind, though, that the above is no longer a reduced expression. Let us also consider the element $v := s_{i_1}w$. The element v has the following immediate properties,

$$v = s_{i_2}s_{i_3}\dots s_{i_r}, \qquad \ell(v)=r-1, \qquad \check\eta \in \check R^+(v) = \check R^+(w)\setminus\{s_{i_r}\dots s_{i_2}(\alpha_{i_1})\}. \tag{3.5}$$

By the induction hypothesis, if $\ell(vs_{\check\eta}) \leq \ell(v)-2$ then we must have that $\check\eta$ is decomposable in $\check R^+(v)$. Therefore by (3.5) it is also decomposable in $\check R^+(w)$ and we are done. So we can now assume that $\ell(vs_{\check\eta}) = \ell(v)-1$. Then in summary we have

$$\ell(vs_{\check\eta}) = \ell(v)-1 = r-2, \qquad \ell(ws_{\check\eta}) = \ell(s_{i_1}vs_{\check\eta}) \overset{(\dagger)}{=} \ell(vs_{\check\eta})-1 = r-3. \tag{3.6}$$

Note that $\ell(s_{i_1}vs_{\check\eta}) = \ell(vs_{\check\eta})\pm 1$, and the equality ($\dagger$) above follows from our assumption that $\ell(ws_{\check\eta}) \leq \ell(w)-2$. We now have as reduced expression for $vs_{\check\eta}$ the following,

$$\hat v := vs_{\check\eta} \overset{\text{red}}{=\!=} s_{i_2}s_{i_3}\dots s_{i_{\ell-1}}s_{i_{\ell+1}}\dots s_{i_r}.$$

Additionally, the equality $\ell(s_{i_1}\hat v) = \ell(\hat v)-1$ implies that $\hat v^{-1}(\check\alpha_{i_1}) < 0$. We now consider the following two positive coroots,

$$\begin{aligned} \check\mu &:= \hat w^{-1}(\check\alpha_{i_1}) = \hat v^{-1}s_{i_1}(\alpha_{i_1}) > 0,\\ \check\mu' &:= w^{-1}s_{i_1}(\check\alpha_{i_1}) = s_{i_r}\dots s_{i_2}(\alpha_{i_1}) > 0. \end{aligned} \tag{3.7}$$

Using $\hat w^{-1} = s_{\check\eta}w^{-1}$, we compute their sum as follows,

$$\begin{aligned} \check\mu+\check\mu' &= s_{\check\eta}w^{-1}(\check\alpha_{i_1}) - w^{-1}(\check\alpha_{i_1})\\ &= \left(w^{-1}(\check\alpha_{i_1}) - \langle\eta, w^{-1}(\check\alpha_{i_1})\rangle\,\check\eta\right) - w^{-1}(\check\alpha_{i_1}) = -\langle\eta, w^{-1}(\check\alpha_{i_1})\rangle\,\check\eta. \end{aligned} \tag{3.8}$$

Since $\check\mu, \check\mu'$ and $\check\eta$ are all positive coroots, we must have that $-\langle\eta, w^{-1}(\check\alpha_{i_1})\rangle = c > 0$ (with $c=1$ in the simply-laced case). We now rewrite (3.8) as

$$\check\eta = \frac{1}{c}(\check\mu+\check\mu'). \tag{3.9}$$

Note that $\check\mu'$ lies in $\check R^+(w)$ by the second expression for it given in (3.7). Let us check that also $\check\mu \in \check R^+(w)$. Using again that $\hat w^{-1} = s_{\check\eta}w^{-1}$, we have

$$\begin{aligned} w(\check\mu) &= ws_{\check\eta}w^{-1}(\check\alpha_{i_1}) = s_{w(\check\eta)}(\check\alpha_{i_1}) = \check\alpha_{i_1} - \langle w(\eta), \check\alpha_{i_1}\rangle w(\check\eta)\\ &= \check\alpha_{i_1} - \langle\eta, w^{-1}(\check\alpha_{i_1})\rangle w(\check\eta) = \check\alpha_{i_1} + c\,w(\check\eta). \end{aligned} \tag{3.10}$$

Since $\check\eta \in \check R^+(w)$ we have $w(\check\eta) < 0$, and since $c>0$ it follows that $\check\alpha_{i_1} + c\,w(\check\eta)$ cannot be a positive coroot. Thus we deduce from (3.10) that $w(\check\mu)$ is a negative coroot and therefore $\check\mu \in \check R^+(w)$. Now (3.9) demonstrates that $\check\eta$ is decomposable in $\check R^+(w)$. This completes the proof the statement ($\star\star$).

We have now proved (2) $\implies$ (1). It remains to show (1) $\implies$ (2). Suppose $\ell(ws_{\check\eta}) = \ell(w)-1$. Thus now the expression in (3.4) is a reduced expression for $ws_{\check\eta}$. Let us assume indirectly that $\check\eta = \frac{1}{c}(\check\mu+\check\mu')$ for some $c \in \mathbb{R}_{>0}$ and distinct elements $\check\mu, \check\mu' \in \check R^+(w)$. Then in any reflection ordering $<_{\mathbf{i}}$ of $\check R^+(w)$ we have that $\check\eta$ appears in the middle between the $\check\mu,\check\mu'$. We may assume that

$$\check\mu' <_{\mathbf{i}} \check\eta <_{\mathbf{i}} \check\mu. \tag{3.11}$$

By truncating w at either end we may additionally assume $\check\mu = s_{i_r}\dots s_{i_2}(\check\alpha_{i_1})$ and $\check\mu' = \check\alpha_{i_r}$, compare the proof of Lemma 3.3. Let us write $i = i_r$ for simplicity. We have $\check\eta = \frac{1}{c}(\check\alpha_i + \check\mu)$. Applying s_η to both sides gives

$$-\check\eta = \frac{1}{c}(s_{\check\eta}(\check\alpha_i) + s_{\check\eta}s_{i_r}\dots s_{i_2}(\check\alpha_{i_1})) = \frac{1}{c}\left(s_{\check\eta}(\check\alpha_i) + s_{i_r}\dots s_{i_{\ell+1}}s_{i_{\ell-1}}\dots s_{i_2}(\check\alpha_{i_1})\right).$$

By our assumption that $\ell(ws_{\check\eta}) = \ell(w)-1$ we have that the right-hand summand in the brackets above is an element of $R^+(ws_{\check\eta})$. In particular it is positive. Rearranging, we have

$$c\,\check\eta + s_{i_r}\dots s_{i_{\ell+1}}s_{i_{\ell-1}}\dots s_{i_2}(\check\alpha_{i_1}) = -s_{\check\eta}(\check\alpha_i) = \langle\eta,\check\alpha_i\rangle\check\eta - \check\alpha_i,$$

and therefore

$$\check\alpha_i + s_{i_r}\dots s_{i_{\ell+1}}s_{i_{\ell-1}}\dots s_{i_2}(\check\alpha_{i_1}) = (\langle\eta,\check\alpha_i\rangle - c)\,\check\eta.$$

Thus, since the left-hand side is a sum of positive roots, this would imply that

$$\langle\eta,\check\alpha_i\rangle > c. \tag{3.12}$$

On the other hand, it follows from $\ell(ws_{\check\eta}) = \ell(w)-1$ that $\check\alpha_i \in \check R^+(ws_{\check\eta})$. Therefore

$$ws_{\check\eta}(\check\alpha_i) \;=\; w\left(\check\alpha_i - \frac{1}{c}\langle\eta,\check\alpha_i\rangle(\check\mu+\check\alpha_i)\right) = \left(1-\frac{1}{c}\langle\eta,\check\alpha_i\rangle\right)w(\check\alpha_i) - \frac{1}{c}\langle\eta,\check\alpha_i\rangle w(\check\mu). \tag{3.13}$$

must be negative. But by (3.12) and since $\check\alpha_i,\check\mu \in \check R^+(w)$, we have that both summands on the right-hand side are positive. Thus we have obtained a contradiction. $\square$

Lemma 3.6. *Let $w \in W^P$ and suppose that $\check\eta \in \check R^+(w)$ and $j \in I_P$. Then $ws_{\check\eta}(\check\alpha_j) < 0$ if and only if $s_{\check\eta}(\check\alpha_j) \in \check R^+(w)$. Moreover*

$$ws_{\check\eta} \in W^P \iff s_{\check\eta}(\check\alpha_j) \in \check R \setminus \check R^+(w) \quad \textit{for all } j \in I_P.$$

Proof. If $s_{\check\eta}(\check\alpha_j) \in \check R^+(w)$, then clearly $ws_{\check\eta}(\check\alpha_j) < 0$, by the definition of $\check R^+(w)$.

Now we assume $ws_{\check\eta}(\check\alpha_j) < 0$. If $s_{\check\eta}(\check\alpha_j) < 0$, then $\langle\eta,\check\alpha_j\rangle > 0$, because

$$s_{\check\eta}(\check\alpha_j) = \check\alpha_j - \langle\eta,\check\alpha_j\rangle\check\eta$$

would otherwise describe $s_{\check\eta}(\check\alpha_j)$ as sum of two positive roots. Since $w \in W^P$, $w(\check\alpha_j) > 0$. Since $\check\eta \in R^+(w)$, $w(\check\eta) < 0$. Hence, $ws_{\check\eta}(\check\alpha_j) = w(\check\alpha_j) + (-\langle\eta,\check\alpha_j\rangle w(\check\eta)) > 0$, resulting in a contradiction. Hence, $s_{\check\eta}(\check\alpha_j) > 0$, so that it is in $R^+(w)$. Hence, the first statement follows.

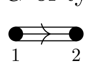
Note that $ws_{\check\eta} \in W^P$ if and only if $ws_{\check\eta}(\check\alpha_j) > 0$ for all $j \in I_P$. The second statement is therefore a direct consequence of the first statement. $\square$

In Proposition 3.4, we have described the set $\check R^+_{w,B}$ directly as natural subset of the set $\check R^+(w)$, without any further reference to w or its reduced expressions. Using Lemma 3.6, we can immediately extend this description to describe the subset $\check R^+_{w,P}$ for general P as follows.

Proposition 3.7. *For any parabolic P and $w \in W^P$ we have*

$\check R^+_{w,P} = \{\check\eta \in \check R^+(w) \mid \check\eta$ *is indecomposable in* $\check R^+(w)$ *and* $s_{\check\eta}(\check\alpha_j) \notin \check R^+(w)$ *whenever* $j \in I_P\}$. $\square$

Example 3.8. Consider G of type G_2, with long root α_1 and short root α_2.



$$\langle\alpha_1,\check\alpha_2\rangle = -3, \quad \langle\alpha_2,\check\alpha_1\rangle = -1.$$

There are two reflection orderings on $R^+ = R^+(w_0)$ corresponding to $w_0 = s_1s_2s_1s_2s_1s_2 = s_2s_1s_2s_1s_2s_1$, or $\mathbf{i} = (1,2,1,2,1,2)$ and $\mathbf{i}' = (2,1,2,1,2,1)$. Namely, these are

$$\check\alpha_2 \;<_{\mathbf{i}}\; \check\alpha_1+\check\alpha_2 \;<_{\mathbf{i}}\; 3\check\alpha_1+2\check\alpha_2 \;<_{\mathbf{i}}\; 2\check\alpha_1+\check\alpha_2 \;<_{\mathbf{i}}\; 3\check\alpha_1+\check\alpha_2 \;<_{\mathbf{i}}\; \check\alpha_1$$

and its reverse,

$$\check\alpha_1 \;<_{\mathbf{i}'}\; 3\check\alpha_1+\check\alpha_2 \;<_{\mathbf{i}'}\; 2\check\alpha_1+\check\alpha_2 \;<_{\mathbf{i}'}\; 3\check\alpha_1+2\check\alpha_2 \;<_{\mathbf{i}'}\; \check\alpha_1+\check\alpha_2 \;<_{\mathbf{i}'}\; \check\alpha_2.$$

If w is the length k element of W that ends s_2 (resp. s_1), then we can read off $\check R^+(w)$ as the first k elements of $\check R^+(w)$ according to $\mathbf{i}$ (resp. $\mathbf{i}'$). Moreover, the indecomposable elements are precisely the first and the last in the ordered set $\check R^+(w)$ (as can be verified by inspection). These indeed coincide with the elements of the subset $\check R^+_{w,B}$.

For example, if $w_1 = s_2s_1s_2s_1$, then we have

$$\check R^+_{w_1,B} = \{\check\alpha_1,\ 3\check\alpha_1+2\check\alpha_2\} \subset \check R^+(w_1) = \{\check\alpha_1,\ \check\alpha_1+\check\alpha_2,\ 2\check\alpha_1+\check\alpha_2,\ 3\check\alpha_1+2\check\alpha_2\},$$

while for $w_2 = s_2s_1s_2s_1s_2$ we have

$$\check R^+_{w_2,B} = \{\check\alpha_2,\ 3\check\alpha_1+\check\alpha_2\} \subset R^+(w_2) = \{\check\alpha_2,\ \check\alpha_1+\check\alpha_2,\ 3\check\alpha_1+2\check\alpha_2,\ 2\check\alpha_1+\check\alpha_2, 3\check\alpha_1+\check\alpha_2\},$$

These are the indecomposable elements of $R^+(w_1)$ and $R^+(w_2)$, respectively.

Note that we also have $w_1 \in W^{P_1}$ and $w_2 \in W^{P_2}$ where P_i denotes the standard parabolic subgroup with $I^{P_i} = \{i\}$. Applying the additional condition $w_i s_{\check\eta} \in W^{P_i}$ to $\check\eta \in \check R^+_{w_i,B}$ gives

$$\check R^+_{w_1,P_1} = \{3\check\alpha_1+2\check\alpha_2\}, \qquad \check R^+_{w_2,P_2} = \{3\check\alpha_1+\check\alpha_2\}.$$

Remark 3.9. By Example 3.8, in type G_2, the integer c arising in Definition 3.2 may be $1, 2$ or 3. In types B_n, C_n and F_4, the integer c may be 1 or 2.

We end this section with the following two conjectures motivated by Proposition 3.4.

Conjecture 3.10. *Suppose W is simply-laced, and $w \in W$ with $\check R^+(w)$ its associated inversion set. If $\check\alpha \in \check R^+(w)$ is decomposable, then there exists a decomposition $\check\alpha = \check\mu + \check\mu'$ such that for two different reduced expressions $\mathbf{i}, \mathbf{i}'$ of w we have $\check\mu <_{\mathbf{i}} \check\mu'$, but $\check\mu' <_{\mathbf{i}'} \check\mu$.*

We note that the statement does not necessarily hold for every decomposition of $\check\alpha$, but the conjecture says it holds for at least one, see Example 3.12 below. Using Proposition 3.4 the above conjecture can be reduced to the following conjecture that is stated directly in terms of the Coxeter group structure of W.

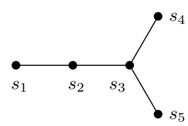
Conjecture 3.11. *Let $w \in W$ and suppose $t \in W$ is a reflection such that $\ell(wt) < \ell(w) - 1$. Then there exists a reduced expression $w = s_{i_1} \dots s_{i_{\ell-1}} s_{i_\ell} s_{i_{\ell+1}} \dots s_{i_r}$ with $i_{\ell-1} = i_{\ell+1} = i$ such that*

$$wt = s_{i_1} \dots s_i \hat s_{i_\ell} s_i \dots s_{i_r},$$

where the factor s_{i_ℓ} has been removed.

Note that this implies the first conjecture as follows. In the situation of Conjecture 3.10, if $\check\alpha \in \check R^+(w)$ is decomposable then $t = s_{\check\alpha}$ has the property $\ell(wt) < \ell(w) - 1$ by Proposition 3.4. Thus Conjecture 3.11 implies the existence of a reduced expression $\mathbf{i} = (i_1, \dots, i, i_\ell, i \dots i_r)$ where $\check\alpha = s_{i_r} \dots s_{i_{\ell+2}} s_i(\check\alpha_j)$, writing $j = i_\ell$. This gives a substring in the associated reflection ordering that is of the form $\check\mu, \check\alpha, \check\mu'$ with $\check\alpha = \check\mu + \check\mu'$, which is reversed in the reflection ordering of $\mathbf{i}' = (i_1, \dots j, i, j, \dots i_r)$, obtained from $\mathbf{i}$ by applying a single braid relation.

Example 3.12. . Consider the minimal coset representative $w^{P_2} = w_0 w_{P_2}$ of w_0 in W^{P_2}, where P_2 is the maximal parabolic subgroup with $I^{P_2} = \{2\}$. Note that $\ell(w_0) = 20$ and $\ell(w_{P_2}) = 7$, so that $\ell(w) = 13$. This element is given in terms of an explicit reduced expression by



$$, \qquad w = w^{P_2} = s_2 s_3 s_4 s_1 s_2 s_3 s_5 s_3 s_4 s_2 s_3 s_1 s_2.$$

In the associated ordering of $\check{R}^+(w)$ the first element is $\check{\alpha}_2$, and the last is

$$\check{\eta}^{\mathbf{i}}_{r=13} = s_2 s_1 s_3 s_2 s_4 s_3 s_5 s_3 s_2 s_1 s_4 s_3(\check{\alpha}_2) = \check{\alpha}_1 + \check{\alpha}_2 + 2\check{\alpha}_3 + \check{\alpha}_4 + \check{\alpha}_5,$$

where $\mathbf{i} = (2, 3, 4, 1, 2, 3, 5, 3, 4, 2, 3, 1, 2)$. The sum of the first and last element of $\check{R}^+(w)$ is still a positive coroot, and thus again in $\check{R}^+(w)$. Namely,

$$\check{\eta}^{\mathbf{i}}_{13} + \check{\alpha}_2 = \check{\alpha}_1 + 2\check{\alpha}_2 + 2\check{\alpha}_3 + \check{\alpha}_4 + \check{\alpha}_5 = \check{\theta},$$

the highest coroot for type D_5. We therefore have in $\check{R}^+(w)$ that

$$\check{\alpha}_2 \ <_{\mathbf{i}} \ \check{\eta}^{\mathbf{i}}_{13} + \check{\alpha}_2 \ <_{\mathbf{i}} \ \check{\eta}^{\mathbf{i}}_{13}.$$

But since $\check{\alpha}_2$ is the only simple coroot in $\check{R}^+(w)$ and every ordering of $\check{R}^+(w)$ coming from a reduced expression must start with $\check{\alpha}_2$, we see that there is no reduced expression of w for which these coroots appear in the opposite order.

However, the decomposable element $\check{\theta}$ can also be described as $\check{\theta} = \check{\eta}^{\mathbf{i}}_7 = s_2 s_1 s_3 s_2 s_4 s_3(\check{\alpha}_5)$. There are two other decompositions of $\check{\theta}$ in $\check{R}^+(w)$,

$$\check{\theta} = \check{\eta}^{\mathbf{i}}_3 + \check{\eta}^{\mathbf{i}}_{11} = \check{\alpha}_{23} + \check{\alpha}_{12345}, \tag{3.14}$$

$$\check{\theta} = \check{\eta}^{\mathbf{i}}_6 + \check{\eta}^{\mathbf{i}}_8 = \check{\alpha}_{1234} + \check{\alpha}_{235}. \tag{3.15}$$

For the second one there is a reduced expression $\mathbf{j}$,

$$w = s_2 s_3 s_4 s_1 s_2 \boxed{s_5 s_3 s_5}\, s_4 s_2 s_3 s_1 s_2,$$

where the summands are in the reverse order.

4. The Picard group of $\mathbb{X}_{w,P}$

In this section, we let $w \in W^P$, and provide precise generators of the Picard group of $\mathbb{X}_{w,P}$ in Proposition 4.2.

Definition 4.1. Recall that $I = I_P \sqcup I^P$ with $I_P = \{k \in I \mid \dot{s}_k \in P\}$. Moreover, $I^B = I$ for the case $P = B$. Associate to w the sets

$$\begin{aligned} I^B_w &:= I_w = \{i \in I \mid s_i < w\}, \\ I^P_w &:= I_w \cap I^P = \{k \in I^P \mid s_k < w\}, \end{aligned}$$

and associate to $i \in I^B_w$ the Weil divisor $\mathbb{X}^{s_i}_{w,P}$ in $\mathbb{X}_{w,P}$. If $k \in I^P_w$ then $\mathbb{X}^{s_k}_{w,P}$ is in fact a Richardson divisor, see Remark 2.1, and we will also use the notation $D_k := \mathbb{X}^{s_k}_{w,P}$ for it, as these divisors will play a special role.

The proposition below is Proposition A from the introduction.

Proposition 4.2. *The divisor $\mathbb{X}^{s_i}_{w,P}$ in $\mathbb{X}_{w,P}$ associated to $i \in I_w$ is Cartier if $i \in I^P_w$. Writing $D_k = \mathbb{X}^{s_k}_{w,P}$ for these Cartier divisors associated to $k \in I^P_w$, we have that the Picard group* $\mathrm{Pic}(\mathbb{X}_{w,P})$ *is freely generated by the line bundles $\mathcal{O}(D_k)$.*

Proof. We recall that restriction of line bundles

$$\mathrm{Res}:\mathrm{Pic}(G/P)\to\mathrm{Pic}(\mathbb{X}_{w,P})$$

is a surjection, by [Ma88, Proposition 6]. We first show that the divisors D_k for $k\in I^P_w$ are Cartier in $\mathbb{X}_{w,P}$ and the line bundles $\mathcal{O}_{\mathbb{X}_{w,P}}(D_k)$ generate the Picard group.

Recall that $P\supseteq B$ is an 'upper-triangular' parabolic subgroup of G. Natural generators of the Picard group of G/P can be constructed in a standard way using Borel-Weil, see [Br05]. Explicitly, in our conventions, for each fundamental weight ω_k with $k\in I^P$ we have a highest weight representation V_{ω_k} and a map

$$G/P\xrightarrow{\phi_k}\mathbb{P}(V_{\omega_k})$$

sending gP to the line $\langle g\cdot v^+_{\omega_k}\rangle_{\mathbb{C}}$ spanned by a highest weight vector. The line bundles $\mathcal{L}_{\omega_k}:=\phi_k^*(\mathcal{O}(1))$ generate the Picard group $\mathrm{Pic}(G/P)\cong\mathbb{Z}^{I^P}$. Namely, the hyperplane in $\mathbb{P}(V_{\omega_k})$ defined by the vanishing of the highest weight component pulls back to the divisor $\mathbb{X}^{s_k,P}$, and we see that $\mathrm{div}(\mathcal{L}_{\omega_i})=[\mathbb{X}^{s_k,P}]$. The divisor class $[\mathbb{X}^{s_k,P}]$ for $k\in I^P$ corresponds via Poincaré duality to the Schubert cohomology class also denoted $\sigma^{s_k}_{G/P}\in H^2(G/P,\mathbb{Z})$. In other words, $c_1(\mathcal{L}_{\omega_k})=\sigma^{s_k}_{G/P}$ describes the standard isomorphism $\mathrm{Pic}(G/P)\cong H^2(G/P,\mathbb{Z})$. If we now restrict $\mathcal{L}_{\omega_k}$ to the Schubert variety $\mathbb{X}_{w,P}$, the associated divisor class $\mathrm{div}(\mathcal{L}_{\omega_k}|_{\mathbb{X}_{w,P}})$ can be computed using intersection theory in G/P via

$$\iota_*(\mathrm{div}(\mathcal{L}_{\omega_k}|_{\mathbb{X}_{w,P}}))=[\mathbb{X}_{w,P}]\cdot\mathrm{div}(\mathcal{L}_{\omega_k})=[\mathbb{X}_{w,P}]\cdot[\mathbb{X}^{s_k,P}]=\sum_{\check\eta\in\check R^+_{w,P}}\langle\omega_k,\check\eta\rangle[\mathbb{X}_{ws_{\check\eta},P}],\tag{4.1}$$

where $\iota:\mathbb{X}_{w,P}\hookrightarrow G/P$ denotes the inclusion of the Schubert variety, and the final equality is by the Chevalley formula in Schubert calculus of G/P [Ch94], see also [FW04, Lemma 8.1]. On the other hand, we have $D_k=\pi_P(\mathbb{X}^{s_k}_{w,B})$, and in G/B,

$$[\mathbb{X}^{s_k}_{w,B}]_{G/B}=[\mathbb{X}^{s_k,B}]_{G/B}\cdot[\mathbb{X}_{w,B}]_{G/B}=\sum_{\check\eta\in\check R^+_{w,B}}\langle\omega_k,\check\eta\rangle[\mathbb{X}_{ws_{\check\eta},B}]_{G/B}.\tag{4.2}$$

Pushing forward via $\pi_P:G/B\to G/P$ and comparing (4.2) with (4.1), we find that

$$[D_k]_{G/P}=(\pi_P)_*([\mathbb{X}^{s_k}_{w,B}]_{G/B})=\iota_*(\mathrm{div}(\mathcal{L}_{\omega_k}|_{\mathbb{X}_{w,P}})),$$

where the extra summands in (4.2) disappear in the pushforward for dimension reasons. Thus, as a divisor in $\mathbb{X}_{w,P}$ we have that D_k is Cartier and $\mathcal{O}_{\mathbb{X}_{w,P}}(D_k)=\mathcal{L}_{\omega_k}|_{\mathbb{X}_{w,P}}$.

Since the line bundles $\mathcal{L}_{\omega_k}$ for $k\in I^P$ generate $\mathrm{Pic}(G/P)$, and Res is a surjection, it follows that the $\mathrm{Res}(\mathcal{L}_{\omega_k})=\mathcal{O}_{\mathbb{X}_{w,P}}(D_k)$, generate $\mathrm{Pic}(\mathbb{X}_{w,P})$. Of course, if $s_k\not<w$, then the divisor class $[D_k]_{\mathbb{X}_{w,P}}=0$ and $\mathcal{L}_{\omega_k}|_{\mathbb{X}_{w,P}}$ is trivial. Thus the line bundles $\mathcal{O}_{\mathbb{X}_{w,P}}(D_k)$ with $k\in I^P_w$ suffice to generate the Picard group of $\mathbb{X}_{w,P}$.

We can check directly that $\mathrm{Pic}(\mathbb{X}_{w,P})$ is freely generated by the $\mathcal{O}_{\mathbb{X}_{w,P}}(D_k)$, equivalently, that there are no non-trivial linear relations between the $[D_k]$ as follows. Suppose we have that $D=\sum_{k\in I^P_w}c_kD_k$ is linearly equivalent to 0. Let $\lambda=\sum_{k\in I^P}c_k\,\omega_k$ and consider $\mathcal{L}_\lambda\in\mathrm{Pic}(G/P)$. Then since $\mathcal{O}_{\mathbb{X}_{w,P}}(D)=\mathrm{Res}(\mathcal{L}_\lambda)$ we have that $\mathcal{L}_\lambda|_{\mathbb{X}_{w,P}}$ is a trivial line bundle. Now for any $s_k<w$ we can restrict $\mathcal{L}_\lambda$ further to the 1-dimensional Schubert variety $\mathbb{X}_{s_k,P}\subseteq\mathbb{X}_{w,P}$. This restriction is still trivial. But the line bundle $\mathcal{L}_\lambda|_{\mathbb{X}_{s_k,P}}$ on the $\mathbb{X}_{s_k,P}$ has degree computed by $\langle\lambda,\check\alpha_k\rangle=c_k$, as in [FW04, Lemma 3.2]. Thus we obtain that $c_k=0$ and the linear relation is trivial. □

Remark 4.3. We have the following commutative diagram,

$$\begin{tikzcd} \mathrm{Pic}(G/P) \arrow[r, two heads, "\mathrm{Res}"] \arrow[d, "c_1"'] & \mathrm{Pic}(\mathbb{X}_{w,P}) \arrow[d, "c_1"'] \arrow[dr, "\mathrm{div}"] & \\ H^2(G/P,\mathbb{Z}) \arrow[r, two heads, "i^*"] & H^2(\mathbb{X}_{w,P},\mathbb{Z}) \arrow[r, "\cap[\mathbb{X}_{w,P}]"] & H_{2\ell(w)-2}(\mathbb{X}_{w,P},\mathbb{Z}). \end{tikzcd}$$

Here the first Chern class maps c_1 are both isomorphisms, using Proposition 4.2 for the right-hand one. We refer to [La04, Section 1.1 C] for the c_1 map for singular Schubert varieties. The divisor class map div is injective, and thus so is the cap-product map

$$H^2(\mathbb{X}_{w,P},\mathbb{Z}) \xrightarrow{\cap[\mathbb{X}_{w,P}]} H_{2\ell(w)-2}(\mathbb{X}_{w,P},\mathbb{Z}). \tag{4.3}$$

We may identify $\mathrm{Pic}(\mathbb{X}_{w,P})$ with its image in $H_{2\ell(w)-2}(\mathbb{X}_{w,P},\mathbb{Z})$, which by Proposition 4.2 is precisely the free span of the Cartier divisor classes $[D_k]$, where $k \in I_w^P$. Moreover, we have

$$[D_k] = \sum_{\check\eta \in \check R^+_{w,P}} \langle \omega_k, \check\eta\rangle [\mathbb{X}_{ws_{\check\eta},P}]. \tag{4.4}$$

From this perspective, Proposition 4.2 implies that the matrix $(\langle \omega_k, \check\eta\rangle)_{\check\eta,k}$ with $\check\eta \in \check R^+_{w,P}$ and $k \in I_w^P$ has full rank equal to $|I_w^P|$. We note that this is a non-trivial statement in terms of Weyl group combinatorics. Moreover, the span $\langle [D_k]_{\mathbb{X}_{w,P}}\rangle_{\mathbb{Z}}$ as a sublattice inside $H_{2\ell(w)-2}(\mathbb{X}_{w,P},\mathbb{Z})$ (the Cartier divisor classes inside the group of Weil divisor classes) encodes more subtle information about the singularities of the Schubert variety $\mathbb{X}_{w,P}$. We make use of this in the following section.

5. Factoriality of Schubert varieties

Let us begin by defining a pair of lattices which will play a central role in analysing factoriality. Recall that $H^2(\mathbb{X}_{w,P},\mathbb{Z})$ can be identified with the Picard group of $\mathbb{X}_{w,P}$, as seen above, and we have a basis given by the Chern classes $c_1(\mathcal{L}_{\omega_k}|\mathbb{X}_{w,P})$ of the generators of $\mathrm{Pic}(\mathbb{X}_{w,P})$. This leads us to identify $H^2(\mathbb{X}_{w,P},\mathbb{Z})$ with the sublattice of the weight lattice of G spanned by the fundamental weights ω_k with $k \in I_w^P$.

Definition 5.1. Let $w \in W^P$. We define a dual pair of $\mathbb{Z}$-lattices in terms of the root datum of G, which are naturally identified with second cohomology and homology of $\mathbb{X}_{w,P}$ as shown.

$$\begin{array}{lcll} \mathcal{P}_{w,P} & := & \langle\ \omega_k \mid k \in I_w^P\ \rangle_{\mathbb{Z}}, & \hat{=}\quad H^2(\mathbb{X}_{w,P},\mathbb{Z}), \\ \mathcal{Q}_{w,P} & := & \langle\ \check\alpha_i \mid i \in I_w \rangle_{\mathbb{Z}}\ /\ \langle \check\alpha_j \mid j \in I_P \cap I_w\rangle_{\mathbb{Z}} & \hat{=}\quad H_2(\mathbb{X}_{w,P},\mathbb{Z}). \end{array}$$

Here $\mathcal{P}_{w,P}$ is a sublattice of the weight lattice of G, and $\mathcal{Q}_{w,P}$ is a subquotient of the coroot lattice of G. The dual pairing is inherited from the dual pairing between the weight and coroot lattices. We also note that $\check R^+_{w,P}$ naturally maps to $\mathcal{Q}_{w,P}$, and we do not distinguish notationally between an element or $\check R^+_{w,P}$ and its image in $\mathcal{Q}_{w,P}$.

We can now state the main results of this section. The following proposition relates properties of the subset $\check R^+_{w,P}$ of $\check R^+(w)$ to the factoriality of $\mathbb{X}_{w,P}$, part (3) of which is **Theorem B**.

Proposition 5.2. *Let $w \in W^P$. The associated subset $\check R^+_{w,P}$ of w 0has the following properties.*

(1) *The $|\check R^+_{w,P}| \times |I_w^P|$ matrix $(\langle \omega_i, \check\eta\rangle)_{\check\eta,k}$, has rank $|I_w^P|$, and the set $\check R^+_{w,P}$ generates a sublattice of full rank inside $\mathcal{Q}_{w,P}$.*
(2) *The Schubert variety $\mathbb{X}_{w,P}$ is $\mathbb{Q}$-factorial if and only if the set $\check R^+_{w,P}$ has cardinality $|I_w^P|$. Equivalently, this is if $\check R^+_{w,P}$ defines a $\mathbb{Q}$-basis of $\mathcal{Q}_{w,P} \otimes_{\mathbb{Z}} \mathbb{Q}$.*

(3) *The Schubert variety* $\mathbb{X}_{w,P}$ *is factorial if and only if* $\check{R}^+_{w,P}$ *is a* $\mathbb{Z}$*-basis of* $\mathcal{Q}_{w,P}$*.*

Then we will prove the theorem below concerning the simply-laced case.

Theorem 5.3. *Suppose* G *is of simply-laced type and* $w \in W^P$*. Then we have the following.*

(1) $\check{R}^+_{w,P}$ *generates* $\mathcal{Q}_{w,P}$*.*
(2) *If* D *is a Weil divisor in* $\mathbb{X}_{w,P}$ *and* D *is* $\mathbb{Q}$*-Cartier, then* D *is in fact Cartier.*
(3) $\mathbb{X}_{w,P}$ *is factorial if and only if* $|\check{R}^+_{w,P}| = |I^P_w|$*.*

Remark 5.4. In type A, Bousquet-Mélou and Butler [BMB07] characterised the permutations giving rise to factorial Schubert varieties in SL_n/B in combinatorial terms and via a pattern-avoidance criterion, proving a conjecture of Woo and Yong from [WY06]. In [En21, Proposition A.6] this criterion was shown to be equivalent to $|\check{R}^+_{w,B}| = |I^B_w|$, which proves Theorem 5.3(3) for Schubert varieties in type A full flag varieties. In [En21, Conjecture A.8] it was conjectured for general type Schubert varieties in G/B that being factorial is equivalent to $|\check{R}^+_{w,B}| = |I^B_w|$. Theorem 5.3 implies that this conjecture holds when G is simply-laced (and even extends to G/P Schubert varieties). We observe that the conjecture is false in non-simply-laced type, though, see Example 3.8 and Remark 5.5. Namely, the condition $|\check{R}^+_{w,B}| = |I^B_w|$ is equivalent to $\mathbb{X}_{w,B}$ being $\mathbb{Q}$-factorial, and there are $\mathbb{Q}$-factorial Schubert varieties that are not factorial in non-simply-laced types.

Remark 5.5. Applying Proposition 5.2 to Example 3.8 we see that all Schubert varieties for G_2 are $\mathbb{Q}$-factorial, since $\check{R}^+_{w,P}$ is always a linearly independent set of cardinality $|I^P_w|$ in the case of G_2. Amongst these, the following list of Schubert varieties are not factorial, again as an application of the proposition:

$$\mathbb{X}_{s_2s_1s_2,P_2}, \quad \mathbb{X}_{s_2s_1s_2,B}, \quad \mathbb{X}_{s_1s_2s_1s_2,B}, \quad \mathbb{X}_{s_2s_1s_2s_1s_2,B},$$

as well as

$$\mathbb{X}_{s_2s_1,P_1}, \quad \mathbb{X}_{s_1s_2s_1,P_1}, \quad \mathbb{X}_{s_2s_1s_2s_1,P_1}, \quad \mathbb{X}_{s_2s_1s_2s_1,B}.$$

Namely, for each of these $\mathbb{X}_{w,P}$ the $\mathbb{Z}$-span of $\check{R}^+_{w,P}$ is a proper sublattice of $\mathcal{Q}_{w,P}$, giving rise to some Weil divisors that are not Cartier. This can be seen directly by inspection of the sets $\check{R}^+_{w,P}$ which are described in Example 3.8.

We may compare the G/B Schubert varieties listed above with the list of singular Schubert varieties for G_2 given in [BP05, Theorem 2.4]. In that list there is one additional singular Schubert variety, $\mathbb{X}_{s_1s_2s_1s_2s_1,B}$. Since $R^+_{s_1s_2s_1s_2s_1,B} = \{\check{\alpha}_1, \check{\alpha}_1+\check{\alpha}_2\}$ this Schubert variety appears to be factorial and in that sense 'less singular' than the others.

Proposition 5.2 is a consequence of the proof of Proposition 4.2 together with known properties of Schubert varieties, and we can give a proof straight away.

Proof of Proposition 5.2. The property (1) is a consequence of Proposition 4.2, see also Remark 4.3. Now recall that $\check{R}^+_{w,P}$ is in bijection with the Schubert divisors of $\mathbb{X}_{w,P}$. The Schubert variety $\mathbb{X}_{w,P}$ is normal and has rational singularities [An85]. Therefore it is $\mathbb{Q}$-factorial if and only if the Betti numbers $b_2 = \dim(H^2(\mathbb{X}_{w,P},\mathbb{Q}))$ and $b_{2\ell(w)-2} = \dim(H_{2\ell(w)-2}(\mathbb{X}_{w,P},\mathbb{Q}))$ agree, by [PP24, Theorem A]. Thus $\mathbb{X}_{w,P}$ is $\mathbb{Q}$-factorial if and only if the order of $\check{R}^+_{w,P}$ is equal to b_2, which is also $|I^P_w|$ and the rank of $\mathcal{Q}_{w,P}$. Since $\check{R}^+_{w,P}$ generates a full rank sublattice of $\mathcal{Q}_{w,P}$ by (1) this is furthermore equivalent to $\check{R}^+_{w,P}$ being a $\mathbb{Z}$-basis of that sublattice, and after tensoring with $\mathbb{Q}$, a basis of $\mathcal{Q}_{w,P}\otimes\mathbb{Q}$. Therefore (2) is proved. If $\check{R}^+_{w,P}$ is a $\mathbb{Z}$-basis of $\mathcal{Q}_{w,P}$ then the matrix from (1) is invertible (over $\mathbb{Z}$) and by (4.4) we have that the Cartier divisor classes $[D_i]$ form a basis of $H_{2\ell(w)-2}(\mathbb{X}_{w,P},\mathbb{Z})$, which implies that $\mathbb{X}_{w,P}$ is factorial and vice versa. Part (3) of the proposition follows. ∎

The proof of Theorem 5.3 will take up the remainder of the section. It involves constructing in simply-laced type a subset of Schubert classes in $H^{2\ell(w)-2}(\mathbb{X}_{w,P},\mathbb{Z})$ that map to a basis of $H_2(\mathbb{X}_{w,P},\mathbb{Z})$ under the dual of the cap product map (4.3), as explained in Remark 5.9.

This concrete construction will in particular imply that the cap product map has saturated image, that is, the Cartier divisor classes form a saturated sublattice in $H_{2\ell(w)-2}(\mathbb{X}_{w,P},\mathbb{Z})$ in simply-laced types.

5.1. Construction of $\mathcal{B}_{w,B}$. We temporarily restrict to the case $P = B$ in this subsection. Our goal is to construct a special subset of $\check{R}^+_{w,B}$ of cardinality $|I^B_w|$ in the simply-laced setting.

Proposition 5.6. *Let G be of simply-laced type and $w \in W$. There exists a subset*

$$\mathcal{B}_{w,B} := \{\check{\eta}_w(k) \mid k \in I^B_w\}$$

of $\check{R}^+_{w,B}$ and an ordering of I^B_w such that the matrix $M = (\langle\omega_i, \check{\eta}_w(k)\rangle)_{k,i\in I^B_w}$ is unipotent lower-triangular. In particular, $\mathcal{B}_{w,B}$ embeds into $\mathcal{Q}_{w,B}$ as a $\mathbb{Z}$-basis.

Remark 5.7. Whenever w has full support, Proposition 5.6 constructs a basis of the coroot lattice. In the example of $w = w_0$, this basis is just the basis of simple coroots.

In preparation for the proof of this proposition we make the following definition.

Definition 5.8. Let $w \in W$ and $s_k < w$. For a reduced expression $w = s_{i_1}s_{i_2}\dots s_{i_r}$, we write $\mathbf{i} = (i_1,\dots,i_r)$, and define $d_{\mathbf{i}}(k)$ to be the distance of the right-most occurrence of s_k in the reduced expression $\mathbf{i}$ from the end of $\mathbf{i}$. In other words, $\ell = r - d_{\mathbf{i}}(k) + 1$ is the right-most position of k in $\mathbf{i}$, meaning $i_\ell = k$ and $i_m \neq k$ whenever $m > \ell$. Now we define the 'absolute minimal distance' from the end,

$$d_w(k) := \min(\{d_{\mathbf{i}}(k) \mid \mathbf{i} \text{ is a reduced expression of } w\}). \tag{5.1}$$

If $d_{\mathbf{i}}(k) = d_w(k)$ we say that the reduced expression $\mathbf{i}$ is 'rightmost' for k. Let $\ell = r+1-d_w(k)$ so that in $i_\ell = k$ is the rightmost occurrence of k in $\mathbf{i}$. We define an element $\check{\alpha}^{\mathbf{i}}_w(k) \in R^+(w)$ by

$$\check{\alpha}^{\mathbf{i}}_w(k) := s_{i_r}s_{i_{r-1}}\dots s_{i_{\ell+1}}(\check{\alpha}_k). \tag{5.2}$$

We often simply write $\check{\alpha}_w(k) := \check{\alpha}^{\mathbf{i}}_w(k)$, namely one reduced expression $\mathbf{i}$ with $d_{\mathbf{i}}(k) = d_w(k)$ is a prior fixed. The coroot $\check{\alpha}_w(k)$ generally depends on the choice of $\mathbf{i}$, however (see Example 5.11). When $\mathbf{i}$ is omitted from the notation, this will indicate that the choice of $\mathbf{i}$ is not relevant to our purpose.

Proof of Proposition 5.6. For every $k \in I^B_w$ let $\check{\alpha}_w(k)$ be as constructed in Definition 5.8. Namely, we have

$$\check{\alpha}_w(k) = s_{i_r}\dots s_{i_{\ell+1}}(\check{\alpha}_k) \tag{5.3}$$

in terms of a right-most reduced expression for k. Note that we can write $\check{\alpha}_w(k)$ as

$$\check{\alpha}_w(k) = \check{\alpha}_k + \sum_{m=\ell+1}^{r} c_m\check{\alpha}_{i_m}, \tag{5.4}$$

for some coefficients $c_m \in \mathbb{Z}_{\geq 0}$. Here the coefficient of $\check{\alpha}_k$ is 1 using the simply-laced assumption and the fact that none of the simple reflections occurring in (5.3) include s_k. So we have $\langle\omega_k, \check{\alpha}_w(k)\rangle = 1$. We now define $\check{\eta}_w(k)$ as follows.

If $\check{\alpha}_w(k) \in \check{R}^+_{w,B}$, then we set $\check{\eta}_w(k) := \check{\alpha}_w(k)$. If not then, by the simply-laced assumption and Proposition 3.4 and Lemma 3.3, we have that $\check{\alpha}_w(k)$ is decomposable into a sum of elements of $\check{R}^+_{w,B}$. Now precisely one of these indecomposable summands $\check{\mu}$ has the property $\langle\omega_k, \check{\mu}\rangle = 1$, and so we set $\check\eta_w(k) = \check\mu$. Note that by construction of $\check\eta_w(k)$ as a summand of $\check\alpha_w(k)$ we obtain from (5.4) an expansion for $\check\eta_w(k)$,

$$\check\eta_w(k) = \check\alpha_k + \sum_{m=\ell+1}^{r} c'_m \check\alpha_{i_m} \tag{5.5}$$

with $c'_m \in \mathbb{Z}_{\geq 0}$.

We now choose a total ordering $\lhd$ on I_w^B, such that $d_w(k_1) < d_w(k_2)$ implies that $k_1 \lhd k_2$. Consider the matrix $M = (\langle \omega_i, \check\eta_w(k)\rangle)_{k,i\in I_w^B}$, with rows and columns ordered according to $(I_w^B, \lhd)$. Note that any $i = i_m$ occurring in the sum in (5.5) has $d_w(i) < d_w(k)$. Thus any simple coroot $\check\alpha_i$ that appears in $\check\eta_w(k)$ has $d_w(i) < d_w(k)$, and therefore $i \lhd k$ in the ordering of I_w^B. Thus, M is lower-triangular. Moreover, as seen in (5.5), $\langle \omega_k, \check\eta_w(k)\rangle = 1$, implying that M has 1's along the diagonal. □

Remark 5.9. Recalling that $\mathcal{Q}_{w,B} = H_2(\mathbb{X}_{w,B}, \mathbb{Z})$, the basis constructed in the lemma can be considered as giving rise to a (nonstandard) basis of the second homology group of $\mathbb{X}_{w,B}$. Namely there is a curve class $[C_{\check\eta}]$ associated to $\check\eta \in \mathcal{B}_{w,B}$, which has the property that for every line bundle $\mathcal{L}_{\omega_i}|_{\mathbb{X}_{w,B}} = \mathcal{O}(D_i)$ we have

$$\langle c_1(\mathcal{O}(D_i)), [C_{\check\eta}]\rangle \overset{1}{=\!=} \langle \omega_i, \check\eta\rangle \overset{2}{=\!=} \langle [D_i], \sigma^{ws_{\check\eta}}\rangle, \tag{5.6}$$

where $\sigma^{ws_{\check\eta}}$ is the Schubert cohomology class in $H^{2\ell(w)-2}(\mathbb{X}_{w,B}, \mathbb{Z})$ associated to $ws_{\check\eta}$ and $i \in I_w^P$. Here 1 can be taken as the definition of $[C_{\check\eta}]$ and 2 follows from (4.4). This identity expresses the fact that the curve class $[C_{\check\eta}]$ is the image of the Schubert class $\sigma^{ws_{\check\eta}}$ under the dual of the cap product map (4.3). In other words, if we think of $\check R^+_{w,B}$ as indexing the Schubert basis of $H^{2\ell(w)-2}(\mathbb{X}_{w,B}, \mathbb{Z})$, then $\mathcal{B}_{w,B}$ indexes a subset of this Schubert basis for which the associated curve classes $[C_{\check\eta}]$ form a basis of $H_2(\mathbb{X}_{w,B}, \mathbb{Z})$. In the smooth case, the elements of $\mathcal{B}_{w,B}$ are of course all the Schubert basis elements σ^w in $H^{2\ell(w)-2}(\mathbb{X}_{w,B}, \mathbb{Z})$, and the $[C_{\check\eta}]$ are their Poincaré dual curve classes. In the general non-simply-laced case such a basis of $H_2(\mathbb{X}_{w,B}, \mathbb{Z})$ may not exist.

Remark 5.10. We expect that a weaker version of Proposition 5.6 with 'unipotent lower-triangular' replaced by 'lower-triangular with diagonal entries in $\mathbb{Z}_{>0}$' holds in the non-simply-laced case. In the G_2 case this can be seen directly, see Example 3.8.

Now we illustrate the construction of $\mathcal{B}_{w,B}$ in some examples.

Example 5.11. Consider type A_4. Consider the following two reduced expressions, $\mathbf{i}$ and $\mathbf{i}'$, of the permutation 53142 in S_5,

$$w = s_2 s_1 s_3 s_4 s_3 s_2 s_1 = s_4 s_3 s_2 s_1 s_3 s_2 s_4. \tag{5.7}$$

We simply denote $\check\alpha_{j_1} + \check\alpha_{j_2} + \cdots + \check\alpha_{j_t}$ as $\check\alpha_{j_1 j_2 \dots j_t}$. Associated to w we have

$$\check R^+_{w,B} = \{\check\alpha_1, \check\alpha_2, \check\alpha_4, \check\alpha_{12}, \check\alpha_{123}, \check\alpha_{234}\} \qquad \text{and} \qquad \check R^+(w) = \check R^+_{w,B} \cup \{\check\alpha_{1234}\}.$$

For $k = 1, 2, 4$ clearly $d_w(k) = 1$ and $\check\alpha_w(k) = \check\alpha_k$. But $d_w(3) = 3$ and there are two options for $\check\alpha_w(3)$, depending on the choice of reduced expression $\mathbf{i}$. Namely, $\check\alpha_w^{\mathbf{i}}(3) = \check\alpha_{123}$, using the first reduced expression in (5.7), and $\check\alpha_w^{\mathbf{i}'}(3) = \check\alpha_{234}$, using the second one. Either of the two reduced expressions gives rise to a subset with expected property in Proposition 5.6, namely

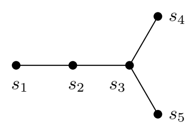
$$\mathcal{B}^{\mathbf{i}}_{w,B} = \{\check\alpha_1, \check\alpha_2, \check\alpha_4, \check\alpha_1 + \check\alpha_2 + \check\alpha_3\}, \qquad \mathcal{B}^{\mathbf{i}'}_{w,B} = \{\check\alpha_1, \check\alpha_2, \check\alpha_4, \check\alpha_2 + \check\alpha_3 + \check\alpha_4\}.$$

Note however, that the dual bases

$$\mathcal{B}^{\mathbf{i},*}_{w,B} = \{\omega_1 - \omega_3, \omega_2 - \omega_3, \omega_4, \omega_3\}, \qquad \mathcal{B}^{\mathbf{i}',*}_{w,B} = \{\omega_1, \omega_2 - \omega_3, \omega_4 - \omega_3, \omega_3\},$$

while they differ, have the same sums: $\omega_1 + \omega_2 - \omega_3 + \omega_4$. This reflects the fact that this Schubert variety is Gorenstein, see Remark 6.11.

Example 5.12. Consider in type D_5 the following element $w \in W^{P_3}$, where P_3 is the maximal parabolic subgroup with $I^{P_3} = \{3\}$. We write down three reduced expressions $\mathbf{i}$, $\mathbf{i}'$ and $\mathbf{i}''$ of w as follows.



$$, \qquad w = s_2s_3s_1s_2s_3s_4s_5s_3 = s_2s_3s_1s_2s_3s_5s_4s_3 = s_1s_2s_3s_5s_4s_1s_2s_3.$$

We have

$$\check{R}^+(w) = \{\check\alpha_3, \check\alpha_{35}, \check\alpha_{34}, \check\alpha_{345}, \check\alpha_{2345} + \check\alpha_3,\ \check\alpha_{12345} + \check\alpha_2,\ \check\alpha_{23}, \check\alpha_{123}\},$$

and the subset of indecomposable elements is.

$$\check{R}^+_{w,B} = \{\check\alpha_3, \check\alpha_{35}, \check\alpha_{34}, \check\alpha_{345}, \check\alpha_{23}, \check\alpha_{123}\}.$$

We can find rightmost elements $\check\alpha_w(k)$ for every index k using $\mathbf{i}$, $\mathbf{i}'$ and $\mathbf{i}''$, and these elements are indecomposable. We therefore have

$$\begin{aligned}
\check\eta_w(3) = \check\alpha^{\mathbf{i}}_w(3) &= \check\alpha_3, \\
\check\eta_w(5) = \check\alpha^{\mathbf{i}}_w(5) &= \check\alpha_3 + \check\alpha_5, \\
\check\eta_w(4) = \check\alpha^{\mathbf{i}'}_w(4) &= \check\alpha_3 + \check\alpha_4, \\
\check\eta_w(2) = \check\alpha^{\mathbf{i}''}_w(2) &= \check\alpha_2 + \check\alpha_3, \\
\check\eta_w(1) = \check\alpha^{\mathbf{i}''}_w(1) &= \check\alpha_1 + \check\alpha_2 + \check\alpha_3.
\end{aligned}$$

Then $\mathcal{B}_{w,B} = \{\check\eta_w(k) \mid k = 1, \ldots, 5\}$ is a subset of $\check{R}^+_{w,B}$ that descends to a basis of $\mathcal{Q}_{w,B}$.

Conjecture 5.13. *Let G be of simply-laced type and $w \in W$. For $k \in I^B_w$ consider a reduced expression $\mathbf{i}$ of w such that $d_{\mathbf{i}}(k) = d_w(k)$ and let $\check\eta_w(k) = \check\alpha^{\mathbf{i}}_w(k)$. Then $\check\eta_w(k)$ is indecomposable in $\check{R}^+(w)$. In particular, $\{\check\eta_w(k) \mid k \in I^B_w\}$ constructed in this way satisfies the expected properties from Proposition 5.6.*

We note that the simply-laced assumption is necessary for this conjecture.

5.2. Construction of $\mathcal{B}_{w,P}$. The generalisation of the basis $\mathcal{B}_{w,B}$ to the parabolic setting is not completely straightforward. It will use a recursive procedure that is based on Lemma 3.6. Namely Lemma 5.15 below provides us with a mechanism for modifying the basis $\mathcal{B}_{w,B}$ in order to ultimately construct our basis of $\check{R}^+_{w,P}$. In the following we fix $\mathcal{B}_{w,B} = \{\check\eta_w(k) \mid k \in I^B_w\}$ constructed as in Proposition 5.6.

Definition 5.14. Let G be of simply-laced type. We call a subset $\mathcal{M} \subseteq \check{R}^+_{w,B}$ a P-adaptation of $\mathcal{B}_{w,B}$, if there is a bijection $\check\mu = \check\mu_w : I^P_w \to \mathcal{M}$ with the property that

$$\check\mu(k) \equiv \check\eta_w(k) \mod \langle \check\alpha_j \mid j \in I_P\rangle \qquad \text{for all } k \in I^P_w.$$

Lemma 5.15. *Let G be of simply-laced type. Let $\mathcal{M}$ be a P-adaptation of $\mathcal{B}_{w,B}$. Suppose that there exist $k_0 \in I^P_w$ and $j \in I_P$ such that $ws_{\check\mu(k_0)}(\check\alpha_j) < 0$. Then $\check\mu'(k_0) := s_{\check\mu(k_0)}(\check\alpha_j)$ satisfies:*

(1) $\check\mu'(k_0) \in \check{R}^+_{w,B}$;
(2) $\check\mu'(k_0) \equiv \check\mu(k_0) \mod \langle \check\alpha_j \mid j \in I_P\rangle_{\mathbb{Z}}$;
(3) *the heights are related by* $\mathrm{ht}(\check\mu'(k_0)) > \mathrm{ht}(\check\mu(k_0))$.

For $k \neq k_0$ let us set $\check\mu'(k) = \check\mu(k)$, then we have another P-adaptation $\mathcal{M}'$ of $\mathcal{B}_{w,B}$ defined by

$$\mathcal{M}' = \{\check\mu'(k) \mid k \in I^P_w\}.$$

Proof. By Lemma 3.6 we have $\check{\mu}'(k_0) \in \check{R}^+(w)$. On the other hand we have

$$\check{\mu}'(k_0) = s_{\check{\mu}(k_0)}(\check{\alpha}_j) = \check{\alpha}_j - \langle \check{\alpha}_j, \check{\mu}(k_0)^\vee \rangle \check{\mu}(k_0). \tag{5.8}$$

Note that $w(\check{\mu}'(k_0)) < 0$ implies $\check{\mu}'(k_0) \neq \check{\alpha}_j$ (indeed, $w(\check{\alpha}_j) > 0$ as $w \in W^P$). Therefore (5.8) together with the fact that $\check{\mu}'(k_0) > 0$ implies that we must have $c' := -\langle \check{\alpha}_j, \check{\mu}(k_0)^\vee \rangle > 0$. Since G is of simply-laced type, the only possible positive factor c' is 1. Therefore we are simply replacing $\check{\mu}(k_0)$ by $\check{\mu}'(k_0) = \check{\mu}(k_0) + \check{\alpha}_j$. This implies (2). To prove (1) it remains to show that $\check{\mu}'(k_0)$ is indecomposable, by Proposition 3.4. Namely, suppose indirectly it has a decomposition $\check{\mu}'(k_0) = \check{\mu}(k_0) + \check{\alpha}_j = \check{\mu}_1 + \check{\mu}_2$. Then $s_j(\check{\mu}'(k_0)) = \check{\mu}(k_0) = \check{\mu}'(k_0) - \alpha_j$ and so we must have that one of the summands, without loss of generality $\check{\mu}_1$, satisfies $s_j(\check{\mu}_1) = \check{\mu}_1 - \alpha_j$ (and the other $s_j(\check{\mu}_2) = \check{\mu}_2$). Therefore, $\check{\mu}_1 = \check{\mu}_1' + \check{\alpha}_j$ for another positive root $\check{\mu}_1'$. Now $\check{\mu}_1 \in \check{R}^+(w)$ implies $w(\check{\mu}_1' + \check{\alpha}_j) = w(\check{\mu}_1') + w(\check{\alpha}_j) < 0$. But $w(\check{\alpha}_j) > 0$ since $j \in I_P$, so that $w(\check{\mu}_1') < 0$ necessarily. Then $\check{\mu}_1' + \check{\mu}_2 = \check{\mu}(k_0)$ gives a decomposition of $\check{\mu}(k_0)$ with summands in $\check{R}^+(w)$ contradicting the assumption that $\check{\mu}(k_0)$ is indecomposable. Thus we have proved (1).

It is clear that $\mathrm{ht}(\check{\mu}'(k_0)) > \mathrm{ht}(\check{\mu}(k_0))$ and (3) is also proved. Finally, $\mathcal{M}'$ is again a P-adaptation of $\mathcal{B}_{w,B}$ by (1) and (2). □

Definition 5.16 (Recursive construction of $\mathcal{B}_{w,P}$)**.** Suppose G is of simply-laced type and $w \in W^P$. Recall $\mathcal{B}_{w,B} = \{\check{\eta}_w(k) \mid k \in I_w^B\}$ was constructed in Proposition 5.6. We construct a subset $\mathcal{B}_{w,P}$ of $\check{R}^+(w)$ as follows. Start with the subset $\mathcal{M}_0 := \{\check{\eta}_w(k) \mid k \in I_w^P\}$ of $\check{R}^+(w)$, which is clearly a P-adaption of $\mathcal{B}_{w,B}$. Suppose $d \in \mathbb{Z}_{\geq 0}$ and we have already constructed a P-adaption $\mathcal{M}_d = \{\check{\mu}(k) \mid k \in I_w^P\}$ of $\mathcal{B}_{w,B}$. If for every $\check{\mu} \in \mathcal{M}_d$ we have that $ws_{\check{\mu}} \in W^P$ then set $\mathcal{B}_{w,P} := \mathcal{M}_d$. Otherwise, pick a $k_0 \in I_w^P$ and $j \in I_P$ such that $ws_{\check{\mu}(k_0)}(\check{\alpha}_j) < 0$ and define

$$\mathcal{M}_{d+1} := (\mathcal{M}_d)'$$

using the construction from Lemma 5.15. This process terminates after finitely many steps since each step increases the height of one element, but the height function is bounded.

Lemma 5.17. *Suppose G is of simply-laced type and $w \in W^P$. The set $\mathcal{B}_{w,P}$ constructed in Definition 5.16 has the following properties.*

1. $\mathcal{B}_{w,P} \subseteq \check{R}^+_{w,P}$.
2. $\mathcal{B}_{w,P} = \{\check{\mu}(k) \mid k \in I_w^P\}$ *with* $\check{\mu}(k) \equiv \check{\eta}_w(k) \mod \langle \check{\alpha}_j \mid j \in I_P \rangle_{\mathbb{Z}}$.
3. *There exists a total ordering on I_w^P for which*
$$\check{\mu}(k) \equiv \check{\alpha}_k + \sum_{k'<k} c_{k,k'} \check{\alpha}_{k'} \mod \langle \check{\alpha}_j \mid j \in I_P \rangle_{\mathbb{Z}},$$
with all coefficients in $\mathbb{Z}_{\geq 0}$.

Proof. For property (1) the fact that the $\check{\mu}_w(k)$ lie in $\check{R}^+_{w,B}$ follows from the recursive construction and Lemma 5.15. Additionally, the recursion defining $\mathcal{B}_{w,P}$ only terminates if all elements $ws_{\check{\mu}(k)}$ lie in W^P. Therefore $\mathcal{B}_{w,P} \subseteq \check{R}^+_{w,P}$. Next, the recursion does not alter the elements modulo $\langle \check{\alpha}_j \mid j \in I_P \rangle$. Therefore (2) and (3) hold as a consequence of Proposition 5.6. □

Remark 5.18. The construction of $\mathcal{B}_{w,P}$ from Definition 5.16 is highly dependent on the choices. However, the image of $\mathcal{B}_{w,P}$ in $H_2(\check{\mathbb{X}}_{w,P}, \mathbb{Z})$ only depends on $\mathcal{B}_{w,P}$ modulo $\langle \check{\alpha}_j \mid j \in I_P \rangle_{\mathbb{Z}}$ and is therefore completely determined by the choice of $\mathcal{B}_{w,B}$. The interpretation of $\mathcal{B}_{w,B}$ given in Remark 5.9 extends verbatim to the simply-laced G/P setting just with P replacing B everywhere. Moreover, the basis of $H_2(\check{\mathbb{X}}_{w,P}, \mathbb{Z})$ given by $\mathcal{B}_{w,P}$ can be constructed directly as an image of the subset $\{\check{\eta}_w(k) \mid k \in I_w^P\}$ of $\mathcal{B}_{w,B}$ via the map

$$H_2(\check{\mathbb{X}}_{w,B}, \mathbb{Z}) \longrightarrow H_2(\check{\mathbb{X}}_{w,P}, \mathbb{Z}),$$

as a consequence of Lemma 5.17(2).

We are now in a position to prove Theorem 5.3.

5.3. Proof of Theorem 5.3 and Theorem A. By Lemma 5.17 there exists a subset $\mathcal{B}_{w,P}$, which is related to the standard basis $\{\check{\alpha}_k \mid k \in I_w^P\}$ of $\mathcal{Q}_{w,P}$ by a lower-triangular integer matrix with diagonal entries 1. It follows that $\mathcal{B}_{w,P}$ is itself a basis of $\mathcal{Q}_{w,P}$, and therefore that $\check{R}^+_{w,P}$ generates the lattice $\mathcal{Q}_{w,P}$. This proves Theorem 5.3(1). Now, for a Weil divisor D we have $[D] = \sum_{\check{\eta}\in\check{R}^+_{w,P}} n_{\check{\eta}}[\mathbb{X}_{ws_{\check{\eta}},P}]$ with coefficients $n_{\check{\eta}} \in \mathbb{Z}$. If D is also $\mathbb{Q}$-Cartier, we may write $[D] = \sum m_k[D_k]$, a $\mathbb{Q}$-linear combination of the generators $[D_k] = \sum_{\check{\eta}\in\check{R}^+_{w,P}} \langle\omega_k, \check{\eta}\rangle[\mathbb{X}_{ws_{\check{\eta}},P}]$ of the Cartier class group from Proposition 4.2. Note also that the m_k are unique since the $[D_k]$ are linearly independent. Or equivalently, the vectors $(\langle\omega_k, \check{\eta}\rangle)_{\check{\eta}\in\check{R}^+_{w,P}}$ are linearly independent, compare Proposition 5.2(1). Thus we have the vector identity $(n_{\check{\eta}})_{\check{\eta}} = \sum_{k\in I_w^P} m_k(\langle\omega_k, \check{\eta}\rangle)_{\check{\eta}}$. Projecting onto the subspace with coordinates indexed by $\mathcal{B}_{w,P}$ we obtain $(n_{\check{\mu}_w(i)})_{i\in I_w^P} = \sum_{k\in I_w^P} m_k(\langle\omega_k, \check{\mu}_w(i)\rangle)_{i\in I_w^P}$. But the matrix $(\langle\omega_k, \check{\mu}_w(i)\rangle)_{i,k\in I_w^P}$ is invertible over $\mathbb{Z}$ by Lemma 5.17(3). Therefore it follows that the m_k all lie in $\mathbb{Z}$. In other words, D is Cartier and we have proved (2). Finally, (3) follows from (1) combined with Proposition 5.2(3).

We have now finished proving Theorem 5.3. We finally observe that Theorem 5.3 implies Theorem A from the introduction using that $\check{R}^+_{w,P}$ indexes the Schubert divisors, so that $|\check{R}^+_{w,P}| = b_{2\ell(w)-2}(\mathbb{X}_{w,P})$, and that $|I_w^P| = b_2(\mathbb{X}_{w,P})$, compare Proposition 4.2.

6. Gorenstein and Fano Schubert varieties

We now use the constructions from the previous section to study the anticanonical divisors of our Schubert varieties $\mathbb{X}_{w,P}$. As a starting point we note that we can make the following explicit choice of an anticanonical divisor. The following proposition was implicit in [Ra87, KLS14], and is explained as [LSZ23, Proposition 2.2 and Lemma 3.8] with a replacement of w_0w_P by w. Here we include the arguments adapted therein verbatim for completeness.

Proposition 6.1. *For a general Schubert variety $\mathbb{X}_{w,P}$ an anticanonical divisor is given by*

$$-K_{\mathbb{X}_{w,P}} = \sum_{j\in I_w^B} \mathbb{X}^{s_j}_{w,P} + \sum_{\check{\eta}\in\check{R}^+_{w,P}} \mathbb{X}_{ws_{\check{\eta}},P},$$

in terms of Schubert and projected Richardson varieties.

Proof. By [KLS14, Lemma 5.4], the sum of all projected Richardson hypersurface in $\mathbb{X}_{w,P} = \mathbb{X}^{\mathrm{id}}_{w,P}$ gives an anticanonical divisor of $\mathbb{X}_{w,P}$. Hence, by [KLS14, Corollary 3.2],

$$-K_{\mathbb{X}_{w,P}} = \sum_{s_j\le_P w} \mathbb{X}^{s_j}_{w,P} + \sum_{\mathrm{id}\le_P x\prec w} \mathbb{X}^{\mathrm{id}}_{x,P}.$$

Here $\le_P$ denotes the P-Bruhat order, namely $u \le_P v$ if and only if there is a chain $u = u_1 \lessdot u_2 \lessdot \cdots \lessdot u_m = v$ of Bruhat covers such that $\pi(u_1) < \pi(u_2) < \cdots < \pi(u_m)$, where $\pi(u_m) \in W^P$ denotes the minimal length representative of the coset u_mW_P. In the second sum, we notice $\mathbb{X}^{\mathrm{id}}_{x,P} = \mathbb{X}_{x,P} = \mathbb{X}_{ws_{\check{\eta}},P}$ for $\check{\eta} \in \check{R}^+_{w,P}$. In the first sum, by definition, $s_j \le_P w$ holds only if $s_j \le w$, namely $j \in I_w^B$.

It remains to verify that $\mathbb{X}^{s_j}_{w,P}$ does indeed have codimension 1 in $\mathbb{X}_{w,P}$. If $j \in I_w^P$ this is clear by Remark 2.1. Now we assume $j \in I_w^B\backslash I_w^P$, and take a reduced expression $w = s_{i_1}\cdots s_{i_r}$

with s_j appearing at the rightmost at ℓ-th position. As in the proof of [LSZ23, Lemma 3.8], we set $u_1 := s_j$ and

$$u_m := \begin{cases} s_{i_{r-m+1}} \cdots s_{i_{r-1}} s_{i_r}, & \text{if } r-\ell+1 \leq m \leq r, \\ s_{i_\ell} s_{i_{r-m+2}} \cdots s_{i_{r-1}} s_{i_r}, & \text{if } 2 \leq m \leq r-\ell. \end{cases}$$

By [LSZ23, Lemma 3.6 (1)], we have $u_m \in W^P$ for $r-\ell+1 \leq m \leq r$, and $s_{i_\ell} u_m \in W^P$ for $2 \leq m \leq r-\ell$.

Let $2 \leq m \leq r-\ell$. If $s_{i_\ell} u_m \neq u_m s_{i_\ell}$, then we have $u_m \in W^P$ by [LSZ23, Lemma 3.6 (2)]. If the inequalities hold for all such m, then we have $\pi(u_m) = u_m$, and notice $\pi(u_1) = \pi(s_j) = \mathrm{id}$. Then $s_j \leq_P w$ by definition. Otherwise, we let a denote the maximal index in $\{2, \cdots, r-\ell\}$ with the equality $s_{i_\ell} u_a = u_a s_{i_\ell}$. Let $v_k = s_{i_\ell} u_k s_{i_\ell}$ for $1 \leq k \leq a$. We have $s_j = v_1 \lessdot v_2 \lessdot \cdots \lessdot v_a \lessdot u_{a+1} \lessdot \cdots \lessdot u_r$, and their projections give $\mathrm{id} \lessdot v_2 s_{i_\ell} \lessdot \cdots \lessdot v_a s_{i_\ell} < u_{a+1} \lessdot u_{a+2} \lessdot \cdots \lessdot u_r$. Hence, we still have $s_j \leq_P w$. $\square$

Now we characterise which Schubert varieties in G/P are Gorenstein in the simply-laced setting, and which of those are Fano. We also describe the anticanonical line bundle for $\mathbb{Q}$-factorial Schubert varieties of arbitrary type, and determine when it is ample. For Schubert varieties in SL_n/B, explicit combinatorial conditions on $w \in S_n$ characterising when $\mathbb{X}_{s,B}$ is Gorenstein were given in [WY06]. The problem to find a characterisation in more general types was posed in [WY06] and again in [WY23].

Definition 6.2. Assume G is simply-laced, and $\mathcal{B}_{w,P} = \{\check{\mu}_w(k) \mid k \in I_w^P\}$ is the subset of $\check{R}^+_{w,P}$ constructed above. By Lemma 5.17 we may consider $\mathcal{B}_{w,P}$ as a basis of $H_2(\mathbb{X}_{w,P}, \mathbb{Z})$. Let

$$\mathcal{B}^*_{w,P} := \{\check{\mu}^*_w(k) \mid k \in I_w^P\} \subset H^2(\mathbb{X}_{w,P}, \mathbb{Z})$$

denote the dual basis. If G is not simply-laced, but $\mathbb{X}_{w,P}$ is $\mathbb{Q}$-factorial, then set $\mathcal{B}_{w,P} := \check{R}^+_{w,P}$. This gives a $\mathbb{Q}$-basis of $H_2(\mathbb{X}_{w,P}, \mathbb{Q})$, see Proposition 5.2, and in the factorial case, $\check{R}^+_{w,P}$ gives a $\mathbb{Z}$-basis of $H_2(\mathbb{X}_{w,P}, \mathbb{Z})$. We again define $\mathcal{B}^*_{w,P}$ as the dual basis to $\check{R}^+_{w,P}$. Recall that we identify $H_2(\mathbb{X}_{w,P}, \mathbb{Z})$ with $\mathcal{Q}_{w,P}$ and $H^2(\mathbb{X}_{w,P}, \mathbb{Z})$ with $\mathcal{P}_{w,P}$, compare Definition 5.1.

Definition 6.3. Consider a Schubert variety $\mathbb{X}_{w,P}$, where $w \in W^P$. If G is simply-laced, we associate to $\mathbb{X}_{w,P}$ the following square matrix

$$M_{w,P} \quad := \quad (\langle \omega_i, \check{\mu}_w(k) \rangle)_{k,i \in I_w^P},$$

where the $\check{\mu}_w(k)$ are the elements of the set $\mathcal{B}_{w,P}$ constructed in Lemma 5.17. We also define

$$N_{w,P} \quad := \quad (\langle \check{\mu}^*_w(k), \check{\alpha}_i \rangle)_{i,k \in I_w^P}.$$

Equivalently, $N_{w,P}$ is the inverse matrix of $M_{w,P}$, which we note is invertible over $\mathbb{Z}$ by Lemma 5.17. The columns of $N_{w,P}$ are the coordinates of $\check{\mu}^*_w(k)$ in terms of the basis $\{\omega_i \mid i \in I_w^P\}$ of $\mathcal{P}_{w,P}$.

Remark 6.4. As a consequence of Lemma 5.17, we have that $M_{w,P}$ is obtained from $M_{w,B}$ simply by deleting the rows and columns indexed by $j \in I_P$.

Definition 6.5 (Simply-laced case $\hat{\mathbf{n}}_{w,P}$)**.** We now use the basis $\mathcal{B}_{w,P} = \{\check{\mu}_w(k) \mid k \in I_w^P\}$ to define a vector $\hat{\mathbf{n}}_{w,P} = (\hat{n}_k)_k$ in $\mathbb{Z}^{I_w^P}$. To $\check{\mu}_w(k)$ associate its height $\mathrm{ht}(\check{\mu}_w(k)) = \langle \rho, \check{\mu}_w(k) \rangle$, and let $\mathbf{h} = (\mathrm{ht}(\check{\mu}_w(k)))_{k \in I_w^P}$. Also let $\mathbf{1} = (1)_{k \in I_w^P}$. Then

$$\hat{\mathbf{n}}_{w,P} := N_{w,P}(\mathbf{h} + \mathbf{1}). \tag{6.1}$$

If $P = B$ the expression for $\hat{\mathbf{n}}_{w,P}$ simplifies. Namely, in this case we have $\sum_{i\in I_w^B}\langle\omega_i,\check\eta\rangle = \langle\rho,\check\eta\rangle$ for any $\check\eta \in \check R^+_{w,B}$, using that $\rho = \sum_{i\in I}\omega_i$ and $\langle\omega_i,\check\eta\rangle = 0$ whenever $i \notin I_w^B$. Therefore $\mathbf{h} = M_{w,B}(\mathbf{1})$ and

$$\hat{\mathbf{n}}_{w,B} = N_{w,B}(M_{w,B}(\mathbf{1}) + \mathbf{1}) = \mathbf{1} + N_{w,B}(\mathbf{1}). \tag{6.2}$$

Thus, for a Schubert variety in G/B, the entry $\hat n_i$ of $\hat{\mathbf{n}}_{w,B}$ is simply 1 plus the sum of the row of $N_{w,B}$ indexed by i. We write $\mathbf{n}_{w,B} = (n_i)_{i\in I_w^B}$ for the vector $N_{w,B}(\mathbf{1}) = \hat{\mathbf{n}}_{w,B} - \mathbf{1}$.

Remark 6.6. In general, for $\check\eta \in \check R^+_{w,P}$ with $P \neq B$ there are two natural notions of 'height'. One is the standard one used above, $\mathrm{ht}(\check\eta) = \langle\rho,\check\eta\rangle$. But the more natural from the geometric perspective is $\mathrm{ht}_P(\check\eta) := \sum_{i\in I_w^P}\langle\omega_i,\check\eta\rangle$. Namely, this one reflects the degree of the curve class in $H_2(\mathbb{X}_{w,P})$ associated to $\check\eta$ in the minimal projective embedding of $\mathbb{X}_{w,P}$. We clearly have $\mathrm{ht}_P(\check\eta) \le \mathrm{ht}(\check\eta)$ and the vector $\mathbf{h}_P = (\mathrm{ht}_P(\check\mu_w(k)))_{k\in I_w^P}$ is related to $M_{w,P}$ by taking row sums, $\mathbf{h}_P = M_{w,P}(\mathbf{1})$. If the difference between these two versions of 'height' for $\check\mu_w(k) \in \mathcal{B}_{w,P}$ is denoted $d_k = \langle\sum_{j\in I_P}\omega_j,\check\mu_w(k)\rangle$ and $\mathbf{d} = (d_k)_{k\in I_w^P}$, then the generalisation of (6.2) using $\mathbf{h} = \mathbf{h}_P + \mathbf{d}$ is

$$\hat{\mathbf{n}}_{w,P} = N_{w,P}(M_{w,P}(\mathbf{1}) + \mathbf{d} + \mathbf{1}) = \mathbf{1} + N_{w,P}(\mathbf{d} + \mathbf{1}).$$

Note that $\mathbf{d}$ captures precisely the part of the height of $\check\eta_w(k)$ which is also affected by the P-adaptation process, compare Lemma 5.15.

Proposition 6.7. *Suppose G is of simply-laced type. The following conditions are equivalent*

(1) $\mathbb{X}_{w,P}$ *is Gorenstein.*

(2) $\left(\sum_{k\in I_w^P}\hat n_k\langle\omega_k,\check\eta\rangle\right) - \mathrm{ht}(\check\eta) = 1$ *for all* $\check\eta \in \check R^+_{w,P}\setminus\mathcal{B}_{w,P}$.

If $\mathbb{X}_{w,P}$ is Gorenstein, then the anticanonical line bundle on $\mathbb{X}_{w,P}$ is given by $\mathcal{L}_\lambda|_{\mathbb{X}_{w,P}}$, where $\lambda = \sum_{i\in I_w^P}\hat n_i\omega_i$. In particular, λ represents the first Chern class $c_1(\mathbb{X}_{w,P})$.

Moreover, in the $P = B$ case, the condition (2) can be replaced by

(2') $\sum_{i\in I_w^B} n_i\langle\omega_i,\check\eta\rangle = 1$ *for all* $\check\eta \in \check R^+_{w,B}\setminus\mathcal{B}_{w,B}$,

where $n_i = \hat n_i - 1$ as in Definition 6.5.

Remark 6.8. For general $w \in W^P$, the numbers $\hat n_i$ are non-canonical, depending on the construction of $\mathcal{B}_{w,P}$. However, whenever $\mathbb{X}_{w,P}$ is Gorenstein the sum $\sum_{i\in I_w^P}\hat n_k\omega_k$ will not depend on the choices made in the construction of the basis $\mathcal{B}_{w,P}$ and $\hat{\mathbf{n}}_{w,P}$. See also Remark 6.11. The condition (2') is the condition (1.3) from the introduction written out in coordinates.

For $\mathbb{Q}$-factorial $\mathbb{X}_{w,P}$ of general Lie type we may describe the Gorenstein condition in the similar terms.

Definition 6.9 ($\mathbb{Q}$-factorial case)**.** Suppose $\mathbb{X}_{w,P}$ is $\mathbb{Q}$-factorial, now of general type. We have a square, integer matrix associated to $\mathbb{X}_{w,P}$,

$$M_{w,P} := (\langle\omega_k,\check\eta\rangle)_{\check\eta\in\mathcal{B}_{w,P},k\in I_w^P},$$

which is invertible over $\mathbb{Q}$, with $\mathcal{B}_{w,P} = \check R^+_{w,P}$ as in Definition 6.2. We denote by

$$N_{w,P} = (n_{k,\check\eta})_{k\in I_w^P,\check\eta\in\check R^+_{w,P}}$$

the inverse of $M_{w,P}$ (over $\mathbb{Q}$ in general). In simply-laced type we may index the elements of $\check R^+_{w,P}$ by $k \in I_w^P$ as in Lemma 5.17 so that $M_{w,P}$ and $N_{w,P}$ agree with the matrices from Definition 6.3 in that case. We let $\hat{\mathbf{n}}_{w,P} := N_{w,P}(\mathbf{h} + \mathbf{1})$, where $\mathbf{h}$ is the vector of heights $(\mathrm{ht}(\check\eta))_{\check\eta}$ and $\mathbf{1} = (1,\dots,1)$, as in Definition 6.5. Here this formula defines an element of $\mathbb{Q}^{\check R^+_{w,P}}$.

Proposition 6.10. *Suppose $\mathbb{X}_{w,P}$ is $\mathbb{Q}$-factorial (of arbitrary Lie type), and let $\hat{\mathbf{n}}_{w,P}$ be the vector from Definition 6.9 with entries $\hat{n}_k \in \mathbb{Q}$. Then the anticanonical class is given by*

$$[-K_{\mathbb{X}_{w,P}}] = \sum_{k \in I_w^P} \hat{n}_k [\mathbb{X}_{w,P}^{s_k}],$$

and thus $\mathbb{X}_{w,P}$ is Gorenstein if and only if $\hat{n}_k \in \mathbb{Z}$ for all $k \in I_w^P$.

Remark 6.11. In the situation of Proposition 6.10 or Proposition 6.7, but with $P = B$, we let $n_i = \sum_{\check{\eta} \in \check{R}^+_{w,B}} n_{i,\check{\eta}}$ be the row sum of $N_{w,B}$ for the row labeled i, as in Definition 6.5. If $\mathbb{X}_{w,P}$ is $\mathbb{Q}$-Gorenstein then the anticanonical class of $\mathbb{X}_{w,B}$ is given by

$$[-K_{\mathbb{X}_{w,B}}] = \sum_{i \in I_w^B} (n_i + 1)[\mathbb{X}_w^{s_i}].$$

Moreover, $\mathbb{X}_{w,B}$ is Gorenstein if and only if the row sums n_i of $N_{w,B}$ lie in $\mathbb{Z}$ for all $i \in I_w^B$. In this case we also have a natural description of the first Chern class of $\mathbb{X}_{w,B}$ under the identification of $H^2(\mathbb{X}_{B,w}, \mathbb{Z})$ with $\mathcal{P}_{w,B}$. Namely, $\mathcal{P}_{w,B}$ has two bases, $\{\omega_i \mid i \in I_w\}$ and the dual basis to $\mathcal{B}_{w,B}$ that we may denote as $\{\check{\eta}_w^*(i) \mid i \in I_w^B\}$. The first Chern class is equal to the sum of these two bases, $c_1(\mathbb{X}_{w,B}) = \sum_{i \in I_w} \omega_i + \sum_{i \in I_w} \check{\eta}_w^*(i)$. Note that here, despite the $\check{\eta}_w^*(i)$ being non-canonical, their sum is canonical. For example the Schubert variety in SL_5/B associated to the permutation 53142 discussed in Example 5.11 has two different possible choices for $\mathcal{B}_{w,B}^*$. But the sum of the dual basis elements is the same in either choice. This Schubert variety is indeed Gorenstein as can be verified using Proposition 6.7, but also independently using [WY06].

The two propositions, Proposition 6.7 and Proposition 6.10, can now be proved together.

Proof of Proposition 6.7 and Proposition 6.10. First, let us express the anticanonical divisor class $[-K_{\mathbb{X}_{w,P}}]$ in terms of the Schubert basis. We have in G/B that for any $j \in I_w^B$,

$$[\mathbb{X}_{w,B}^{s_j}] = [\mathbb{X}^{s_j,B}].[\mathbb{X}_{w,B}] = \sum_{\check{\eta} \in \check{R}^+_{w,B}} \langle \omega_j, \check{\eta} \rangle [\mathbb{X}_{ws_{\check{\eta}},B}],$$

using the Chevalley formula as in the proof of Proposition 4.2, see (4.2). Thus, for the projected Richardson variety, $\mathbb{X}_{w,P}^{s_j} = \pi(\mathbb{X}_{w,B}^{s_j})$ we have in $H_{2\ell(w)-2}(\mathbb{X}_{w,P}, \mathbb{Z})$ that

$$[\mathbb{X}_{w,P}^{s_j}] = \pi_*[\mathbb{X}_{w,B}^{s_j}] = \sum_{\check{\eta} \in \check{R}^+_{w,P}} \langle \omega_j, \check{\eta} \rangle [\mathbb{X}_{ws_{\check{\eta}},P}],$$

using that $\pi_*[\mathbb{X}_{ws_{\check{\eta}},B}] = 0$ if $\check{\eta} \in \check{R}^+_{w,B} \setminus \check{R}^+_{w,P}$ for dimension reasons. Note however that unlike in Proposition 4.2, the divisor $\mathbb{X}_{w,P}^{s_j}$ may not be a Richardson variety in $\mathbb{X}_{w,P}$ (or Cartier), since j is now not assumed to lie in I_w^P.

Recall that $[-K_{\mathbb{X}_{w,P}}]$ involves the sum of all $[\mathbb{X}_{w,P}^{s_j}]$ for $j \in I_w^B$. We have that $\sum_{j \in I_w^B} \langle \omega_j, \check{\eta} \rangle = \langle \rho, \check{\eta} \rangle = \mathrm{ht}(\check{\eta})$, for any $\check{\eta} \in \check{R}^+_{w,B}$. Therefore

$$[-K_{\mathbb{X}_{w,P}}] = \sum_{j \in I_w^B} [\mathbb{X}_{w,P}^{s_j}] + \sum_{\check{\eta} \in \check{R}^+_{w,P}} [\mathbb{X}_{ws_{\check{\eta}},P}] = \sum_{\check{\eta} \in \check{R}^+_{w,B}} (\mathrm{ht}(\check{\eta}) + 1)[\mathbb{X}_{ws_{\check{\eta}},P}]. \tag{6.3}$$

We have that $\mathbb{X}_{w,P}$ is Gorenstein if and only if $-K_{\mathbb{X}_{w,P}}$ is Cartier. Assuming $-K_{\mathbb{X}_{w,P}}$ is Cartier we make an ansatz for the associated line bundle. Namely we set $\mathcal{O}(-K_{\mathbb{X}_{w,P}}) = \mathcal{L}_\mu|_{\mathbb{X}_{w,P}}$ where

$\mu = \sum_{k \in I_w^P} m_k \omega_k$ is to be determined. Then we compute the associated divisor class as in the proof of Proposition 4.2 and separate out the summands corresponding to $\check{\mu}_w(k) \in \mathcal{B}_{w,P}$,

$$\operatorname{div}(\mathcal{L}_\mu|_{\mathbb{X}_{w,P}}) = \sum_{\check{\eta} \in \check{R}^+_{w,P}} \langle \mu, \check{\eta} \rangle [\mathbb{X}_{ws_{\check{\eta}},P}] = \sum_{k \in I_w^P} \langle \mu, \check{\mu}_w(k) \rangle [\mathbb{X}_{ws_{\check{\mu}_w(k)},P}] + \sum_{\check{\eta} \in \check{R}^+_{w,P} \setminus \mathcal{B}_{w,P}} \langle \mu, \check{\eta} \rangle [\mathbb{X}_{ws_{\check{\eta}},P}].$$

Comparing the formula for $\operatorname{div}(\mathcal{L}_\mu|_{\mathbb{X}_{w,P}})$ with the one for $-K_{\mathbb{X}_{w,P}}$ in (6.3) and projecting onto the span of the Schubert basis elements $[\mathbb{X}_{ws_{\check{\mu}_w(k)},P}]$ corresponding the $\check{\mu}_w(k) \in \mathcal{B}_{w,P}$, we have

$$\sum_{k \in I_w^P} \langle \mu, \check{\mu}_w(k) \rangle [\mathbb{X}_{ws_{\check{\mu}_w(k)},P}] = \sum_{k \in I_w^P} (\operatorname{ht}(\check{\mu}_w(k)) + 1) [\mathbb{X}_{ws_{\check{\mu}_w(k)},P}].$$

In terms of the coefficient vector $\mathbf{m} = (m_k)_{k \in I_w^P}$ of $\mu = \sum_{k \in I_w^P} m_k \omega_k$, this equation can be rephrased as $M_{w,P}(\mathbf{m}) = \mathbf{h} + \mathbf{1}$, compare Definition 6.5. Thus μ is uniquely determined by just these summands, and has coefficients given by $\mathbf{m} = N_{w,P}(\mathbf{h} + \mathbf{1}) = \hat{\mathbf{n}}_{w,P} = (\hat{n}_k)_k$.

The condition for $\mathcal{L}_\mu|_{\mathbb{X}_{w,P}}$ to be the anticanonical line bundle is now that for any remaining $\check{\eta} \in \check{R}^+_{w,P} \setminus \mathcal{B}_{w,P}$ we have

$$\sum_{k \in I_w^P} \hat{n}_k \langle \omega_k, \check{\eta} \rangle = \operatorname{ht}(\check{\eta}) + 1. \tag{6.4}$$

This translates to the condition (2) in Proposition 6.7. Moreover, condition (2) is equivalent to (2') in the $P = B$-case by (6.2). The formula for $[-K_{\mathbb{X}_{w,P}}]$ and the characterisation of the Gorenstein property in Proposition 6.7 and Proposition 6.10 also follow. ☐

Corollary 6.12. *Let $\mathbb{X}_{w,P}$ be either a factorial Schubert variety of arbitrary Lie type or a simply-laced Gorenstein Schubert variety, and let $N_{w,P}$ and $\hat{\mathbf{n}}_{w,P} = (\hat{n}_k)_{k \in I_w^P}$ be as in Definition 6.3, Definition 6.5 and Definition 6.9. The following are equivalent.*

1. *$\mathbb{X}_{w,P}$ is a Gorenstein Fano variety.*
2. *$\hat{\mathbf{n}}_{w,P} \in \mathbb{Z}_{>0}^{I_w^P}$.*

Moreover if $P = B$ then $\mathbb{X}_{w,B}$ is Fano if and only if the row-sum vector $N_{w,B}(\mathbf{1}) \geq 0$.

Proof. By Proposition 6.7 and Proposition 6.10 the anticanonical line bundle is $\mathcal{L}_\lambda|_{\mathbb{X}_{w,P}}$ with $\lambda = \sum_{k \in I_w^P} \hat{n}_k \omega_k$. Let us assume that w has full support, so that $I_w^P = I^P$. Otherwise we may replace G with a Levi subgroup. If all $\hat{n}_k > 0$ then $\mathcal{L}_\lambda$ is ample on G/P, and therefore its restriction $\mathcal{L}_\lambda|_{\mathbb{X}_{w,P}}$ is also ample. Conversely, if $\mathcal{L}_\lambda|_{\mathbb{X}_{w,P}}$ is ample, then for any $k \in I^P$ the restriction to $\mathbb{X}_{s_k,P} \subset \mathbb{X}_{w,P}$ is ample (and equals $\mathcal{O}_{\mathbb{X}_{s_k,P}}(\hat{n}_k)$). This implies $\hat{n}_k > 0$. The characterisation in the $P = B$ case follows from the second half of Definition 6.5. ☐

We note that **Theorem D** and **Theorem C** are the combination of Proposition 6.7, Proposition 6.10 and Corollary 6.12.

7. Examples

Recall that we simply denote a coroot $\check{\alpha}_{i_1} + \check{\alpha}_{i_2} + \cdots + \check{\alpha}_{i_k}$ as $\check{\alpha}_{i_1 i_2 \ldots i_k}$.

Example 7.1. Consider G/B of type A_4 and let $w = s_3 s_4 s_1 s_2 s_3$. Below we show that the (singular) Schubert variety $\mathbb{X}_{w,B}$ in G/B is Gorenstein Fano but is not factorial.

We have that every element of $\check{R}^+(w)$ is indecomposable, so that

$$\check{R}^+_{w,B} = \check{R}^+(w) = \{\check{\alpha}_3, \check{\alpha}_{23}, \check{\alpha}_{123}, \check{\alpha}_{34}, \check{\alpha}_{234}\}.$$

Meanwhile $I_w^B = I = [4]$ and $\operatorname{Pic}(\mathbb{X}_{w,B}) \cong \mathbb{Z}^4$. So $\mathbb{X}_{w,B}$ is not factorial by Theorem B.

We now construct the subset $\mathcal{B}_{w,P}$ of $\check{R}^+_{w,P}$, by for each $i \in I^B_w$ selecting the rightmost occurrence of s_i in any reduced expression of w and letting $\check{\mu}_w(i)$ be the associated positive root. This gives us

$$\mathcal{B}_{w,B} = \{\check{\mu}_w(3) = \check{\alpha}_3, \check{\mu}_w(2) = \check{\alpha}_{23}, \check{\mu}_w(1) = \check{\alpha}_{123}, \check{\mu}_w(4) = \check{\alpha}_{34}\} \tag{7.1}$$

We now order the indexing set as follows $I^B_w = \{3, 2, 1, 4\}$. This ordering is compatible with the position of the root $\check{\mu}_w(k)$ in its optimal reflection ordering. The entries of the matrix $M_{w,B} = (\langle \omega_i, \check{\mu}_w(k)\rangle)_{k,i}$ can be read off row by row, directly from (7.1), and we have

$$M_{w,B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \qquad N_{w,B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix},$$

where $N_{w,B}$ is the inverse of $M_{w,B}$. We may read off the dual basis $\mathcal{B}^*_{w,B}$ from the columns of $N_{w,B}$ (keeping in mind the ordering $3, 2, 1, 4$ of I^B_w). Namely,

$$\mathcal{B}^*_{w,B} = \{\check{\mu}^*_w(3) = \omega_3 - \omega_2 - \omega_4,\ \check{\mu}^*_w(2) = \omega_2 - \omega_3,\ \check{\mu}^*_w(1) = \omega_1,\ \check{\mu}^*_w(4) = \omega_4\}.$$

We now have $\sum_k \check{\mu}^*_w(k) = \omega_3$ and we see that $\mathbb{X}_{w,B}$ is Gorenstein, because $\check{R}^+_{w,B} \backslash \mathcal{B}_{w,B} = \{\check{\alpha}_{234}\}$ and $\langle \omega_3, \check{\alpha}_{234}\rangle = 1$. We have $\hat{\mathbf{n}}_{w,B} = \mathbf{1} + N_{w,B}(\mathbf{1}) = (2, 1, 1, 1)$ implying Fano by Corollary 6.12. The anticanonical line bundle is given by

$$\mathcal{L}_{2\omega_3+\omega_2+\omega_1+\omega_4}|_{\mathbb{X}_{w,B}}.$$

Example 7.2. Consider G/P of type A_4 with $I^P = \{1, 2, 3\}$ and $I_P = \{4\}$. Then we consider the same Weyl group element $w = s_3s_4s_1s_2s_3$, now as element of W^P, and below we show that the Schubert variety $\mathbb{X}_{w,P}$ is both Gorenstein Fano and factorial. Since $ws_3 \notin W^P$ and $ws_3s_2s_3 \notin W^P$, we must remove the elements $\check{\alpha}_3, \check{\alpha}_{23}$ from $\check{R}^+_{w,B}$ to obtain

$$\check{R}^+_{w,P} = \{\check{\alpha}_{123}, \check{\alpha}_{34}, \check{\alpha}_{234}\}.$$

The two elements $\check{\mu}^B_w(1) = \check{\alpha}_{123}$ and $\check{\mu}^B_w(4) = \check{\alpha}_{34}$ of $\check{R}^+_{w,P}$ appeared in $\mathcal{B}_{w,B}$ in Example 7.1, but the third didn't. We now demonstrate how the recursive construction of $\mathcal{B}_{w,P}$ from Definition 5.16 gives us $\check{R}^+_{w,P}$ (as it must).

We begin with $\check{\mu} = \check{\mu}^B_w(3)$ which does not have $ws_{\check{\mu}}(\check{\alpha}_4) > 0$ (and therefore $ws_{\check{\mu}} \notin W^P$). The recursion tells us to replace $\check{\mu}^B_w(3)$ by $s_{\check{\mu}^B_w(3)}(\check{\alpha}_4) = \check{\mu}^B_w(3) + \check{\alpha}_4 = \check{\alpha}_3 + \check{\alpha}_4$, and this now lies in $\check{R}^+_{w,P}$. So we set

$$\check{\mu}^P_w(3) := \check{\alpha}_3 + \check{\alpha}_4.$$

Next we note that $\check{\mu} = \check{\mu}^B_w(2) = \check{\alpha}_{23}$ does not have $ws_{\check{\mu}}(\check{\alpha}_4) > 0$, so we analogously replace it by

$$\check{\mu}^P_w(2) := \check{\alpha}_2 + \check{\alpha}_3 + \check{\alpha}_4.$$

The remaining element (indexed by I^P_w) is $\check{\mu}^B_w(1)$, which does lie in $\check{R}^+_{w,P}$ already, and so does not need to be replaced. Thus altogether we have

$$\mathcal{B}_{w,P} = \{\, \check{\mu}^P_w(3) = \check{\alpha}_{34},\ \check{\mu}^P_w(2) = \check{\alpha}_{234},\ \check{\mu}^P_w(1) = \check{\alpha}_{123}\,\}. \tag{7.2}$$

This illustrates the recursive procedure. Note that in this example the cardinality of $\check{R}^+_{w,P}$ was already minimal, equal to the rank $|I^P_w| = 3$ of $\mathrm{Pic}(\mathbb{X}_{w,P})$. Therefore, as a set we were bound to arrive at $\mathcal{B}_{w,P} = \check{R}^+_{w,P}$ in this case. Unlike the full flag variety example $\mathbb{X}_{w,B}$ from above, the Schubert variety $\mathbb{X}_{w,P}$ in G/P is factorial. In fact, $\mathbb{X}_{w,P}$ in G/P is a smooth Schubert variety by [GR02].

We read off the entries of the matrix $M_{w,P} = (\langle \omega_i, \check\mu_w(k)\rangle)_{k,i}$ now with $i,k \in I_w^P$, row by row directly from (7.2). This gives

$$M_{w,P} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \qquad N_{w,P} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix},$$

where $N_{w,P}$ is the inverse of $M_{w,P}$. We may read off the dual basis $\mathcal{B}^*_{w,P}$ from the columns of $N_{w,P}$ (keeping in mind the ordering $3,2,1$ of I_w^P). Namely,

$$\mathcal{B}^*_{w,P} = \{\check\mu^*_w(3) = \omega_3 - \omega_2,\ \check\mu^*_w(2) = \omega_2 - \omega_1,\ \check\mu^*_w(1) = \omega_1\}.$$

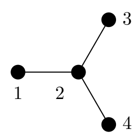
We also have $\hat{\mathbf{n}}_{w,P} = N_{w,P}(\mathbf{h} + \mathbf{1}) = (3,1,0)$. Using $\hat{\mathbf{n}}_{w,P}$ and Proposition 6.7, or equivalently using (1.1), we find that

$$c_1(\mathbb{X}_{w,P}) = \sum_{k=1}^{3} (\mathrm{ht}(\check\mu_w(k)) + 1)\mu^*_w(k) = 3(\omega_3 - \omega_2) + 4(\omega_2 - \omega_1) + 4\omega_1 = 3\omega_3 + \omega_2.$$

Hence $\mathbb{X}_{w,P}$ is semi-Fano with anticanonical line bundle given by

$$\mathcal{L}_{3\omega_3+\omega_2}|_{\mathbb{X}_{w,P}}.$$

Example 7.3. The following Schubert variety in G/B of type D_4 is not Gorenstein.



$$w = s_1 s_3 s_4 s_2 s_1 s_3 s_4 s_2$$

We compute first the set $\check R^+_{w,B}$ to be

$$\check R^+_{w,B} = \{\check\alpha_2, \check\alpha_{24}, \check\alpha_{23}, \check\alpha_{12}, \check\alpha_1 + 2\check\alpha_2 + \check\alpha_3 + \check\alpha_4, \check\alpha_{123}, \check\alpha_{124}, \check\alpha_{234}\}.$$

Within that we have the basis

$$\mathcal{B}_{w,B} = \{\check\mu_w(2) = \check\alpha_2,\ \check\mu_w(1) = \check\alpha_{12},\ \check\mu_w(3) = \check\alpha_{23},\ \check\mu_w(4) = \check\alpha_{24}\}.$$

The dual basis elements are given by

$$\check\mu^*_w(2) = \omega_2 - \omega_1 - \omega_3 - \omega_4, \quad \check\mu^*_w(1) = \omega_1, \quad \check\mu^*_w(3) = \omega_3, \quad \check\mu^*_w(4) = \omega_4$$

The sum $\sum_k \check\mu^*_w(k) = \omega_2$. However, the highest coweight, $\check\theta = \check\alpha_1 + 2\check\alpha_2 + \check\alpha_3 + \check\alpha_4$ is an element of $\check R^+_{w,B} \setminus \mathcal{B}_{w,B}$, and it has the property $\langle \sum_k \check\mu^*_w(k), \check\theta\rangle = \langle \omega_2, \check\theta\rangle = 2$. Therefore the Schubert variety $\mathbb{X}_{w,B}$ is not Gorenstein by the formulation (1.3) of Theorem D. Note that the condition (2') from Proposition 6.7 amounts to the exact same calculation.

Example 7.4 (Schubert surfaces and 3-folds in G/B for type G_2)**.** Let G be of type G_2 with α_1 being the long simple root as in Example 3.8. Recall that $M_{w,B} = \big(\langle \omega_i, \check\eta\rangle\big)$ has columns corresponding to ω_i indexed by I_w^B, and rows corresponding to $\check\eta$ from $\check R^+_{w,B}$. Both of these sets have cardinality 2, where we are assuming w is of length 2 or 3 in W of type G_2. Recall that $N_{w,B}$ denotes the inverse of $M_{w,B}$.

1.) $\mathbb{X}_{s_1s_2,B}$. For $w = s_1 s_2$, we have $\check R^+_{w,B} = \{\check\alpha_2, \check\alpha_1 + \check\alpha_2\}$. We order $I_w^B = \{2,1\}$.

$$M_{s_1s_2,B} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad N_{s_1s_2,B} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad N_{s_1s_2,B}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Thus, this Schubert variety is Fano. Keeping in mind the ordering of I_w^B, we have $n_2 = 1$ and $n_1 = 0$, and the first Chern class of the anticanonical line bundle is $c_1(\mathbb{X}_{s_1s_2,B}) = \omega_1 + 2\omega_2$.

2.) $\mathbb{X}_{s_2s_1,B}$. For $w = s_2s_1$, we have $\check{R}^+_{w,B} = \{\check{\alpha}_1, 3\check{\alpha}_1 + \check{\alpha}_2\}$. We order $I^B_w = \{1, 2\}$.

$$M_{s_2s_1,B} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \quad N_{s_2s_1,B} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}, \quad N_{s_2s_1,B} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Thus $\mathbb{X}_{s_2s_1,B}$ is no longer Fano. Indeed, we have $c_1(\mathbb{X}_{s_2s_1,B}) = 2\omega_1 - \omega_2$.

3.) $\mathbb{X}_{s_1s_2s_1,B}$. For $w = s_1s_2s_1$, we order $I_{w,B} = \{1, 2\}$. In this case,

$$\check{R}^+_{s_1s_2s_1,B} = \{\check{\alpha}_1,\ 2\check{\alpha}_1 + \check{\alpha}_2\} \subset \check{R}^+(s_1s_2s_1) = \{\check{\alpha}_1,\ 3\check{\alpha}_1 + \check{\alpha}_2,\ 2\check{\alpha}_1 + \check{\alpha}_2\},$$

$$M_{s_1s_2s_1,B} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad N_{s_1s_2s_1,B} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \quad N_{s_1s_2s_1,B} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Hence, $\mathbb{X}_{s_1s_2s_1,B}$ is factorial, for $M_{s_1s_2s_1,B}$ being invertible over $\mathbb{Z}$. Moreover, $\mathbb{X}_{s_1s_2s_1,B}$ is not Fano (but only semi-Fano), with $c_1(\mathbb{X}_{s_1s_2s_1,B}) = 2\omega_1$.

4.) $\mathbb{X}_{s_2s_1s_2,B}$. For $w = s_2s_1s_2$, we order $I_{w,B} = \{2, 1\}$. In this case,

$$\check{R}^+_{s_2s_1s_2,B} = \{\check{\alpha}_2,\ 3\check{\alpha}_1 + 2\check{\alpha}_2\} \subset \check{R}^+(s_2s_1s_2) = \{\check{\alpha}_2,\ \check{\alpha}_1 + \check{\alpha}_2,\ 3\check{\alpha}_1 + 2\check{\alpha}_2\}.$$

$$M_{s_2s_1s_2,B} = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}, \quad N_{s_2s_1s_2,B} = \begin{pmatrix} 1 & 0 \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}, \quad N_{s_2s_1s_2,B} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{3} \end{pmatrix}.$$

Hence, $\mathbb{X}_{s_2s_1s_2,B}$ is $\mathbb{Q}$-factorial but not factorial, for $M_{s_2s_1s_2,B}$ being invertible over $\mathbb{Q}$ other than $\mathbb{Z}$. Moreover, it is $\mathbb{Q}$-Gorenstein Fano, with the first Chern class as an element of $H^2(\mathbb{X}_{s_2s_1s_2,B}, \mathbb{Q})$ given by $c_1(\mathbb{X}_{s_2s_1s_2,B}) = 2\omega_2 + \frac{2}{3}\omega_1$. In particular, $\mathbb{X}_{s_2s_1s_2,B}$ is singular.

We note that $\mathbb{X}_{s_1s_2s_1,B}$ (resp. $\mathbb{X}_{s_2s_1s_2,B}$) was known to be smooth (resp. singular) by [BP05, Theorem 2.4].

Remark 7.5. It was shown in [Fa98, Ka13] that a general Schubert variety $X_{w,B}$ in G/B is a (smooth) toric variety if and only if w is a product of distinct simple reflections. A characterisation of the (weak) Fano property of toric Schubert varieties was provided in [LMP23]. In Remark 1.5 of loc. cit., we also see that the Schubert variety $\mathbb{X}_{s_2s_1,B}$ of type G_2 is isomorphic to the Hirzebruch surface $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(3))$.

In addition, the Schubert variety $\mathbb{X}_{s_1s_2s_1,B}$ of type G_2 is an example of lowest dimension among those smooth non-Fano, non-toric Schubert varieties in a partial flag variety G/P. For Schubert varieties in the full flag variety G/B of type A, $X_{s_4s_2s_3s_2s_1,B}$ is a smooth non-Fano, non-toric Schubert variety $\mathbb{X}_{w,B}$, by using similar calculations to the above examples together with the criterion of the smoothness in [LSa90].

Example 7.6 (Schubert surfaces in G_2-Grassmannians)**.** Let G be of type G_2 and P_k denote the maximal parabolic subgroup with $I^{P_k} = \{k\}$. Notice that the natural projection $G/B \to G/P_k$ is a $\mathbb{P}^1$-bundle, and the restriction of the base G/P_k to the Schubert surface gives a $\mathbb{P}^1$-bundle $X_{s_2s_1s_2,B} \to X_{s_2s_1,P_1}$ (resp. $X_{s_1s_2s_1,B} \to X_{s_1s_2,P_2}$) when $k = 1$ (resp. 2).

Notice that G/P_2 is isomorphic to the smooth quadratic hypersurface in $\mathbb{P}^6$. Hence, the Schubert surface $X_{s_1s_2,P_2}$ in G/P_2 is isomophic to $\mathbb{P}^2$, being smooth Fano with $c_1(\mathbb{X}_{s_1s_2,P_2}) = 3\omega_2$. Nevertheless, its preimage $X_{s_1s_2s_1,B}$ is smooth but non-Fano by Example 7.4.

It follows directly from the properties of $X_{s_2s_1s_2,B}$ in Example 7.4 that the Schubert surface $X_{s_2s_1,P_1}$ in G/P_1 is singular, $\mathbb{Q}$-Gorenstein and $\mathbb{Q}$-factorial (since these are local properties and $X_{s_2s_1s_2,B} \to X_{s_2s_1,P_1}$ is a $\mathbb{P}^1$-bundle). Indeed, we can also see these properties from $R^+_{s_2s_1,P_1} = \{3\check{\alpha}_1 + \check{\alpha}_2\}$, which leads to $M_{s_2s_1,P_1} = (3)$ and $N_{s_2s_1,P_1} = (\frac{1}{3})$. As a consequence, $c_1(\mathbb{X}_{s_1s_2,P_2}) = \frac{5}{3}\omega_1$, so that $X_{s_2s_1,P_1}$ is $\mathbb{Q}$-Gorenstein Fano.

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