PRIMITIVE FORMS VIA POLYVECTOR FIELDS

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ABSTRACT. We develop a complex differential geometric approach to the theory of higher residues and primitive forms from the viewpoint of Kodaira-Spencer gauge theory, unifying the semi-infinite period maps for Calabi-Yau models and Landau-Ginzburg models. We give an explicit perturbative construction of primitive forms with respect to opposite filtrations and primitive elements. This leads to a concrete algorithm to compute the Taylor expansions of primitive forms as well as the description of their moduli space for all weighted homogenous cases. As an application, we show the uniqueness of primitive forms for exceptional unimodular singularities, and illustrate our perturbative formula by presenting unknown expressions for the primitive form of E_{12}-singularity.

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1. Introduction

1.1. **Motivations.** Around early 1980's, the third author introduced *primitive forms* [43–46] in his study of periods as to generalize the elliptic period integral theory to higher dimensions. One important consequence of a primitive form is the *flat structure* associated to the universal unfolding of a singularity. Later Dubrovin [18] introduced the notion of *Frobenius manifold structure* to axiomatize two-dimensional topological field theories on the sphere. It was soon realized that the aforementioned flat structure describes precisely the Frobenius manifold structure of topological Landau-Ginzburg models. The birth of mirror symmetry explodes the context of singularity or Landau-Ginzburg theory both in the mathematics and physics literature. As a consequence, primitive forms not only produce intrinsic structures associated to singularities, but also predict geometric quantities as periods on Calabi-Yau manifolds, Gromov-Witten type invariants, etc.

The concept of higher residues [43, 45] plays a key role in the theory of primitive forms. Motivated by mirror symmetry, Barannikov and Kontsevich [5] constructed a large class of Frobenius manifold structures on the formal extended moduli space of complex structures on Calabi-Yau manifolds. In the Calabi-Yau case, the structures of higher residues were formulated by Barannikov as the notion of *variation of semi-infinite Hodge structures* [2, 4], within which the *semi-infinite*

period map plays the role of a primitive form. Related concepts on semi-infinite Hodge structures are further interpreted by Givental as the symplectic geometry of Langrangian cone in a loop space formalism [23]. In [3], Barannikov used oscillating integrals to obtain the variation of semi-infinite Hodge structures for the singularity describing the mirror of projective spaces, and proved the corresponding mirror symmetry. This construction is generalized by Sabbah [40] and Douai-Sabbah [16, 17] to wider examples of Laurent polynomials. We refer to [27] for an introduction to this subject.

The Barannikov-Kontsevich construction is closely modeled on the Kodaira-Spencer gauge theory introduced by the physicists Bershadsky, Cecotti, Ooguri and Vafa [6] (which we will call BCOV theory). The systematic mathematical treatment of BCOV theory has been developed recently by Costello and the second author [14] based on the effective quantization of Givental's symplectic formalism. The precise relation between BCOV theory and the variation of semi-infinite Hodge structures is explained in [35]. See [34] also for a review.

The BCOV theory is a gauge theory of polyvector fields on Calabi-Yau manifolds, describing the closed string field theory of the B-twisted topological string. It has a natural extension [36] to Landau-Ginzburg model for the pair (X, f), where X is a Stein manifold and f is a holomorphic function on X with finite critical set. The gauge field is given by $\mathrm{PV}_c(X)[[t]]$, where $\mathrm{PV}_c(X)$ is the space of smooth polyvector fields with compact support and t is a formal variable. The action functional is constructed with the help of the *trace map* (See section 2.3)

$$\mathrm{Tr} : \mathrm{PV}_c(X)[[t]] \to \mathbb{C}[[t]]$$

by integrating out the polyvector fields with respect to a choice of holomorphic volume form. Surprisingly, this simple integration map lifts the third author's higher residue map at the cochain level (Section 2.3). In particular, the ordinary residue is lifted at the cochain level as the leading order of the trace map (Proposition 2.5). It is then an extremely intriguing question whether there is a deep connection between the theory of primitive forms and BCOV theory. A related work from string theoretical viewpoint was provided by Losev [37].

The initial purpose of the current paper is to reveal this connection and develop a complex differential geometric approach to the theory of higher residues and primitive forms from the viewpoint of BCOV theory. It turns out that such viewpoint leads to an algebraic perturbative formula of primitive forms, unifying the well-known results on ADE and simple elliptic singularities. In particular, this allows us to compute the potential function of the associated Frobenius manifold structure up to any finite order for arbitrary weighted homogeneous singularities. This is applied in [32] to prove the mirror symmetry for exceptional unimodular singularities.

Another Hodge theoretical aspect is called the tt^*-geometry discovered by Cecotti and Vafa [10] , whose integrability structure was studied by Dubrovin [19]. The algebraic formulation of tt^*-geometry was due to Hertling [28], and recently Fan [21] presented an approach through harmonic analysis in the spirit of $N = 2$ supersymmetry [9]. One essential aspect of tt^*-geometry is

about the real structure, which is naturally built into the smooth forms and their Hodge theoretical aspects. This is another motivation for us to develop the theory of higher residues inside the smooth category.

1.2. **Higher residue and primitive forms.** To explain primitive forms, we start with a pair (X, f), where $X \subset \mathbb{C}^n$ is a Stein domain, and $f : X \to \mathbb{C}$ is a holomorphic function with finite critical set. With respect to the choice of a nonwhere vanishing holomorphic volume form Ω_X on X, we can identify the space $\mathrm{PV}(X)$ of smooth polyvector fields with the space $\mathcal{A}(X)$ of smooth complex differential forms on X, by using contraction operator $\vdash$ with Ω_X. This gives an isomorphism of cochain complexes (Definition 2.7)

$$(\mathrm{PV}(X)((t)), Q_f) \to (\mathcal{A}(X)((t)), d + \frac{df}{t}\wedge),$$

where Q_f is the coboundary operator on $\mathrm{PV}(X)((t))$ that is defined by using $d + \frac{df}{t}\wedge$ together with the isomorphism $\mathrm{PV}(X)((t)) \cong \mathcal{A}(X)((t))$ of vector spaces. Let $\mathrm{PV}_c(X) \subset \mathrm{PV}(X)$ be the subspace of smooth polyvector fields with compact support. It is easy to observe that the natural embedding of complexes

$$\iota : (\mathrm{PV}_c(X)[[t]], Q_f) \hookrightarrow (\mathrm{PV}(X)[[t]], Q_f)$$

is in fact a quasi-isomorphism. Therefore, we obtain a canonical isomorphism

$$\mathcal{H}^{f,\Omega}_{(0)} := H^*(\mathrm{PV}(X)[[t]], Q_f) \overset{\iota^{-1}}{\to} H^*(\mathrm{PV}_c(X)[[t]], Q_f).$$

Combining this with the trace map defined by

$$\mathrm{Tr} : \mathrm{PV}_c(X) \to \mathbb{C};\ \alpha \mapsto \mathrm{Tr}(\alpha) := \int_X (\alpha \vdash \Omega_X) \wedge \Omega_X,$$

we obtain a map $\widehat{\mathrm{Res}}_f$ as the composition of Tr and the inverse of ι in the following diagram

$$\begin{array}{ccc} \mathcal{H}^{f,\Omega}_{(0)} := H^*(\mathrm{PV}(X)[[t]], Q_f) & \xrightarrow{\iota^{-1}} & H^*(\mathrm{PV}_c(X)[[t]], Q_f) \\ & \searrow^{\widehat{\mathrm{Res}}^f} & \downarrow \mathrm{Tr} \\ & & \mathbb{C}[[t]] \end{array}$$

One of our main observations is that $\widehat{\mathrm{Res}}^f$ realizes the original higher residue map in our smooth set-up of polyvector fields. More precisely, every holomorphic function g on X represents a cohomology class $[g]$ in $\mathcal{H}^{f,\Omega}_{(0)}$. If we write

$$\widehat{\mathrm{Res}}^f([g]) = \sum_{k\geq 0} \widehat{\mathrm{Res}}_{(k)}([g])t^k,$$

then $\widehat{\mathrm{Res}}_{(0)}(g)$ recognizes the ordinary residue (Proposition 2.5), and $\{\widehat{\mathrm{Res}}_{(k)}\}_k$ constitutes the tower of higher residues. The higher residue pairing

$$\mathcal{K}^f_\Omega(\text{-},\text{-}) : \mathcal{H}^{f,\Omega}_{(0)} \times \mathcal{H}^{f,\Omega}_{(0)} \to \mathbb{C}[[t]]$$

is constructed in a similar fashion such that $\widehat{\mathrm{Res}}(\text{-}) = \mathcal{K}^f_\Omega(\text{-}, [1])$.

Primitive forms are defined with respect to the universal folding F of f parametrizes by S. We assume that $o \in S$ is the reference point such that $F|_o = f$, and fix a family of holomorphic volume form Ω. The cohomology $\mathcal{H}^{f,\Omega}_{(0)}$ extends to an $\mathcal{O}_S[[t]]$-module $\mathcal{H}^{F,\Omega}_{(0)}$ on S, and the higher residue pairing extends to

$$\mathcal{K}^F_\Omega(\text{-}, \text{-}) : \mathcal{H}^{F,\Omega}_{(0)} \times \mathcal{H}^{F,\Omega}_{(0)} \to \mathcal{O}_S[[t]]$$

with a compatible Gauss-Manin connection

$$\nabla^\Omega : \mathcal{H}^{F,\Omega}_{(0)} \to \Omega^1_S \otimes t^{-1}\mathcal{H}^{F,\Omega}_{(0)}$$

The basic properties of the triple $(\mathcal{H}^{F,\Omega}_{(0)}, \mathcal{K}^F_\Omega(\text{-}, \text{-}), \nabla^\Omega)$ are summarized to define a variation of semi-infinite Hodge structures (see section 3.3). A primitive form is a section $\zeta \in \Gamma(S, \mathcal{H}^{F,\Omega}_{(0)})$ that satisfies the following four properties: *Primitivity*, *Orthogonality*, *Holonomicity* and *Homogeneity* (see Definition 3.21). Such definition is equivalent to the original description of primitive forms (see Theorem 3.27).

Since we are working with polyvector fields, we use the superscript notation to distinguish from the original notation of $\mathcal{H}_f$, $\mathcal{K}_f$, etc. We have made a choice of the family of holomorphic volume form Ω. However, different choices are essentially equivalent (see section 3.5). The combined volume form $\zeta\Omega$ represents a primitive form in the original approach [46]. Therefore it is equivalent to work inside differential forms as in [46]. We have made extra efforts to present in the current form in order to compare with BCOV theory on compact Calabi-Yau manifolds [4, 14, 35]. This motivates the perturbative formula for primitive forms (Theorem 5.16) as a unification of Calabi-Yau and Laudan-Ginzburg models, and it sheds lights on the Landau-Ginzburg/Calabi-Yau correspondence in the B-model.

1.3. **Construction of primitive forms and applications.** In [46], the third author introduced the notion of *good sections*, which we will call *good opposite filtrations*, to produce analytic primitive forms by solving a version of Riemann-Hilbert-Birkhoff problem. It leads to M. Saito's general solution on the existence of primitive forms [50] . This rather delicate construction provides a complete description of the space of primitive forms locally, but on the other hand makes the computation of primitive forms very difficult in general. Explicit expressions of primitive forms were only known, when the holomorphic function f is given by either (1) two classes of weighted homogeneous polynomials (namely ADE-singularities and simple elliptic singularities) [46] or (2) a few Laurent polynomials [3, 16, 17, 54]. It has also been known very recently, when f is an affine cusp polynomial [30, 52].

In the present paper, we develop new techniques to describe primitive forms locally with respect to good opposite filtrations. The basic idea is to apply Barannikov's construction of semi-infinite period map for compact Calabi-Yaus to the Landau-Ginzburg case once we have

completely similar settings via polyvector fields. In the formal neighborhood of $o \in S$, this leads to explicit expressions for the primitive forms which we now briefly describe. Let

$$\mathcal{H}^{f,\Omega} := H^*(\mathrm{PV}(X)((t)), Q_f)$$

and $\mathcal{H}^{F,\Omega}$ be the corresponding extension to the universal unfolding. The vector space $\mathcal{H}^{f,\Omega}$ is equipped with a symplectic structure, with $\mathcal{H}^{f,\Omega}_{(0)}$ being an isotropic linear subspace. A good opposite filtration $\mathcal{L}$ is given by a splitting

$$\mathcal{H}^{f,\Omega} = \mathcal{H}^{f,\Omega}_{(0)} \oplus \mathcal{L}$$

such that $\mathcal{L}$ is (1) isotropic, (2) preserved by multiplying by t^{-1} and (3) a version of $\mathbb{C}^*$-invariance (Definition 4.12). We will be working in the formal neighborhood of $o \in S$, and there exists a similar notion of formal primitive forms (Definition 4.6). Let $\check{\mathcal{H}}^{F,\Omega}$ ($\check{\mathcal{H}}^{F,\Omega}_{(0)}$) denote the pull-back of $\mathcal{H}^{F,\Omega}$ ($\mathcal{H}^{F,\Omega}_{(0)}$) to the formal neighborhood. The advantage of formal setting is that the Gauss-Manin connection ∇^Ω on $\check{\mathcal{H}}^{F,\Omega}$ is easily trivialized via the transformation $e^{(f-F)/t}$ (Lemma 4.7). Let $\mathcal{L}_{\mathcal{R}} \subset \check{\mathcal{H}}^{F,\Omega}$ be the extension of $\mathcal{L}$ with respect to ∇^Ω (see Definition 4.9 for precise descriptions). Then there is an induced splitting

$$\check{\mathcal{H}}^{F,\Omega} = \check{\mathcal{H}}^{F,\Omega}_{(0)} \oplus \mathcal{L}_{\mathcal{R}}$$

Our main result is the following

Theorem 1.1 (see Theorem 4.15 and Theorem 4.17)**.**

(1) *There is a bijection between analytic primitive forms in the germ of the universal unfolding and formal primitive forms, both of which are in one-to-one correspondence with pairs $(\mathcal{L}, \zeta_0)$, where $\mathcal{L} \subset \mathcal{H}^{f,\Omega}$ is a good opposite filtration and $\zeta_0 \in \mathcal{H}^{f,\Omega}_{(0)}$ is a primitive element with respect to $\mathcal{L}$.*
(2) *The Taylor series expansion of the primitive form corresponding to $(\mathcal{L}, \zeta_0)$ is given by the projection of the formal section $e^{\frac{f-F}{t}}\zeta_0$ of $\check{\mathcal{H}}^{F,\Omega} = \check{\mathcal{H}}^{F,\Omega}_{(0)} \oplus \mathcal{L}_{\mathcal{R}}$ to $\check{\mathcal{H}}^{F,\Omega}_{(0)}$.*

The above theorem unifies the semi-infinite period maps for Calabi-Yau models and Landau-Ginzburg models. It has particularly nice applications when f is a weighted homogenous polynomial (of weight degree 1). In this case, we always take the standard fixed family of volume form $\Omega = dz_1 \wedge \cdots \wedge dz_n$, where $(z_1, \cdots, z_n)$ denotes the coordinates of $X = \mathbb{C}^n$. Take a set $\{\phi_i\}_{i=1}^{\mu}$ of weighted homogeneous polynomials that represent a basis of the Jacobian ring, with the order of their weight degrees ascending: $\deg \phi_1 \leq \deg \phi_2 \leq \cdots \leq \deg \phi_\mu$. Then we can completely determine the space of good opposite filtrations and primitive elements with elementary methods. Denote $r(i,j) := \deg \phi_i - \deg \phi_j$ and set

$$D := \sharp\{(i,j) \mid r(i,j) \in \mathbb{Z}_{>0}, i+j < \mu+1\} + \sharp\{(i,j) \mid r(i,j) \in \mathbb{Z}^{\mathrm{odd}}_{>0}, i+j = \mu+1\}.$$

Applying the first part of Theorem 1.1 to the weighted homogeneous polynomial f, we obtain

Theorem 1.2 (Theorem 5.14). *The moduli space of primitives forms,*

$$\mathcal{M} := \{\textit{primitive forms of } \mathcal{H}^{F,\Omega}_{(0)}\}/\sim ,$$

is parametrized by $\mathbb{C}^D$. Here we say two primitive forms ζ_1, ζ_2 to be equivalent, denoted as $\zeta_1 \sim \zeta_2$, if $\zeta_1 = c\zeta_2$ in the germ of the universal unfolding, for some nonzero constant $c \in \mathbb{C}^*$.

There are related statements for Brieskorn lattices [26] (see also chapter 10 of [27]). As a direct consequence of the theorem, there exists a unique primitive form (up to a scalar) for ADE-singularities [46] and exceptional unimodular singularities (Corollary 5.15).

Applying the second part of Theorem 1.1 to f, we obtain a concrete algorithm to compute the Taylor series expansions of primitive forms up to an arbitrary finite order. The precise formula is explained in **Theorem 5.16**. Actually, such perturbative formula unifies the following known results [46]. For ADE-singularities, the unique primitive form is given by $\zeta = 1$; for simple elliptic singularities, there is a one-parameter family of primitive forms (up to a nonzero constant scalar), given by the period integrals on the corresponding elliptic curves. Beyond this, our perturbative formula also give precise expressions of primitive forms for other cases up to a finite order. As an example, we consider $f = x^3 + y^7$, which is an exceptional unimodular singularity of type E_{12}. The primitive forms for it were unknown, due to the difficulty of the generally expected phenomenon of mixing between deformation parameters of positive and negative degrees. As an application of Theorem 5.16, the unique primitive ζ has the leading expression

$$\zeta = 1 + \frac{4}{3\cdot 7^2}u_{11}u_{12}^2 - \frac{64}{3\cdot 7^4}u_{11}^2u_{12}^4 - \frac{76}{3^2\cdot 7^4}u_{10}u_{12}^5 + (\frac{1}{7^2}u_{12}^3 - \frac{101}{5\cdot 7^4}u_{11}u_{12}^5)x - \frac{53u_{12}^6}{3^2\cdot 7^4}x^2 \mod \mathfrak{m}^7,$$

where $\mathfrak{m}$ denotes the maximal ideal of the reference point $0 \in S$, and u_{10}, u_{11}, u_{12} are part of the coordinates of S appearing in the coefficients of the deformations by $\phi_{10} = xy^3$, $\phi_{11} = xy^4$ and $\phi_{12} = xy^5$, respectively. It is clear from this result that the mixing phenomenon mentioned above appears in a nontrivial way, and non-trivial elements in the Jacobian ring are also generated along deformation, unlike the ADE and simple elliptic singularities. This strongly suggests that the primitive form not only favors for special flat coordinates on the deformation space, but also for the fibration itself. All these cases are discussed in detail in the last section of examples.

Our perturbative formula for primitive forms are extremely useful for applications in mirror symmetry. The theory of primitive forms presents the mathematical context of Landau-Ginzburg B-model in physics. The mirror mathematical theory of Landau-Ginzburg A-model is established by Fan, Jarvis and Ruan [20] following Witten's proposal [57], which is nowadays popularized as the FJRW-theory. As a direct application of our method, it is proved in [32] that Landau-Ginzburg B-models for exceptional unimodular singularities are equivalent to the mirror FJRW-theory. Our current construction is mainly focused on the local theory, and it would be extremely interesting to understand the global behavior of primitive forms. Related works on global mirror symmetry along this line have appeared in [11, 12, 29, 39].

The present paper is organized as follows: Section 2 is devoted to the basic set-up of polyvector fields. We formulate the complex differential geometric description of higher residues via the trace map on compactly supported polyvector fields. In section 3, we explain the variation of semi-infinite Hodge structures associated with our model and define the corresponding primitive forms. In section 4, we describe the general constructions of primitive forms in the formal neighborhood of the universal unfolding and explain our perturbative formula. In section 5, we describe the moduli space of primitive forms for weighted homogeneous polynomials, and provide a concrete algorithm to compute the Taylor series expansions of the primitive form. Finally in section 6, we apply our method to explicit examples.

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2. Polyvector fields

Throughout the present paper, we will assume X to be a Stein domain in $\mathbb{C}^n$, and assume $f : X \to \mathbb{C}$ to be a holomorphic function with finite critical points $\mathrm{Crit}(f)$. We will fix a nowhere vanishing holomorphic volume form Ω_X on X once and for all. As we will see in section 3.5 (Proposition 3.26), the choice of Ω_X is not essential.

As a notation convention, we will always denote by [-,-] the graded-commutator. That is, for operators A, B of degree $|A|, |B|$, the bracket is given by

$$[A, B] := AB - (-1)^{|A|\cdot|B|}BA. \tag{2.1}$$

2.1. **Compactly supported polyvector fields.** Let T_X denote the holomorphic tangent bundle of X, and $\mathcal{A}_c^{0,j}(X, \wedge^i T_X)$ denote the space of compactly supported smooth $(0, j)$-forms valued in $\wedge^i T_X$. We consider the space of compactly supported smooth polyvector fields on X:

$$\mathrm{PV}_c(X) = \bigoplus_{0\le i,j\le n} \mathrm{PV}_c^{i,j}(X) \quad \text{with} \quad \mathrm{PV}_c^{i,j}(X) := \mathcal{A}_c^{0,j}(X, \wedge^i T_X).$$

Clearly, $\mathrm{PV}_c(X)$ is a bi-graded $C^\infty(X)$-module. As we will see below, the space $\mathrm{PV}_c(X)$ is a double complex: the total degree of $\alpha \in \mathrm{PV}_c^{i,j}(X)$ is defined[1] to be $|\alpha| := j - i$. There is a natural wedge product structure (see formula (2.4)) on polyvector fields, which makes $\mathrm{PV}_c(X)$ a graded-commutative algebra.

Using the contraction $\vdash$ with Ω_X, we obtain an isomorphism of $C^\infty(X)$-modules between compactly supported polyvector fields and compactly supported differential forms

$$\Gamma_{\Omega_X} : \mathrm{PV}_c^{i,j}(X) \overset{\vdash \Omega_X}{\cong} \mathcal{A}_c^{n-i,j}(X).$$

As a consequence, every operator P on $\mathcal{A}_c(X) := \mathcal{A}_c^{*,*}(X)$ induces an operator P_{Ω_X} on $\mathrm{PV}_c(X)$:

$$P_{\Omega_X}(\alpha) := \Gamma_{\Omega_X}^{-1}\big(P(\Gamma_{\Omega_X}(\alpha))\big), \quad \forall \alpha \in \mathrm{PV}_c(X).$$

We will simply denote P_{Ω_X} as P_Ω. Furthermore, if the induced operator P_{Ω_X} is independent of choices of Ω_X, we simply denote it as P by abuse of notation. For instance, the differential operators $\bar\partial, \partial$ on differential forms, respectively, induce operators on polyvector fields

$$\bar\partial_{\Omega_X} : \mathrm{PV}_c^{*,*}(X) \to \mathrm{PV}_c^{*,*+1}(X) \quad \text{and} \quad \partial_{\Omega_X} : \mathrm{PV}_c^{*,*}(X) \to \mathrm{PV}_c^{*-1,*}(X).$$

The operator $\bar\partial_{\Omega_X}$ is simply denoted as $\bar\partial$, while the operator ∂_{Ω_X} that depends on Ω_X is denoted by ∂_Ω. In other words,

$$(\bar\partial\alpha) \vdash \Omega_X = \bar\partial(\alpha \vdash \Omega_X), \text{ and } (\partial_\Omega\alpha) \vdash \Omega_X = \partial(\alpha \vdash \Omega_X)$$

for any $\alpha \in \mathrm{PV}_c(X)$. Both $\bar\partial$ and ∂_Ω are of cohomology degree 1.

The operator ∂_Ω further defines a bracket

$$\{\text{-},\text{-}\} : \mathrm{PV}_c(X) \times \mathrm{PV}_c(X) \to \mathrm{PV}_c(X),$$

$$\{\alpha, \beta\} := \partial_\Omega(\alpha \wedge \beta) - (\partial_\Omega\alpha) \wedge \beta - (-1)^{|\alpha|}\alpha \wedge \partial_\Omega\beta \tag{2.2}$$

for $\alpha \in \mathrm{PV}_c^{i,j}(X)$ and $\beta \in \mathrm{PV}_c^{k,l}(X)$. Here $\wedge$ is the wedge product (see formula (2.4)). Unless otherwise stated, we will simply denote $\alpha\beta := \alpha \wedge \beta$. In the special case when the polyvector field $\alpha = g$ is a function, one can easily check that

$$\{g, \beta\} \vdash \Omega_X = \partial g \wedge (\beta \vdash \Omega_X). \tag{2.3}$$

The bracket $\{\text{-},\text{-}\}$ corresponds to the Schouten-Nijenhuis bracket (up to a sign), which is independent of the choice of Ω_X.

We also consider the larger space

$$\mathrm{PV}(X) = \bigoplus_{0 \le i,j \le n} \mathrm{PV}^{i,j}(X)$$

[1]The convention for the degree of α is $j + i$ in [14].

of smooth polyvector fields (not necessarily compactly supported), which is naturally identified with the smooth differential forms by extending Γ_{Ω_X} above. The operators $\bar{\partial}$, ∂_Ω and $\{\text{-},\text{-}\}$ are all defined on $\mathrm{PV}(X)$.

Remark 2.1. *The triple* $\big\{\{\text{-},\text{-}\},\wedge,\partial_\Omega\big\}$ *make* $\mathrm{PV}(X)$ *a* Batalin-Vilkovisky algebra, *which is used in* [5,7,55,56] *to prove the Formality Theorem for polyvector fields in Calabi-Yau geometry.*

We describe the above constructions in coordinates $\mathbf{z} = (z_1, \cdots, z_n)$ of $\mathbb{C}^n$ and fix our conventions. We will write

$$\Omega_X = \frac{dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n}{\lambda(\mathbf{z})}$$

where $\lambda : X \to \mathbb{C}$ is a nowhere vanishing holomorphic function on X. For an ordered subset $I = \{a_1, a_2, \cdots, a_i\}$ of $\{1, 2, \cdots, n\}$ with $|I| = i$, we use the following notations:

$$d\bar{z}_I := d\bar{z}_{a_1} \wedge \cdots \wedge d\bar{z}_{a_i}, \quad \partial_I := \frac{\partial}{\partial z_{a_1}} \wedge \frac{\partial}{\partial z_{a_2}} \wedge \cdots \wedge \frac{\partial}{\partial z_{a_i}}.$$

In particular, we will use ∂_k for the polyvector field $\frac{\partial}{\partial z_k} \in \mathrm{PV}^{1,0}(X)$ in order to distinguish it from the partial derivative $\partial_{z_k} := \frac{\partial}{\partial z_k}$ on $C^\infty(X)$. We will occasionally use $\frac{\partial}{\partial z_k}$ when its meaning is self-evident.

For $\alpha \in \mathrm{PV}^{i,j}(X)$ and $\beta \in \mathrm{PV}^{k,l}(X)$,

$$\alpha = \sum_{|I|=i,|J|=j} \alpha_J^I d\bar{z}_J \otimes \partial_I, \quad \beta = \sum_{|K|=k,|L|=l} \beta_L^K d\bar{z}_L \otimes \partial_K,$$

where α_J^I, $\beta_L^K \in C^\infty(X)$, the wedge product is given by

$$\alpha \wedge \beta := \sum_{I,J,K,L} (-1)^{il} \alpha_J^I \beta_L^K (d\bar{z}_J \wedge d\bar{z}_L) \otimes (\partial_I \wedge \partial_K), \tag{2.4}$$

which satisfies the graded-commutativity

$$\alpha \wedge \beta = (-1)^{|\alpha|\cdot|\beta|} \beta \wedge \alpha.$$

We will use $\frac{\partial}{\partial \partial_r}$ for the derivation with respect to the polyvector field ∂_r:

$$\frac{\partial}{\partial \partial_r}\left(\alpha_J^I d\bar{z}_J \otimes \partial_I\right) := \begin{cases} (-1)^{|J|+m-1} \alpha_J^I d\bar{z}_J \otimes \partial_{a_1} \cdots \partial_{a_{m-1}} \partial_{a_{m+1}} \cdots \partial_{a_i}, & \text{if } a_m = r \text{ for some } m, \\ 0, & \text{otherwise.} \end{cases}$$

These operators are of degree 1, satisfying the graded-commutativity relations

$$\frac{\partial}{\partial \partial_k} \frac{\partial}{\partial \partial_l} = -\frac{\partial}{\partial \partial_l} \frac{\partial}{\partial \partial_k}, \quad \forall 1 \le k, l \le n.$$

Under the above conventions, we have

$$\bar{\partial}\alpha = \sum_{I,J} (\bar{\partial}\alpha_J^I) \wedge d\bar{z}_J \otimes \partial_I, \quad \partial_\Omega \alpha = \sum_{I,J} \sum_{r=1}^n \lambda \partial_{z_r}\left(\frac{\alpha_J^I}{\lambda}\right) \frac{\partial}{\partial \partial_r}(d\bar{z}_J \otimes \partial_I),$$

and

$$\{\alpha,\beta\}=\sum_{r=1}^{n}\Big(\frac{\partial}{\partial\partial_r}\alpha\Big)\wedge\Big(\sum_{K,L}(\partial_{z_r}\beta_L^K)d\bar{z}_L\otimes\partial_K\Big)+(-1)^{|\alpha|}\sum_{r=1}^{n}\Big(\sum_{I,J}(\partial_{z_r}\alpha_J^I)d\bar{z}_J\otimes\partial_I\Big)\wedge\frac{\partial}{\partial\partial_r}\beta.$$

2.2. **Cohomology of polyvector fields.** In this subsection, we will study the cohomology of (compactly supported) polyvector fields with respect to the following coboundary operator

$$\bar{\partial}_f:=\bar{\partial}+\{f,\text{-}\},$$

acting on $\mathrm{PV}(X)$ and preserving $\mathrm{PV}_c(X)$. It is easy to see that $\bar{\partial}_f^2=0$ and $\bar{\partial}_f g=0,\forall g\in\Gamma(X,\mathcal{O}_X)$. Let $\mathrm{Jac}(f):=\Gamma(X,\mathcal{O}_X)/(\partial_{z_1}f,\cdots,\partial_{z_n}f)$, which is called the *Jacobian ring* or *Chiral ring*.

Lemma 2.2. *$H^k(\mathrm{PV}(X),\bar{\partial}_f)$ vanishes unless $k=0$. Furthermore, there is a natural isomorphism $\mathrm{Jac}(f)\xrightarrow{\cong}H^0(\mathrm{PV}(X),\bar{\partial}_f)$ sending $[g]$ in $\mathrm{Jac}(f)$, $g\in\Gamma(X,\mathcal{O}_X)$, to the cohomology class $[g]$.*

Proof. $\mathrm{PV}(X)$ is a double complex with horizontal operator $\{f,\text{-}\}$ and vertical operator $\bar{\partial}$. We consider the spectral sequence associated to the descending filtration

$$\mathcal{F}^k\,\mathrm{PV}(X)=\bigoplus_{i\leq n-k}\mathrm{PV}^{i,*}(X)$$

for $k\in\mathbb{Z}$. Since X is Stein, the E_1-term is given by holomorphic sections $H^*(\mathrm{PV}(X),\bar{\partial})=\bigoplus_k H^0(X,\wedge^k T_X)$, with the differential $\{f,\text{-}\}$. Since $\mathrm{Crit}(f)$ is finite, $\partial_{z_1}f,\cdots,\partial_{z_n}f$ forms a regular sequence on X. Therefore the cohomology of the holomorphic Koszul complex

$$0\xrightarrow{\{f,\text{-}\}}H^0(X,\wedge^n T_X)\xrightarrow{\{f,\text{-}\}}H^0(X,\wedge^{n-1}T_X)\xrightarrow{\{f,\text{-}\}}\cdots\xrightarrow{\{f,\text{-}\}}H^0(X,\mathcal{O}_X)\xrightarrow{\{f,\text{-}\}}0,$$

which is E_2, is concentrated at the top term $\frac{H^0(X,\mathcal{O}_X)}{\mathrm{Im}\left(\{f,\text{-}\}:H^0(X,T_X)\to H^0(X,\mathcal{O}_X)\right)}=\mathrm{Jac}(f)$. Clearly, the spectral sequence degenerates at E_2-stage, and the statement follows. □

In order to represent cohomology classes by compactly supported polyvector fields, we introduce some operators. Let

$$\|df\|^2:=\sum_{k=1}^{n}|\partial_{z_k}f|^2,$$

and we consider the following operator

$$V_f=\sum_{i=1}^{n}\frac{\overline{\partial_{z_i}f}}{\|df\|^2}\partial_i\wedge:\ \ \mathrm{PV}^{*,*}(X\setminus\mathrm{Crit}(f))\longrightarrow\mathrm{PV}^{*+1,*}(X\setminus\mathrm{Crit}(f)).$$

Let ρ be a smooth cut-off function on X, such that $\rho|_{U_1}\equiv 1$ and $\rho|_{X\setminus U_2}\equiv 0$ for some relatively compact open neighborhoods $U_1\subset\overline{U_1}\subset U_2$ of $\mathrm{Crit}(f)$ in X.

The operators $\bar{\partial}$ and V_f are of degree 1 and -1 respectively (where defined), and $[\bar{\partial},V_f]=\bar{\partial}V_f+V_f\bar{\partial}$ is of degree 0. We define

$$T_\rho:\mathrm{PV}(X)\to\mathrm{PV}_c(X),\quad\text{and}\quad R_\rho:\mathrm{PV}(X)\to\mathrm{PV}(X)$$

by

$$T_\rho(\alpha) := \rho\alpha + (\bar\partial\rho)V_f\frac{1}{1+[\bar\partial, V_f]}(\alpha), \quad R_\rho(\alpha) := (1-\rho)V_f\frac{1}{1+[\bar\partial, V_f]}(\alpha), \quad \alpha \in \mathrm{PV}(X).$$

Here as an operator

$$\frac{1}{1+[\bar\partial, V_f]} := \sum_{k=0}^{\infty}(-1)^k[\bar\partial, V_f]^k,$$

which is well-defined since $[\bar\partial, V_f]^k(\alpha) = 0$ for any $k > n$.

Lemma 2.3. *$[\bar\partial_f, R_\rho] = 1 - T_\rho$ as operators on $\mathrm{PV}(X)$.*

Proof. It is easy to see that

$$[\{f, \text{-}\}, V_f] = 1 \quad \text{on } \mathrm{PV}(X \setminus \mathrm{Crit}(f)).$$

Moreover,

$$[P, [\bar\partial, V_f]] = 0$$

for P being $\{f, \text{-}\}, \bar\partial$ or V_f. Therefore, where defined,

$$\begin{aligned}[\bar\partial_f, R_\rho] &= [\bar\partial_f, 1-\rho]V_f\frac{1}{1+[\bar\partial, V_f]} + (1-\rho)[\bar\partial_f, V_f]\frac{1}{1+[\bar\partial, V_f]} \\ &= -(\bar\partial\rho)V_f\frac{1}{1+[\bar\partial, V_f]} + (1-\rho) \\ &= 1 - T_\rho.\end{aligned}$$

□

Corollary 2.4. *The embedding $\left(\mathrm{PV}_c(X), \bar\partial_f\right) \hookrightarrow \left(\mathrm{PV}(X), \bar\partial_f\right)$ is a quasi-isomorphism.*

Proof. This follows from Lemma 2.3 that $H^*(\mathrm{PV}(X)/\mathrm{PV}_c(X), \bar\partial_f) \equiv 0$. □

We will let ι be the natural embedding

$$\iota : \mathrm{PV}_c(X) \hookrightarrow \mathrm{PV}(X).$$

Corollary 2.4 and Lemma 2.2 lead to a canonical isomorphism

$$\iota : H^*\left(\mathrm{PV}_c(X), \bar\partial_f\right) \xrightarrow{\cong} \mathrm{Jac}(f),$$

which we still denote by ι. By Lemma 2.3, a representative of $\iota^{-1}([g])$ for $[g] \in \mathrm{Jac}(f)$, where $g \in \Gamma(X, \mathcal{O}_X)$, is given by the compactly supported polyvector field $T_\rho(g)$. In particular, the class $[T_\rho(g)]$ represented by $T_\rho(g)$ depends only on the class $[g] \in \mathrm{Jac}(f)$, not on the choice of the coordinates or the cut-off function ρ.

For compactly supported polyvector fields, we can define the *trace map*

$$\mathrm{Tr} : \mathrm{PV}_c(X) \to \mathbb{C}, \ \mathrm{Tr}(\alpha) := \int_X (\alpha \vdash \Omega_X) \wedge \Omega_X.$$

This induces a non-degenerate pairing $\langle \text{-}, \text{-} \rangle : \mathrm{PV}_c(X) \otimes \mathrm{PV}_c(X) \to \mathbb{C}$ defined by

$$\alpha \otimes \beta \mapsto \langle \alpha, \beta \rangle := \mathrm{Tr}\,(\alpha\beta)\,.$$

For any $\alpha \in \mathrm{PV}_c^{i,j}(X)$ and $\beta \in \mathrm{PV}_c^{k,l}(X)$, it is easy to see that

$$\langle \bar{\partial}\alpha, \beta \rangle = -(-1)^{|\alpha|}\langle \alpha, \bar{\partial}\beta \rangle \quad \text{and} \quad \langle \partial_\Omega \alpha, \beta \rangle = (-1)^{|\alpha|}\langle \alpha, \partial_\Omega \beta \rangle. \tag{2.5}$$

Furthermore, we have

$$\mathrm{Tr}(\bar{\partial}\alpha) = 0 \quad \text{and} \quad \mathrm{Tr}(\{f, \alpha\}) = 0,$$

which imply that the trace map is well defined on the cohomology

$$\mathrm{Tr} : H^*(\mathrm{PV}_c(X), \bar{\partial}_f) \to \mathbb{C}.$$

The above pairing descends to a pairing $\langle \text{-}, \text{-} \rangle$ on the cohomology which will be non-degenerate as well (Proposition 2.5).

The trace map is closely related to the residue as follows. Let $\{V_1, \cdots, V_n\}$ be a $\mathcal{O}_X$-basis of holomorphic vector fields on X such that $(V_n \wedge V_{n-1} \wedge \cdots \wedge V_1) \vdash \Omega_X \equiv 1$. Then $V_1(f), \cdots, V_n(f)$ form a regular sequence on X, and we have the concept of residue map (see e.g. [25], [24] for details)

$$\mathrm{Res} : \mathrm{Jac}(f) \to \mathbb{C}, \quad g \to \mathrm{Res}(g) = \sum_{x \in \mathrm{Crit}(f)} \mathrm{Res}_x \frac{g\Omega_X}{V_1(f)V_2(f)\cdots V_n(f)}.$$

which is independent of the choice of $\{V_1, \cdots, V_n\}$ (but depends on the choice of Ω_X). In coordinates $\mathbf{z}$ of $\mathbb{C}^n$, we have

$$\mathrm{Res}(g) = \sum_{x \in \mathrm{Crit}(f)} \mathrm{Res}_x \frac{g(z)dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n}{\lambda^2(\partial_{z_1} f)\cdots(\partial_{z_n} f)}$$

for $\Omega_X = \frac{dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n}{\lambda(\mathbf{z})}$.

One of the key observations is that the trace map is compatible with the residue map in the following sense.

Proposition 2.5. *There exists a constant C_n, depending only on $n = \dim X$, such that the following diagram commutes*

$$\begin{array}{ccc} \mathrm{Jac}(f) & \xrightarrow{\iota^{-1}} & H^*(\mathrm{PV}_c(X), \bar{\partial}_f) \\ & \searrow^{C_n\,\mathrm{Res}} & \downarrow \mathrm{Tr} \\ & & \mathbb{C} \end{array}$$

Proof. The statement follows from similar calculations as in [24] page 654. The proof is given below, while our reader may skip the details.

Let $\mathrm{Crit}(f) = \{x_1, \cdots, x_r\}$. For $0 < \varepsilon << 1$, we denote by B_ε a closed ball of radius ε contained in X, S_ε being the boundary of B_ε. We choose $\varepsilon_1, \cdots, \varepsilon_r$ and small disjoint balls $B_{2\varepsilon_i}$ such that each

$\mathrm{Int}(B_{\varepsilon_i})$ contains precisely x_i in $\mathrm{Crit}(f)$, hence $\mathrm{Crit}(f) \subset \bigcup_{i=1}^r \mathrm{Int}(B_{\varepsilon_i})$. Take a smooth cut-off function ρ such that

$$\rho|_{\bigcup_{i=1}^r B_{\varepsilon_i}} \equiv 1, \quad \rho|_{X\setminus\bigcup_{i=1}^r B_{2\varepsilon_i}} \equiv 0.$$

Let $X' = \bigcup_{i=1}^r \mathrm{Int}(B_{2\varepsilon_i}) - \bigcup_{i=1}^r B_{\varepsilon_i}$. For $1 \le k \le n$, we denote $f_k := \partial_{z_k} f$ and $h_k := \frac{\bar{f}_k}{\|df\|^2}$, where we recall $\|df\|^2 = \sum_{j=1}^n |f_j|^2$.

The operator T_ρ gives a realization of ι^{-1}. Let g be a holomorphic function on X representing an element of $\mathrm{Jac}(f)$. Let

$$A_j := V_f[\bar\partial, V_f]^j(g) \in \mathrm{PV}^{j+1,j}(X).$$

Claim: $\bar\partial A_{n-1} = 0$ and

$$A_{n-1} = C_n' g \sum_{i=1}^n (-1)^{i-1} \frac{\bar{f}_i}{\|df\|^{2n}} d\bar{f}_1 \wedge \cdots \wedge \widehat{d\bar{f}_i} \wedge \cdots \wedge d\bar{f}_n \otimes (\partial_1 \cdots \partial_n)$$

for some nonzero constant C_n' depending only on n.

Let us assume the Claim first, and let $\Omega_{X'} := \Omega_X|_{X'}$. Then

$$\begin{aligned}
&\mathrm{Tr}(\iota^{-1}(g)) = \mathrm{Tr}(T_\rho g) = \mathrm{Tr}((1-\rho)g + (\bar\partial\rho)\sum_{k=1}^n (-1)^k A_k) \\
&= (-1)^{n-1}\int_X \big(((\bar\partial\rho)A_{n-1}) \vdash \Omega_X\big) \wedge \Omega_X = (-1)^{n-1}\int_{X'} \big(((\bar\partial\rho)A_{n-1}) \vdash \Omega_{X'}\big) \wedge \Omega_{X'} \\
&= (-1)^{n-1}\int_{X'} d\big((\rho A_{n-1} \vdash \Omega_{X'}) \wedge \Omega_{X'}\big) = (-1)^{n-1}\int_{\partial X'} (\rho A_{n-1} \vdash \Omega_{X'}) \wedge \Omega_{X'} \\
&= (-1)^n \sum_{j=1}^r \oint_{S_{\varepsilon_j}} (A_{n-1} \vdash \Omega_{X'}) \wedge \Omega_{X'}.
\end{aligned}$$

Hence, there exists nonzero constant C_n depending only on n such that

$$\begin{aligned}
\mathrm{Tr}(T_\rho g) &= C_n \sum_{j=1}^r \oint_{S_{\varepsilon_j}} \frac{1}{\lambda^2} g \sum_{i=1}^n (-1)^{i-1} \frac{\bar{f}_i}{\|df\|^{2n}} d\bar{f}_1 \wedge \cdots \widehat{d\bar{f}_i} \wedge \cdots \wedge d\bar{f}_n \wedge dz_1 \cdots \wedge dz_n \\
&\overset{(*)}{=} C_n \sum_{x_j} \mathrm{Res}_{x_j} \frac{g}{(\partial_{z_1} f)\cdots(\partial_{z_n} f)} \cdot \frac{1}{\lambda^2} dz_1 \wedge \cdots \wedge dz_n \\
&= C_n \,\mathrm{Res}(g).
\end{aligned}$$

The equality $(*)$ follows from the Residue Theorem (see e.g. section 5.1 of [24]).

It remains to show our claim. We have

$$V_f = \sum_i h_i \partial_i, \quad \big[\bar\partial, V_f\big]^k(g) = \left(\sum_i \bar\partial h_i \partial_i\right)^k g$$

where $\bar\partial h_i = \frac{\bar\partial \bar f_i}{\|df\|^2} - \frac{\bar f_i}{\|df\|^4}\bar\partial\|df\|^2$. Using

$$\bar\partial\|df\|^2 \wedge \bar\partial\|df\|^2 = 0 \text{ and } \left(\sum_i h_i\partial_i\right)\left(\sum_i \frac{\bar f_i}{\|df\|^4}\bar\partial\|df\|^2\partial_i\right) = -\frac{\bar\partial\|df\|^2}{\|df\|^2}\left(\sum_i h_i\partial_i\right)^2 = 0,$$

it follows easily that

$$\begin{aligned} A_{n-1} &= \left(\sum_i h_i\partial_i\right)\left(\sum_i \frac{\bar\partial \bar f_i}{\|df\|^2}\partial_i\right)^{n-1} g - (n-1)\left(\sum_i h_i\partial_i\right)\left(\sum_i \frac{\bar\partial \bar f_i}{\|df\|^2}\partial_i\right)^{n-2}\left(\sum_i \frac{\bar f_i}{\|df\|^4}\bar\partial\|df\|^2\partial_i\right) g \\ &= \left(\sum_i h_i\partial_i\right)\left(\sum_i \frac{\bar\partial \bar f_i}{\|df\|^2}\partial_i\right)^{n-1} g \\ &= (-1)^{n(n-1)/2}(n-1)!g\sum_{i=1}^n (-1)^{i-1}\frac{\bar f_i}{\|df\|^{2n}} d\bar f_1 \wedge \cdots \widehat{d\bar f_i} \wedge \cdots \wedge d\bar f_n \otimes (\partial_1 \cdots \partial_n). \end{aligned}$$

The equation $\bar\partial A_{n-1} = 0$ follows from the above formula. This proves the Claim. □

2.3. **Polyvector fields with a descendant variable.** Following [43, 45], we consider the vector space of $\mathrm{PV}(X)$ (resp. $\mathrm{PV}_c(X)$) valued Laurent series in t:

$$\mathrm{PV}(X)((t)) = \mathrm{PV}(X)[[t]][t^{-1}] \quad (\text{resp.} \quad \mathrm{PV}_c(X)((t)) = \mathrm{PV}_c(X)[[t]][t^{-1}]).$$

Here t is a formal variable of cohomology degree zero, called descendant variable.

Remark 2.6. *We are following the notation t in* [13, 14] *for the descendant variable, which is also denoted by various notations like δ^{-1}* [43–46]*, $\hbar$* [3, 4]*, u* [31]*, z* [23]*, etc.*

Both $\mathrm{PV}(X)((t))$ and $\mathrm{PV}_c(X)((t))$ are complexes with respect to the following twisted coboundary operator

$$Q_f := \bar\partial_f + t\partial_\Omega. \tag{2.6}$$

The complex $\mathrm{PV}(X)((t))$ has a natural decreasing filtration $\mathcal{F}^\bullet \mathrm{PV}(X)((t))$ preserved by Q_f, which is defined by

$$\mathcal{F}^k(\mathrm{PV}(X)((t))) := t^k \mathrm{PV}(X)[[t]]. \tag{2.7}$$

On the other hand, the space $\mathcal{A}(X)((t))$ (resp. $\mathcal{A}_c(X)((t))$ of $\mathcal{A}(X)$ (resp. $\mathcal{A}_c(X)$) valued Laurent series is a complex with respect to either of the following twisted coboundary operator:

$$d_f^+ := d + \frac{df}{t}\wedge, \qquad d_f^- := td + (df)\wedge. \tag{2.8}$$

Proposition-Definition 2.7. *There are two isomorphisms $\hat\Gamma^\pm_{\Omega_X}$ of filtered complexes,*

$$\hat\Gamma^+_{\Omega_X} : (\mathrm{PV}(X)((t)), Q_f) \to (\mathcal{A}(X)((t)), d_f^+), \qquad \hat\Gamma^-_{\Omega_X} : (\mathrm{PV}(X)((t)), Q_f) \xrightarrow{\cong} (\mathcal{A}(X)((t)), d_f^-),$$

which, for $t^k\alpha \in t^k \operatorname{PV}^{i,j}(X)$, are respectively defined by

$$\hat{\Gamma}^+_{\Omega_X}(t^k\alpha) := t^{k+i-1}\,(\alpha \vdash \Omega_X) \in t^{k+i-1}\mathcal{A}^{n-i,j}(X) \quad \textit{and} \quad \hat{\Gamma}^-_{\Omega_X}(t^k\alpha) := t^{k+j}\,(\alpha \vdash \Omega_X) \in t^{k+j}\mathcal{A}^{n-i,j}(X).$$

Here the filtered complex structure of $(\mathcal{A}(X)((t)), d^{\pm})$ are respectively given by

$$\hat{\Gamma}^+_{\Omega_X}(t^k \operatorname{PV}(X)[[t]]) = \prod_{m\in\mathbb{Z}} \bigoplus_{i\geq m+k, j\in\mathbb{Z}} \mathcal{A}^{i,j}(X)\otimes t^{n-m-1}, \tag{2.9}$$

$$\hat{\Gamma}^-_{\Omega_X}(t^k \operatorname{PV}(X)[[t]]) = \prod_{m\in\mathbb{Z}} \bigoplus_{j\leq m-k, i\in\mathbb{Z}} \mathcal{A}^{i,j}(X)\otimes t^{m}, \tag{2.10}$$

Proof. Clearly, $\hat{\Gamma}^+_{\Omega_X}$ and $\Gamma^-_{\Omega_X}$ are both isomorphisms of $C^\infty(X)((t))$-modules. Furthermore, it is easy to check $\hat{\Gamma}^+_{\Omega_X}\circ Q_f = d_f^+ \circ \hat{\Gamma}^+_{\Omega_X}$ and $\hat{\Gamma}^-_{\Omega_X}\circ Q_f = d_f^- \circ \hat{\Gamma}^-_{\Omega_X}$. Therefore $\hat{\Gamma}^{\pm}_{\Omega_X}$ are isomorphisms of complexes. It is direct to check that $\hat{\Gamma}^{\pm}_{\Omega_X}(\mathcal{F}^\bullet \operatorname{PV}(X)((t)))$ makes $(\mathcal{A}(X)((t)), d_f^{\pm})$ a filtered-complex as desired. □

We remark that the filtration $\mathcal{F}^\bullet \operatorname{PV}(X)((t))$ plays the role of the Hodge filtration under the isomorphism $\hat{\Gamma}^+_{\Omega_X}$. The restriction of $\hat{\Gamma}^-_{\Omega_X}$ to the holomorphic data $\Gamma(X, \wedge^* T_X)[[t]]$ gives the same filtration $\hat{\Gamma}^-_{\Omega_X}\big(t^k\Gamma(X,\wedge^* T_X)[[t]]\big) = t^k\Omega^*_X[[t]]$ as in [46]. The coboundary operator $d_f^- = td + (df)\wedge$ was introduced by the third author in [43, 45]. As we will see in Proposition-Definition 2.10, $\hat{\Gamma}^-_{\Omega_X}$ provides a precise identification of the filtrations of cohomology between our approach of using polyvector fields and the original setting up via differential forms. However, we have chosen to work with polyvector fields, Q_f and $\hat{\Gamma}^+_{\Omega_X}$ in order to compare with the compact Calabi-Yau manifolds [14] and illustrate aspects of mirror symmetry. The variable t is related to the gravitational descendant in physics, mirror to the descendant variable in Gromov-Witten theory. The complex $(\operatorname{PV}(X)[[t]], Q_f)$ realizes a natural deformation of $(\operatorname{PV}(X), \bar{\partial}_f)$, capturing the central aspects of the non-commutative Hodge structures [31].

2.3.1. *Cohomology of polyvector fields with descendants.*

Definition 2.8. *We define the following vector spaces $\mathcal{H}^{f,\Omega}_{(0)}, \mathcal{H}^{f,\Omega}$ associated to f*

$$\mathcal{H}^{f,\Omega}_{(0)} := \frac{\Gamma(X,\mathcal{O}_X)[[t]]}{\operatorname{Im}\left(Q_f : \Gamma(X,T_X)[[t]] \to \Gamma(X,\mathcal{O}_X)[[t]]\right)},\ \mathcal{H}^{f,\Omega} := \mathcal{H}^{f,\Omega}_{(0)} \otimes_{\mathbb{C}[[t]]} \mathbb{C}((t)).$$

where $\Gamma(X, T_X)$ is the space of holomorphic vector fields on X. There is a canonical isomorphism

$$\mathcal{H}^{f,\Omega}_{(0)}/t\mathcal{H}^{f,\Omega}_{(0)} \cong \operatorname{Jac}(f).$$

Proposition 2.9. *There are canonical isomorphisms*

$$\mathcal{H}^{f,\Omega}_{(0)} \xrightarrow{\cong} H^*(\operatorname{PV}(X)[[t]], Q_f) \quad \textit{and} \quad \mathcal{H}^{f,\Omega} \xrightarrow{\cong} H^*(\operatorname{PV}(X)((t)), Q_f),$$

both defined by $[\alpha] \mapsto [\alpha]$, where $\alpha \in \Gamma(X,\mathcal{O}_X)[[t]]$ represents a class on both sides of the first isomorphism, and similarly for the second isomorphism.

Proof. This is essentially an application of the spectral sequence associated to the filtration (2.7). We present the details here. Denote $A^k = \bigoplus_{j-i=k} \mathrm{PV}^{i,j}(X)$. Since $Q_f(\Gamma(X,\mathcal{O}_X)[[t]]) = 0$, we have a well-defined morphism φ of abelian groups,

$$\varphi : \Gamma(X,\mathcal{O}_X)[[t]] \longrightarrow H^0(\mathrm{PV}(X)[[t]], Q_f) = \frac{\mathrm{Ker}(Q_f : A^0[[t]] \to A^1[[t]])}{\mathrm{Im}(Q_f : A^{-1}[[t]] \to A^0[[t]])},$$

given by $\alpha \mapsto [\alpha]$. Let $\sum_{m=0}^{\infty} a_m t^m \in \mathrm{Ker}(Q_f : A^0[[t]] \to A^1[[t]])$ and $a_{-1} := 0$. We have

$$\sum_{m=0}^{\infty} (\bar{\partial}_f a_m + \partial_\Omega a_{m-1}) t^m = 0. \tag{2.11}$$

Since $[a_0] \in H^0(\mathrm{PV}(X), \bar{\partial}_f)$, by Lemma 2.2 there exist $b_0 \in \Gamma(X,\mathcal{O}_X)$ and $c_0 \in A^{-1}$ such that $a_0 = b_0 + \bar{\partial}_f c_0$. Using (2.11) and Lemma 2.2, we can inductively define sequences $b_0, b_1, \cdots$ of holomorphic functions on X and sequences $c_0, c_1, \cdots$ of elements in A^{-1} such that

$$\sum_{m=0}^{\infty} a_m t^m = \sum_{m=0}^{\infty} b_m t^m + Q_f(\sum_{m=0}^{\infty} c_m t^m).$$

Therefore φ is surjective.

Similarly, let $\alpha = \sum_{m\geq 0} a_m t^m \in \ker(\varphi)$. Then $\alpha = Q_f(\beta)$ for $\beta = \sum_{m\geq 0} b_m t^m \in A^{-1}[[t]]$. The leading term gives the relation $a_0 = \bar{\partial}_f b_0$ which implies that $[a_0] = 0$ in $\mathrm{Jac}(f)$ by Lemma 2.2. Therefore we can find $c_0 \in \Gamma(X, T_X)$ such that $\bar{\partial}_f c_0 = a_0$, and

$$\alpha - Q_f(c_0) \in \ker(\varphi) \cap t\Gamma(X,\mathcal{O}_X)[[t]]$$

Inductively, we find $c = \sum_{m\geq 0} c_m t^m \in \Gamma(X, T_X)[[t]]$ such that $\alpha = Q_f(c)$. Thus $\mathrm{Ker}(\varphi) = Q_f(\Gamma(X, T_X)[[t]])$.

By similar but easier arguments, we conclude $H^k(\mathrm{PV}(X)((t)), Q_f) = 0$ if $k \neq 0$.

The proof for $\mathcal{H}^{f,\Omega} \xrightarrow{\cong} H^*(\mathrm{PV}(X)((t)), Q_f)$ is completely similar. □

Proposition-Definition 2.10. *For any $k \in \mathbb{Z}$, there are canonical isomorphisms of $\mathbb{C}[[t]]$-modules:*

$$\mathcal{H}^{f,\Omega}_{(-k)} := t^k \mathcal{H}^{f,\Omega}_{(0)} \xrightarrow{\cong} H^*\left(t^k \mathrm{PV}(X)[[t]], Q_f\right) \xrightarrow{\cong} H^*\left(t^k \Omega^*_X[[t]], d_f^-\right) =: \mathcal{H}_f^{(-k)},$$

where the second isomorphism is induced by $\hat{\Gamma}^-_{\Omega_X}$. The composition of the two isomorphisms is given by $[\alpha] \mapsto [\alpha \vdash \Omega_X]$. Here we have used the notation $\mathcal{H}^{f,\Omega}_{(-k)}$ to distinguish it from $\mathcal{H}_f^{(-k)}$. The family $\{\mathcal{H}^{f,\Omega}_{(-k)}\}_{k\in\mathbb{Z}_{\geq 0}}$ form a filtration of $\mathcal{H}^{f,\Omega}_{(0)}$, called the (semi-infinite) Hodge filtration following [46]. *The associated graded piece of the filtration $\mathcal{F}^k\mathcal{H}^{f,\Omega}_{(0)} = \mathcal{H}^{f,\Omega}_{(-k)}$ is given by*

$$Gr^k_{\mathcal{F}}(\mathcal{H}^{f,\Omega}_{(0)}) \cong t^k \mathrm{Jac}(f).$$

Proof. By Lemma 2.7, we have isomorphic filtered complexes

$$\hat{\Gamma}^-_{\Omega_X} : \left(\mathrm{PV}(X)[[t]], Q_f\right) \to \prod_{m\in\mathbb{Z}} \bigoplus_{j\leq m, i\in\mathbb{Z}} \mathcal{A}^{i,j}(X) \otimes t^m, d_f^-).$$

It is easy to see that it identifies the filtered sub-complexes

$$\hat{\Gamma}^{-}_{\Omega_X} : (\Gamma(X, \wedge^* T_X)[[t]], Q_f) \to (\Omega^*_X[[t]], d_f^-)$$

passing to the cohomology yields the proposition.

□

Using the same V_f and ρ as in the previous subsection, we can define the following operators, in analogy with T_ρ, R_ρ.

Definition 2.11. *Let $Q := \bar{\partial} + t\partial_\Omega$. We define $T^t_\rho : \mathrm{PV}(X)((t)) \to \mathrm{PV}_c(X)((t))$ and $R^t_\rho : \mathrm{PV}(X)((t)) \to \mathrm{PV}(X)((t))$ by*

$$T^t_\rho := \rho + [Q, \rho] V_f \frac{1}{1 + [Q, V_f]} \quad \text{and} \quad R^t_\rho := (1 - \rho) V_f \frac{1}{1 + [Q, V_f]}. \tag{2.12}$$

Clearly, R^t_ρ preserves $\mathrm{PV}_c(X)[[t]]$, and $T^t_\rho(\mathrm{PV}(X)[[t]]) \subset \mathrm{PV}_c(X)[[t]]$. Furthermore, Lemma 2.3 has the following generalization whose proof is completely similar.

Lemma 2.12. *$[Q_f, R^t_\rho] = 1 - T^t_\rho$ as operators on $\mathrm{PV}(X)((t))$.*

As a direct consequence of Proposition 2.9 and Lemma 2.12, we have

Corollary 2.13. *The embeddings*

$$\left(\mathrm{PV}_c(X)[[t]], Q_f\right) \hookrightarrow \left(\mathrm{PV}(X)[[t]], Q_f\right), \quad \left(\mathrm{PV}_c(X)((t)), Q_f\right) \hookrightarrow \left(\mathrm{PV}(X)((t)), Q_f\right)$$

are both quasi-isomorphisms.

Similarly, we will denote by ι_t the induced canonical isomorphisms

$$\iota_t : H^*(\mathrm{PV}_c(X)[[t]], Q_f) \xrightarrow{\cong} \mathcal{H}^{f,\Omega}_{(0)}, \quad \iota_t : H^*(\mathrm{PV}_c(X)((t)), Q_f) \xrightarrow{\cong} \mathcal{H}^{f,\Omega}$$

whose inverse can be realized by

$$\iota_t^{-1} = T^t_\rho.$$

The trace map $\mathrm{Tr} : \mathrm{PV}_c(X) \to \mathbb{C}$ is naturally extended to $\mathrm{PV}_c(X)[[t]] \to \mathbb{C}[[t]]$. It further descends to cohomologies, still denoted as Tr,

$$\mathrm{Tr} : H^*(\mathrm{PV}_c(X)[[t]], Q_f) \to \mathbb{C}[[t]]. \tag{2.13}$$

The composition

$$\widehat{\mathrm{Res}}^f = \mathrm{Tr} \circ \iota_t^{-1} : \mathcal{H}^{f,\Omega}_{(0)} \to \mathbb{C}[[t]] \tag{2.14}$$

provides a complex differential geometric interpretation of the *higher residue map* introduced by the third author [43, 45].

Moreover, we can use the trace map $\mathrm{Tr} : \mathrm{PV}_c(X) \to \mathbb{C}$ to define a pairing

$$\mathrm{PV}_c(X)[[t]] \times \mathrm{PV}_c(X)[[t]] \to \mathbb{C}[[t]] \quad \text{given by} \quad (\alpha_1 \nu_1(t), \alpha_2 \nu_2(t)) \mapsto \nu_1(t)\nu_2(-t)\,\mathrm{Tr}(\alpha_1\alpha_2),$$

where $\alpha_1, \alpha_2 \in \mathrm{PV}_c(X)$ and $\nu_1, \nu_2 \in \mathbb{C}[[t]]$. It follows from Equations (2.5) that the differential Q_f is graded skew-symmetric: for $\alpha_1 \in \mathrm{PV}_c^{i,j}(X)$,

$$\mathrm{Tr}\left(Q_f(\alpha_1\nu_1(t)) \wedge \overline{\alpha_2\nu_2(t)}\right) = -(-1)^{|\alpha_1|}\,\mathrm{Tr}\left(\alpha_1\nu_1(t) \wedge \overline{Q_f(\alpha_2\nu_2(t))}\right),$$

where the operator $^- : \mathrm{PV}(X)((t)) \to \mathrm{PV}(X)((t))$ is defined by

$$\overline{\nu(t)\alpha} := \nu(-t)\alpha, \quad \text{where } \alpha \in \mathrm{PV}(X), \nu(t) \in \mathbb{C}((t))$$

Hence, the above pairing gives rise to a paring on the cohomology:

$$H^*(\mathrm{PV}_c(X)[[t]], Q_f) \times H^*(\mathrm{PV}_c(X)[[t]], Q_f) \to \mathbb{C}[[t]].$$

Combining it with the isomorphism ι_t, we obtain a pairing

$$\mathcal{K}_\Omega^f : \mathcal{H}_{(0)}^{f,\Omega} \times \mathcal{H}_{(0)}^{f,\Omega} \to \mathbb{C}[[t]], \tag{2.15}$$

called the *higher residue pairing*. We remark that $\widehat{\mathrm{Res}}^f = \mathcal{K}_\Omega^f(\cdot, 1)$.

The above constructions are naturally extended to $\mathrm{PV}_c(X)((t))$ and $\mathcal{H}^{f,\Omega}$ by replacing $\mathbb{C}[[t]]$ with $\mathbb{C}((t))$, and we have

$$\mathcal{K}_\Omega^f : \mathcal{H}^{f,\Omega} \times \mathcal{H}^{f,\Omega} \to \mathbb{C}((t)).$$

Remark 2.14. *The isomorphisms $\hat{\Gamma}_{\Omega_X}^+$ and $\hat{\Gamma}_{\Omega_X}^-$ are related via a pairing as follows. Consider*

$$\mathrm{PV}_c(X)((t)) \times \mathrm{PV}_c(X)((t)) \to \mathbb{C}((t))$$

defined by $(\alpha_1\nu_1(t), \alpha_2\nu_2(t)) \mapsto t^{1-n}\int_X \hat{\Gamma}_{\Omega_X}^+(\alpha_1\nu_1(t)) \wedge \overline{\Gamma_{\Omega_X}^-(\alpha_2\nu_2(t))}$, which differs slightly from the pairing defined above by a sign. It is easy to check that this pairing descends to the cohomology, which coincides with $\mathcal{K}_\Omega^f$.

Example 2.15 (A_m-singularity)**.** *Let $X = \mathbb{C}$, $f = \frac{z^{m+1}}{m+1}$, and $\Omega_X = dz$. It is easy to compute that $V_f = \frac{1}{f'}\partial_z\wedge$. Let $h(z)$ be a holomorphic function, and C be a small circle centered at the origin inside which the value of the cut-off function ρ is identically equal to 1. Then*

$$\begin{aligned}
\widehat{\mathrm{Res}}^f(h) &= \sum_{r\geq 0}(-t)^r\,\mathrm{Tr}\left((\bar\partial\rho)\,V_f\,[Q, V_f]^r\,h\right) = \sum_{r\geq 0}(-t)^r\,\mathrm{Tr}\left((\bar\partial\rho)\,\frac{\partial_z\wedge}{f'}\left(\frac{\partial}{\partial z}\frac{1}{f'}\right)^r h\right)\\
&= \sum_{r\geq 0}(-t)^r\int_{\mathbb{C}}\left((\bar\partial\rho)\,\frac{dz}{f'}\left(\frac{\partial}{\partial z}\frac{1}{f'}\right)^r h\right) = \sum_{r\geq 0}(-t)^r\oint_C\left(\frac{dz}{f'}\left(\frac{\partial}{\partial z}\frac{1}{f'}\right)^r h\right)\\
&= \sum_{r\geq 0}(-t)^r\,\mathrm{Res}_{z=0}\left(\frac{1}{f'}\left(\frac{\partial}{\partial z}\frac{1}{f'}\right)^r h dz\right)\\
&= \sum_{r\geq 0}(-t)^r\prod_{k=0}^{r-1}(m+k(m+1))\,\mathrm{Res}_{z=0}\left(\frac{h dz}{z^{r(m+1)+m}}\right).
\end{aligned}$$

In particular, the leading term with $r = 0$ gives the ordinary residue $\mathrm{Res}_{z=0}\frac{h dz}{f'}$.

2.3.2. *Opposite filtrations.* Following Givental [22], we equip $\mathrm{PV}_c(X)((t))$ with a (graded) symplectic pairing

$$\omega(\alpha_1\nu_1(t),\alpha_2\nu_2(t)) := \mathrm{Res}_{t=0}(\nu_1(t)\nu_2(-t)dt)\,\mathrm{Tr}(\alpha_1\alpha_2),$$

where $\alpha_1,\alpha_2 \in \mathrm{PV}_c(X)$ and $\nu_1,\nu_2 \in \mathbb{C}((t))$. Clearly, $t^k\,\mathrm{PV}_c(X)[[t]]$ is an isotropic subspace of $\mathrm{PV}_c(X)((t))$ with respect to ω, for any $k \in \mathbb{Z}_{\geq 0}$. This gives rise to a paring on the cohomology. Combing it with the isomorphism ι_t, we obtain a symplectic paring:

$$\omega : \mathcal{H}^{f,\Omega} \times \mathcal{H}^{f,\Omega} \to \mathbb{C} \quad \text{given by} \quad \omega(s,s') := \mathrm{Res}_{t=0}\,\mathcal{K}^f_\Omega(s,s')dt.$$

With respect to this pairing, all the Hodge filtered pieces $\mathcal{H}^{f,\Omega}_{(-k)}$ are isotropic subspaces of $\mathcal{H}^{f,\Omega}$.

Lemma/Definition 2.16. *Let $\mathcal{L}$ be a linear subspace of $\mathcal{H}^{f,\Omega}$ that satisfies both (1) $\mathcal{H}^{f,\Omega} = \mathcal{H}^{f,\Omega}_{(0)} \oplus \mathcal{L}$ and (2) $t^{-1}\mathcal{L} \subset \mathcal{L}$. Let $B := \mathcal{H}^{f,\Omega}_{(0)} \cap t\mathcal{L}$. Then the following are equivalent:*

$$(3)\ \ \omega(\mathcal{L},\mathcal{L}) = 0; \qquad (3)'\ \ \mathcal{K}^f_\Omega(\mathcal{L},\mathcal{L}) \subset t^{-2}\mathbb{C}[t^{-1}]; \qquad (3)''\ \ \mathcal{K}^f_\Omega(B,B) \subset \mathbb{C}.$$

If $\mathcal{L}$ further satisfies (3), (3)′ or (3)″, we call it an opposite filtration *of $\mathcal{H}^{f,\Omega}$.*

Proof. Since $\mathcal{L}$ satisfies (1) and (2), we have $\mathcal{H}^{f,\Omega} = t^k\mathcal{H}^{f,\Omega} = \mathcal{H}^{f,\Omega}_{(-k)} \oplus t^k\mathcal{L}, \forall k \in \mathbb{Z}_{\geq 0}$. It leads to the decomposition $\mathcal{H}^{f,\Omega}_{(-k)} = \mathcal{H}^{f,\Omega}_{(-k-1)} \oplus (\mathcal{H}^{f,\Omega}_{(-k)} \cap t^{k+1}\mathcal{L})$ and induces an isomorphism of vector spaces

$$\mathcal{H}^{f,\Omega}_{(-k)}/\mathcal{H}^{f,\Omega}_{(-k-1)} \cong \mathcal{H}^{f,\Omega}_{(-k)} \cap t^{k+1}\mathcal{L}$$

for any $k \in \mathbb{Z}_{\geq 0}$. In particular, we have $B = \mathcal{H}^{f,\Omega}_{(0)} \cap t\mathcal{L} \subset \mathcal{H}^{f,\Omega}_{(0)}$ and $B \cong \mathcal{H}^{f,\Omega}_{(0)}/\mathcal{H}^{f,\Omega}_{(-1)} \cong \mathrm{Jac}(f)$. As a consequence, we have

$$\mathcal{H}^{f,\Omega} = B((t)), \quad \mathcal{H}^{f,\Omega}_{(0)} = B[[t]], \quad \text{and } \mathcal{L} = t^{-1}B[t^{-1}].$$

Clearly, $(3)'' \Rightarrow (3)'$, and $(3)' \Rightarrow (3)$. We show that $(3) \Rightarrow (3)''$.

Assuming (3), we only need to show that $\mathcal{K}^f_\Omega(B,B) \subset \mathbb{C}[t^{-1}]$ since $\mathcal{K}^f_\Omega(B,B) \subset \mathbb{C}[[t]]$. We prove it by contradiction. Let $s,s' \in B$ such that $\mathcal{K}^f_\Omega(s,s') = \sum_{i=-N}^{\infty} a_i t^i$ with $a_k \neq 0$ for some $k > 0$. This would imply that $t^{-k}s, t^{-1}s' \in \mathcal{L}$ and $\omega(t^{-k}s, t^{-1}s') = -a_k \neq 0$, contradicting to the hypothesis (3). □

Remark 2.17. *The notion of opposite filtration, introduced by Deligne* [15]*, was used by M. Saito for the Hodge structures of the singularity* [51, 53] *to construct primitive forms. However, in the present paper, we will use it for Denition 2.16 since in the parallel compact Calabi-Yau case, the analogue opposite filtration is equivalent to a usual splitting of the Hodge filtration on $H^*(X)$ (see for example* [14, Lemma 5.2.2]*). Therefore we will keep the terminology of opposite filtration with respect to the semi-infinite Hodge filtration $\{\mathcal{H}^{f,\Omega}_{(-k)}\}$, which is sometimes called a polarization in the context of symplectic geometry* [23].

As we will see in section 4, the space of primitive forms is essentially identified with the space of appropriate opposite filtrations.

3. PRIMITIVE FORMS

3.1. **Frame setup.** We recall some notions from [43], [48]. Recall that $X \subset \mathbb{C}^n$ is a Stein domain and $f : X \to \mathbb{C}$ is a holomorphic function with finite critical points.

Definition 3.1. *We call a 4-tuple (Z, S, p, F) a frame associated to (X, f), if the following are satisfied.*

(1) *The space Z is a Stein domain in $\mathbb{C}^{n+\mu}$, S is a Stein open neighborhood of the origin $\mathbf{0}$ of $\mathbb{C}^\mu$, $p : Z \to S$ is a Stein map with $p^{-1}(\mathbf{0}) \cong X$, and $F : Z \to \mathbb{C}$ is a holomorphic function on Z.*
(2) *The restriction $p|_{\mathcal{C}(F)}$ of p to the relative critical set $\mathcal{C}(F)$ (see Equation* (3.2)*) is a proper morphism.*
(3) *The function F is a universal unfolding of f in the following sense.*
   (a) *The restriction $F|_{p^{-1}(\mathbf{0})}$ coincides with the composition $p^{-1}(\mathbf{0}) \xrightarrow{\cong} X \xrightarrow{f} \mathbb{C}$.*
   (b) *The Kodaira-Spencer map*

$$\mathrm{KS} : \mathcal{T}_S \to p_*\mathcal{O}_{\mathcal{C}(F)}, \tag{3.1}$$

to be described below, is an isomorphism of sheaves of $\mathcal{O}_S$-modules. Let $\mathcal{T}_S$ (resp. $\mathcal{T}_Z$) denote the holomorphic tangent sheaf of S (resp. Z). For any open set $U \subset S$ and any $V \in \Gamma(U, \mathcal{T}_S)$, we take a lifting $\widetilde{V} \in \Gamma\left(p^{-1}(U), \mathcal{T}_Z\right)$ with $p_*(\widetilde{V}) = V$. Then the Kodaira-Spencer map is defined by $\mathrm{KS}|_U(V) := \widetilde{V}(F)|_{\mathcal{C}(F)}$, which is independent of the choice of the lifting $\widetilde{V}$.

In practice, it is convenient to fix a projection $\pi_X : Z \to X$ (where we make Z smaller if necessary), so that there is an embedding $Z \hookrightarrow X \times S \subset \mathbb{C}^n \times \mathbb{C}^\mu$. In particular, the fiber $p^{-1}(\mathbf{0})$ is given by $X \times \{0\}$. With respect to such embedding, the Stein map p and the projection π_X are the restriction of the natural projections $X \times S \to S$ and $X \times S \to X$ to Z, respectively. In terms of coordinates $(\mathbf{z}, \mathbf{u})$ of $\mathbb{C}^n \times \mathbb{C}^\mu$, the relative critical set $\mathcal{C}(F)$ is a subvariety of Z defined by

$$\mathcal{C}(F) := \{\partial_{z_1} F = \cdots = \partial_{z_n} F = 0\}, \quad \text{and } \mathcal{O}_{\mathcal{C}(F)} = \mathcal{O}_Z \big/ (\partial_{z_1} F, \cdots, \partial_{z_n} F). \tag{3.2}$$

We can always achieve a frame (Z, S, p, F) (see e.g. [48] for a precise choice when $f : X \to \mathbb{C}$ has an isolated critical point). Furthermore, we have

Proposition 3.2 ([43])**.** *The structure sheaf $\mathcal{O}_{\mathcal{C}(F)}$ is flat over $\mathcal{O}_S$, so that $p_*\mathcal{O}_{\mathcal{C}(F)}$ is a locally free sheaf on S of rank $\dim_{\mathbb{C}} \mathrm{Jac}(f)$.*

Definition 3.3. *The* Euler vector field *associated to (Z, S, p, F) is defined by*

$$E := \mathrm{KS}^{-1}([F]) \in \Gamma(S, \mathcal{T}_S). \tag{3.3}$$

Notations: In this section, we will study properties in the unfolding case parallel to those in the previous section. Therefore, we will use the same notations as in section 2 to denote the relevant operators in the unfolding case, such as ρ, T_ρ, R_ρ, T_ρ^t, ι, ι_t, Tr, etc..

3.2. **Relative polyvector fields.** Given a frame (Z, S, p, F), we denote by $T_{Z/S}$ the fiberwise holomorphic tangent bundle on Z relative to S, and consider the space of smooth relative polyvector fields

$$\mathrm{PV}(Z/S) = \bigoplus_{\substack{0\le i\le n\\ 0\le j\le n+\mu}} \mathrm{PV}^{i,j}(Z/S), \quad \text{where} \quad \mathrm{PV}^{i,j}(Z/S) := \mathcal{A}^{0,j}\left(Z, \wedge^i T_{Z/S}\right).$$

Similarly, we consider the space of smooth relative differential forms

$$\mathcal{A}(Z/S) = \bigoplus_{\substack{0\le i\le n\\ 0\le j\le n+\mu}} \mathcal{A}^{i,j}(Z/S), \quad \mathcal{A}^{i,j}(Z/S) = \mathcal{A}^{0,j}\left(Z, \Omega^i_{Z/S}\right).$$

We note that $T_{Z/S}$ contains n-directions tangent to the fiber of p, whereas $\mathcal{A}^{0,1}$ contains $(n+\mu)$-directions of anti-holomorphic cotangent vectors. We fix a family of holomorphic volume form (which is nowhere vanishing)

$$\Omega_{Z/S} \in H^0(Z, \Omega^n_{Z/S}). \tag{3.4}$$

Contraction with $\Omega_{Z/S}$ induces an isomorphism

$$\Gamma_\Omega : \mathrm{PV}^{i,j}(Z/S) \to \mathcal{A}^{n-i,j}(Z/S)$$

of $C^\infty(Z)$-modules. Consequently, we have a relative version of the operators

$$\bar\partial : \mathrm{PV}^{i,j}(Z/S) \to \mathrm{PV}^{i,j+1}(Z/S), \quad \partial_\Omega : \mathrm{PV}^{i,j}(Z/S) \to \mathrm{PV}^{i-1,j}(Z/S),$$

where ∂_Ω is defined by

$$\partial_\Omega(\alpha) := \Gamma_\Omega^{-1}\big(\partial(\Gamma_\Omega(\alpha))\big).$$

Example 3.4. *In coordinates $(\mathbf{z}, \mathbf{u})$ of $\mathbb{C}^n \times \mathbb{C}^\mu$, every $\alpha \in \mathrm{PV}^{i,j}(Z/S)$ is of the form*

$$\alpha = \sum_{I, J_1, J_2} \alpha^I_{J_1,J_2}(\mathbf{z}, \mathbf{u}) d\bar z_{J_1} d\bar u_{J_2} \otimes \frac{\partial}{\partial z_I}$$

where the summation is over subsequences I, J_1, J_2 of $[1, \cdots, n]$, $[1, \cdots, n]$ and $[1, \cdots, \mu]$, respectively, with $|I| = i$ and $|J_1| + |J_2| = j$. The volume form can be expressed by

$$\Omega_{Z/S} = \frac{1}{\lambda(\mathbf{z}, \mathbf{u})} dz_1 \wedge \cdots \wedge dz_n.$$

Then we have

$$\partial_\Omega \alpha = \sum_{k=1}^n \sum_{I, J_1, J_2} \lambda \partial_{z_k}\Big(\frac{\alpha^I_{J_1,J_2}}{\lambda}\Big) \frac{\partial}{\partial \partial_k}\big(d\bar z_{J_1} d\bar u_{J_2} \otimes \frac{\partial}{\partial z_I}\big) \quad \textit{and} \quad \bar\partial \alpha = \sum_{I, J_1, J_2} (\bar\partial \alpha^I_{J_1,J_2}) d\bar z_{J_1} d\bar u_{J_2} \otimes \frac{\partial}{\partial z_I}.$$

The operator ∂_Ω induces a bracket $\{\text{-},\text{-}\}$ on $\mathrm{PV}(Z/S)$, defined in the same way as formula (2.2). We define the twisted coboundary operator in the relative setting by

$$\bar\partial_F = \bar\partial + \{F, \text{-}\} : \mathrm{PV}(Z/S) \to \mathrm{PV}(Z/S).$$

We also consider the subspace

$$\mathrm{PV}_c(Z/S) := \{\alpha \in \mathrm{PV}(Z/S) \mid p|_{\mathrm{supp}(\alpha)} \text{ is proper}\}.$$

Clearly, all the operators $\bar{\partial}, \partial_\Omega, \{\text{-},\text{-}\}, \bar{\partial}_F$ preserve the subspace $\mathrm{PV}_c(Z/S)$. Since $p : Z \to S$ is a Stein map, we can obtain sheaf-theoretic version of similar propositions in the previous section as follows.

Let $(\mathbf{z}, \mathbf{u})$ be the coordinates of $\mathbb{C}^n \times \mathbb{C}^\mu \supset Z$. Consider the following smooth operator V_F of degree -1 on $\mathrm{PV}((Z \setminus \mathcal{C}(F))/S)$:

$$V_F = \frac{1}{\sum\limits_{j=1}^{n} |\partial_{z_j} F|^2} \sum_{i=1}^{n} (\overline{\partial_{z_i} F}) \partial_i \wedge .$$

Choose a smooth function $\rho \in C^\infty(Z)$ such that $\rho|_{U_1} \equiv 1$ and $\rho|_{Z \setminus U_2} \equiv 0$ for some open neighborhoods U_1, U_2 of $\mathcal{C}(F)$ in Z with the properties (a) $U_1 \subset \overline{U_1} \subset U_2 \subset \overline{U_2} \subset Z$, and (b) $p|_{\overline{U_2}}$ is proper. Then the operators

$$T_\rho := \rho + (\bar{\partial}\rho) V_F \frac{1}{1 + [\bar{\partial}, V_F]} \quad \text{and} \quad R_\rho := (1 - \rho) V_F \frac{1}{1 + [\bar{\partial}, V_F]}$$

are both defined on the whole space $\mathrm{PV}(Z/S)$. Furthermore, we have

$$T_\rho(\mathrm{PV}(Z/S)) \subset \mathrm{PV}_c(Z/S),$$

such that

$$\left[\bar{\partial}_F, R_\rho\right] = 1 - T_\rho. \tag{3.5}$$

Let $R^\bullet p_* \left(\mathrm{PV}_c(Z/S), \bar{\partial}_F\right)$ be the sheaf of $\mathcal{O}_S$-modules on S obtained by the sheafification of $\{H^* \left(\mathrm{PV}_c(p^{-1}(U)/U), \bar{\partial}_F\right) \mid U \subset S \text{ is open Stein}\}$. Similarly, we obtain $R^\bullet p_* \left(\mathrm{PV}(Z/S), \bar{\partial}_F\right)$.

Proposition 3.5. *There is a natural isomorphism of sheaves*

$$p_* \mathcal{O}_{\mathcal{C}(F)} \xrightarrow{\cong} R^\bullet p_* \left(\mathrm{PV}(Z/S), \bar{\partial}_F\right).$$

Proof. The statement follows immediately from the sheafification of Lemma 2.2. □

Proposition 3.6. *The natural embedding* $\left(\mathrm{PV}_c(Z/S), \bar{\partial}_F\right) \hookrightarrow \left(\mathrm{PV}(Z/S), \bar{\partial}_F\right)$ *is quasi-isomorphic, inducing a canonical isomorphism of sheaves*

$$\iota : R^\bullet p_* \left(\mathrm{PV}_c(Z/S), \bar{\partial}_F\right) \xrightarrow{\cong} p_* \mathcal{O}_{\mathcal{C}(F)}$$

whose inverse morphism can be represented by T_ρ.

Proof. Formula (3.5) implies that $H^*\big(\mathrm{PV}(p^{-1}(U)/U)/\mathrm{PV}_c(p^{-1}(U)/U), \bar{\partial}_F\big) = 0$ for any open (Stein) subset U of S. The first statement follows, and the second statement becomes a direct consequence of Proposition 3.5. □

In analogy with the descriptions in section 2.3, we consider the complex $\mathrm{PV}(Z/S)((t))$ (resp. $\mathrm{PV}_c(Z/S)((t))$) of $\mathrm{PV}(Z/S)$ (resp. $\mathrm{PV}_c(Z/S)$)-valued Laurent series in the descendant variable t, with respect to the following twisted coboundary operator

$$Q_F := \bar{\partial}_F + t\partial_\Omega.$$

Proposition-Definition 3.7. *There are isomorphisms of sheaves of cochain complexes*

$$\hat{\Gamma}^{\pm}_{\Omega} : (\mathrm{PV}(Z/S)((t)), Q_F) \to \big(\mathcal{A}(Z/S)((t)), d_F^{\pm}\big) \quad \textit{with} \quad d_F^{+} := d + \frac{dF}{t}\wedge,\ d_F^{-} = td + (dF)\wedge,$$

defined by

$$t^k\alpha \in t^k\,\mathrm{PV}^{i,j}(Z/S) \mapsto \hat{\Gamma}^{+}_{\Omega}(t^k\alpha) := t^{k+i-1}\alpha \vdash \Omega_{Z/S} \in t^{k+i-1}\mathcal{A}^{n-i,j}(Z/S),$$
$$t^k\alpha \in t^k\,\mathrm{PV}^{i,j}(Z/S) \mapsto \hat{\Gamma}^{-}_{\Omega}(t^k\alpha) := t^{k+j}\alpha \vdash \Omega_{Z/S} \in t^{k+j}\mathcal{A}^{n-i,j}(Z/S).$$

Proof. The proof is the same as Proposition-Definition 2.7. □

The following operators are defined on $\mathrm{PV}(Z/S)((t))$:

$$T^t_\rho := \rho + [Q,\rho]V_F\frac{1}{1+[Q,V_F]} \quad \text{and} \quad R^t_\rho := (\mathrm{id}-\rho)V_F\frac{1}{1+[Q,V_F]},$$

where $Q := \bar{\partial} + t\partial_\Omega$. The relation

$$[Q_F, R^t_\rho] = 1 - T^t_\rho$$

still holds where defined. In addition, we have

$$T^t_\rho(\mathrm{PV}(Z/S)[[t]]) \subset \mathrm{PV}_c(Z/S)[[t]], \quad T^t_\rho(\mathrm{PV}(Z/S)((t))) \subset \mathrm{PV}_c(Z/S)((t)).$$

Consider the sheaves $\mathcal{H}^{F,\Omega}_{(0)}$ and $\mathcal{H}^{F,\Omega} := \mathcal{H}^{F,\Omega}_{(0)} \otimes_{\mathbb{C}[[t]]} \mathbb{C}((t))$, where

$$\mathcal{H}^{F,\Omega}_{(0)}(U) := \frac{\Gamma(p^{-1}(U),\mathcal{O}_Z)[[t]]}{\mathrm{Im}\left(Q_F : \Gamma(p^{-1}(U),\mathcal{T}_{Z/S})[[t]] \to \Gamma(p^{-1}(U),\mathcal{O}_Z)[[t]]\right)}, \text{ for Stein } U \subset S.$$

Proposition 3.8. *There are canonical isomorphisms of sheaves of $\mathcal{O}_S$-modules*

$$\iota_t : R^\bullet p_*\big(\mathrm{PV}_c(Z/S)[[t]], Q_F\big) \xrightarrow{\cong} R^\bullet p_*(\mathrm{PV}(Z/S)[[t]], Q_F) \cong \mathcal{H}^{F,\Omega}_{(0)}.$$

The inverse of the composition ι_t of the above isomorphism can be represented by T^t_ρ.

Proof. The proof is completely similar to the proofs of Proposition 2.9 and Corollary 2.13. □

Proposition-Definition 3.9. *For any $k \in \mathbb{Z}$, we define $\mathcal{H}^{F,\Omega}_{(-k)} := t^k\mathcal{H}^{F,\Omega}_{(0)}$. The family version of the filtered complex isomorphism $\hat{\Gamma}^{-}_{\Omega_X}$ gives a canonical isomorphism of sheaves of $\mathcal{O}_S[[t]]$-modules:*

$$\hat{\Gamma}^{-}_{\Omega} : \mathcal{H}^{F,\Omega}_{(-k)} \xrightarrow{\cong} R^\bullet p_*(t^k\Omega^*_{Z/S}[[t]], d_F^{-}) =: \mathcal{H}_F^{(-k)}$$

defined by $t^k[\alpha] \mapsto [t^k\alpha \vdash \Omega_{Z/S}]$. The (semi-infinite) Hodge filtration of $\mathcal{H}^{F,\Omega}_{(0)}$ is defined by $\{\mathcal{H}^{F,\Omega}_{(-k)}\}_{k\in\mathbb{Z}_{\geq 0}}$.

Proof. The argument is completely similar to the proof of Proposition-Definition 2.10. □

3.3. **Variation of semi-infinite Hodge structures.** The general structures of higher residues and primitive forms developed in [45, 46] were geometrically reformulated by Barannikov as the notion of *variation of semi-infinite Hodge structures* [2, 4]. We will adopt this geometric notion and present the corresponding constructions in our case.

3.3.1. *Gauss-Manin connection.* The sheaf $\mathcal{H}^{F,\Omega}$ inherits a flat Gauss-Manin connection [8, 46]

$$\nabla^{\Omega} : \mathcal{H}^{F,\Omega} \to \Omega_S^1 \otimes \mathcal{H}^{F,\Omega}$$

from its identification with differential forms via $\Omega_{Z/S}$. To describe this, we consider the isomorphism of cochain complexes in Proposition-Definition 3.7

$$\hat{\Gamma}_{\Omega}^{+} : (\mathrm{PV}(Z/S)((t)), Q_F) \to \left(\mathcal{A}(Z/S)((t)), d_F^{+}\right) \quad \text{with} \quad d_F^{+} := d + \frac{dF}{t}\wedge,$$

which induces an isomorphism of sheaves of $\mathcal{O}_S$-modules, still denoted as

$$\hat{\Gamma}_{\Omega}^{+} : \mathcal{H}^{F,\Omega} \cong R^{\bullet}p_* \left(\mathrm{PV}(Z/S)((t)), Q_F\right) \xrightarrow{\cong} R^{\bullet}p_* \left(\mathcal{A}(Z/S)((t)), d_F^{+}\right).$$

Lemma/Definition 3.10. *The sheaf $R^{\bullet}p_* \left(\mathcal{A}(Z/S)((t)), d_F^{+}\right)$ naturally carries the Gauss-Manin connection*

$$\nabla^{\mathrm{GM}} : R^{\bullet}p_* \left(\mathcal{A}(Z/S)((t)), d_F^{+}\right) \to \Omega_S^1 \otimes R^{\bullet}p_* \left(\mathcal{A}(Z/S)((t)), d_F^{+}\right).$$

Proof. This is a twisted version of the usual differential geometric construction.

Let $V \in \mathcal{T}_S$ be a holomorphic vector field on S, and we choose an arbitrary smooth lifting $\tilde{V}$ to a vector field of type $(1,0)$ on Z such that

$$p_*(\tilde{V}) = V.$$

Let $[\alpha] \in R^{\bullet}p_* \left(\mathcal{A}(Z/S)((t)), d_F^{+}\right)$ be a section represented by $\alpha \in \mathcal{A}(Z/S)((t))$. Then the Gauss-Manin connection is defined by

$$\nabla_V^{\mathrm{GM}}[\alpha] = \left[\mathcal{L}_{\tilde{V}}\alpha + \frac{\partial_{\tilde{V}}(F)}{t}\alpha\right] \tag{3.6}$$

where $\mathcal{L}_{\tilde{V}}$ is the Lie derivative with respect to $\tilde{V}$. It is routine to check that ∇^{GM} is well-defined and gives a flat connection. □

Definition 3.11. *We define the Gauss-Manin connection ∇^{Ω} on $\mathcal{H}^{F,\Omega}$ associated with $\Omega_{Z/S}$ by*

$$\nabla^{\Omega} := \left(\hat{\Gamma}_{\Omega}^{+}\right)^{-1} \circ \left(\nabla^{\mathrm{GM}}\right) \circ \hat{\Gamma}_{\Omega}^{+}.$$

The next properties follow directly from the definition of the connection ∇^{Ω}:

Lemma 3.12 (Transversality). $\nabla^{\Omega}|_{\mathcal{H}_{(0)}^{F,\Omega}} : \mathcal{H}_{(0)}^{F,\Omega} \to \Omega_S^1 \otimes t^{-1}\mathcal{H}_{(0)}^{F,\Omega} = \Omega_S^1 \otimes \mathcal{H}_{(1)}^{F,\Omega}$.

Lemma 3.13 (Flatness). $\nabla^{\Omega} \circ \nabla^{\Omega} = 0$.

Now we describe the Gauss-Manin connection ∇^Ω in coordinates $(\mathbf{z}, \mathbf{u})$ of $\mathbb{C}^n \times \mathbb{C}^\mu$. Since $p : Z \to S$ is the restriction of the natural projection $\mathbb{C}^n \times \mathbb{C}^\mu \to \mathbb{C}^\mu$, a special lifting of a local section $V = \sum_{j=1}^{\mu} g_j(\mathbf{u}) \frac{\partial}{\partial u_j}$ of $\mathcal{T}_S$ is given by

$$\hat{V} := \sum_{j=1}^{\mu} (p^* g_j) \frac{\partial}{\partial u_j}, \tag{3.7}$$

where $\frac{\partial}{\partial u_j}$ is defined using the projection $\pi_X : Z \to X$.

Proposition 3.14. *If $\Omega_{Z/S} = \frac{1}{\lambda(\mathbf{z})} dz_1 \wedge \cdots \wedge dz_n \in \pi_X^*(\Gamma(X, \Omega_X^n))$, where λ is a nonwhere vanishing holomorphic function on Z depending only on the fiber direction, then*

$$\nabla_V^\Omega [s] = \left[\partial_{\hat{V}} s + \frac{\partial_{\hat{V}} F}{t} s \right], \tag{3.8}$$

where $[s] \in \mathcal{H}^{F,\Omega}$ is represented by $s \in \Gamma(Z, \mathcal{O}_Z)((t))$.

As we will see in section 3.5, the choice of $\Omega_{Z/S}$ is not essential. Hence, in this paper we will always choose the volume form $\Omega_{Z/S}$ to be independent of the deformation parameters. In such case, the calculation of the connection ∇^Ω is simplified as in the above proposition.

3.3.2. *Extended Gauss-Manin connection.* We would like to think about t as a coordinate on the formal punctured disk $\hat{\Delta}^* = \operatorname{Spec} \mathbb{C}((t))$, and $\mathcal{H}^{F,\Omega}$ is naturally a locally free sheaf on $S \times \hat{\Delta}^*$. Then we can extend the flat connection ∇^Ω to $S \times \hat{\Delta}^*$ as follows. For $s \in \mathrm{PV}^{i,*}(Z/S)((t))$, we define

$$\nabla^\Omega_{t\partial_t} s = t \nabla^\Omega_{\partial_t} s := \left(t \frac{\partial}{\partial t} + i - \frac{F}{t} \right) s. \tag{3.9}$$

Clearly, $\nabla^\Omega_{t\partial_t}$ preserves $\mathrm{PV}_c(Z/S)((t))$. The next lemma follows from direct calculations

Lemma 3.15. *$[\nabla^\Omega_{t\partial_t}, Q_F] = 0$ holds as operators on $\mathrm{PV}(Z/S)((t))$, hence $\nabla^\Omega_{t\partial_t}$ descends to an operator on $\mathcal{H}^{F,\Omega}$. Furthermore for any $V \in \Gamma(S, \mathcal{T}_S)$, we have $[\nabla^\Omega_{t\partial_t}, \nabla^\Omega_V] = 0$.*

It follows that ∇^Ω is extended to a well-defined flat connection on $S \times \hat{\Delta}^*$ for the sheaf $\mathcal{H}^{F,\Omega}$ viewed as $\mathcal{O}_{S \times \hat{\Delta}^*}$-module, which we will call the *extended Gauss-Manin connection*. The originally defined connection ∇^Ω on S will be referred to as the *non-extended* connection.

3.3.3. *Higher residues.* We define a cochain map Tr of complexes of sheaves on S,

$$\mathrm{Tr} : (\mathrm{PV}_c(p^{-1}(U)/U)((t)), Q_F) \to (\mathcal{A}^{0,*}(U)((t)), \bar{\partial}_S),$$

by

$$\mathrm{Tr}(\nu(t)\alpha) := \nu(t) \int_{p^{-1}(U)/U} (\alpha \vdash \Omega_{Z/S}) \wedge \Omega_{Z/S}, \quad \nu(t) \in \mathbb{C}((t)), \alpha \in \mathrm{PV}_c(p^{-1}(U)/U)$$

where $\bar{\partial}_S$ denotes the usual $\bar{\partial}$ operator on differential forms on S, $\int_{p^{-1}(U)/U}$ denotes the fiberwise integration, and $U \subset S$ is a Stein open subset. It gives rise to a map of sheaves of cohomologies

$$\mathrm{Tr} : R^\bullet p_*(\mathrm{PV}_c(Z/S)((t))) \to \mathcal{O}_S((t)), \tag{3.10}$$

as the family version of the trace map in formula (2.13).

Similarly, we define a pairing

$$\mathcal{K}_\Omega^{F,c}(\text{-},\text{-}) : \mathrm{PV}_c(p^{-1}(U)/U)((t)) \otimes \mathrm{PV}_c(p^{-1}(U)/U)((t)) \to \mathcal{A}^{0,*}(U)((t)),$$

by

$$\mathcal{K}_\Omega^{F,c}(\nu_1(t)\alpha_1, \nu_2(t)\alpha_2) := \nu_1(t)\nu_2(-t)\,\mathrm{Tr}\,(\alpha_1\alpha_2)$$

where $\nu_1, \nu_2 \in \mathbb{C}((t))$ and $\alpha_1, \alpha_2 \in \mathrm{PV}_c(p^{-1}(U)/U)$.

Lemma 3.16. *The pairing $\mathcal{K}_\Omega^{F,c}(\text{-},\text{-})$ on $\mathrm{PV}_c(p^{-1}(U)/U)$ gives rise to that on cohomology:*

$$\mathcal{K}_\Omega^{F,c}(\text{-},\text{-}) : R^\bullet p_*(\mathrm{PV}_c(Z/S)((t))) \otimes R^\bullet p_*(\mathrm{PV}_c(Z/S)((t))) \to \mathcal{O}_S((t)).$$

Proof. The relative version of formula (2.5) says that $\mathcal{K}_\Omega^{F,c}$ is a cochain map. Let $\alpha, \beta \in \mathrm{PV}_c(Z/S)((t))$ be Q_F-closed, representing sections of $R^\bullet p_*(\mathrm{PV}_c(Z/S)((t)))$. We can assume that both α, β have degree 0, hence $\mathcal{K}_\Omega^{F,c}(\alpha,\beta) \in C^\infty(S)((t))$. Moreover,

$$\bar{\partial}_S \mathcal{K}_\Omega^{F,c}(\alpha,\beta) = \mathcal{K}_\Omega^{F,c}(Q_F\alpha,\beta) + \mathcal{K}_\Omega^{F,c}(\alpha, Q_F\beta) = 0,$$

i.e., $\mathcal{K}_\Omega^{F,c}(\alpha,\beta) \in \mathcal{O}_S((t))$. □

This allows us to give the complex differential geometric construction of the higher residue pairing [45].

Definition 3.17. *We define the* ***higher residue pairing***

$$\mathcal{K}_\Omega^F(\text{-},\text{-}) : \mathcal{H}^{F,\Omega} \otimes \mathcal{H}^{F,\Omega} \to \mathcal{O}_S((t))$$

by $\mathcal{K}_\Omega^F([s],[s']) := \mathcal{K}_\Omega^{F,c}\left(\iota_t^{-1}([s]), \iota_t^{-1}([s'])\right)$ for sections $[s],[s']$ of $\mathcal{H}^{F,\Omega}$ (see Proposition 3.8 for the isomorphism ι_t).

Every local section $\iota_t^{-1}[s]$ of $R^\bullet p_*(\mathrm{PV}_c(Z/S)((t)))$ has a representative $T_\rho^t(s)$. Hence we have the following explicit formula

$$\mathcal{K}_\Omega^F([s],[s']) = \mathrm{Tr}\left(T_\rho^t(s)\overline{T_\rho^t(s')}\right)$$

where the $^-$ operator is defined by

$$^- : \mathrm{PV}(Z/S)((t)) \to \mathrm{PV}(Z/S)((t)), \quad \overline{\nu(t)\alpha} := \nu(-t)\alpha \quad \text{for } \alpha \in \mathrm{PV}(Z/S).$$

Remark 3.18. *Since both $T_\rho^t(s)$ and $T_\rho^t(s')$ have compactly support along the fiber direction,*

$$\mathcal{K}_\Omega^F([s],[s']) = \mathrm{Tr}\left(T_\rho^t(s)\overline{s'}\right) = \mathrm{Tr}\left(s\,\overline{T_\rho^t(s')}\right).$$

Definition 3.19. *The higher residue map associated to (Z, S, p, F) and Ω is defined by the composition*

$$\widehat{\mathrm{Res}}^F := \mathrm{Tr} \circ \iota_t^{-1} : \mathcal{H}^{F,\Omega} \to \mathcal{O}_S((t)).$$

As before, if we restrict $\mathcal{K}_\Omega^F$ to $\mathcal{H}_{(0)}^{F,\Omega}$, it takes values in $\mathcal{O}_S[[t]]$. That is,

$$\mathcal{K}_\Omega^F(\text{-},\text{-}) : \mathcal{H}_{(0)}^{F,\Omega} \otimes \mathcal{H}_{(0)}^{F,\Omega} \to \mathcal{O}_S[[t]].$$

The triple $\{\mathcal{H}_{(0)}^{F,\Omega}, \mathcal{K}_\Omega^F(\text{-},\text{-}), \nabla^\Omega\}$ satisfy the following properties.

Proposition 3.20. *Let s_1, s_2 be local sections of $\mathcal{H}_{(0)}^{F,\Omega}$.*

(1) $\mathcal{K}_\Omega^F(s_1, s_2) = \overline{\mathcal{K}_\Omega^F(s_2, s_1)}$.
(2) $\mathcal{K}_\Omega^F(\nu(t)s_1, s_2) = \mathcal{K}_\Omega^F(s_1, \nu(-t)s_2) = \nu(t)\mathcal{K}_\Omega^F(s_1, s_2)$ *for any* $\nu(t) \in \mathcal{O}_S[[t]]$.
(3) $\partial_V \mathcal{K}_\Omega^F(s_1, s_2) = \mathcal{K}_\Omega^F(\nabla_V^\Omega s_1, s_2) + \mathcal{K}_\Omega^F(s_1, \nabla_V^\Omega s_2)$ *for any local section V of $\mathcal{T}_S$.*
(4) $(t\partial_t + n)\,\mathcal{K}_\Omega^F(s_1, s_2) = \mathcal{K}_\Omega^F(\nabla_{t\partial_t}^\Omega s_1, s_2) + \mathcal{K}_\Omega^F(s_1, \nabla_{t\partial_t}^\Omega s_2)$.
(5) *The higher residue pairing induces a pairing on*

$$\mathcal{H}_{(0)}^{F,\Omega}/t\mathcal{H}_{(0)}^{F,\Omega} \otimes_{\mathcal{O}_S} \mathcal{H}_{(0)}^{F,\Omega}/t\mathcal{H}_{(0)}^{F,\Omega} \to \mathcal{O}_S,$$

which coincides with the classical residue pairing $p_*\mathcal{O}_{\mathcal{C}(F)} \otimes_{\mathcal{O}_S} p_*\mathcal{O}_{\mathcal{C}(F)} \to \mathcal{O}_S$. In particular, the induced pairing is non-degenerate.

Proof. The first four statements follow from direct calculations and the definition of the pairing $\mathcal{K}_\Omega^F(\text{-},\text{-})$.

The fiber of $\mathcal{H}_{(0)}^{F,\Omega}/t\mathcal{H}_{(0)}^{F,\Omega}$ at $u \in S$ is given by $H^*(\mathrm{PV}_c(Z_u), \bar{\partial}_{F|_{Z_u}}) = \mathrm{Jac}(F|_{Z_u})$, where $Z_u := p^{-1}(u)$. By Proposition 2.5, the pairing at u coincides with the residue pairing $\mathrm{Jac}(F|_{Z_u}) \otimes \mathrm{Jac}(F|_{Z_u}) \to \mathbb{C}$, which is known to be non-degenerate [25]. Hence (5) follows. □

The triple $\{\mathcal{H}_{(0)}^{F,\Omega}, \mathcal{K}_\Omega^F(\text{-},\text{-}), \nabla^\Omega\}$ satisfying properties of Proposition 3.20 and the Transversality (Lemma 3.12) is called a *variation of semi-infinite Hodge structure*, a notion due to Barannikov [4].

3.4. **Primitive forms.** With the triple $\{\mathcal{H}_{(0)}^{F,\Omega}, \mathcal{K}_\Omega^F(\text{-},\text{-}), \nabla^\Omega\}$, we can define the notion of primitive forms [46]. In terms of variation of semi-infinite Hodge structures, the primitive forms correspond geometrically to the semi-infinite period maps [4].

Definition 3.21. *A section $\zeta \in \Gamma(S, \mathcal{H}_{(0)}^{F,\Omega})$ is called a* primitive form *if it satisfies the following conditions:*

(P1) (Primitivity) *The section ζ induces an $\mathcal{O}_S$-module isomorphism*

$$t\nabla^\Omega\zeta : \mathcal{T}_S \to \mathcal{H}_{(0)}^{F,\Omega}/t\mathcal{H}_{(0)}^{F,\Omega} = p_*\mathcal{O}_{\mathcal{C}(F)}; \quad V \mapsto t\nabla_V^\Omega\zeta.$$

(P2) (Orthogonality) *For any local sections V_1, V_2 of $\mathcal{T}_S$,*

$$\mathcal{K}^F_\Omega\left(\nabla^\Omega_{V_1}\zeta, \nabla^\Omega_{V_2}\zeta\right) \in t^{-2}\mathcal{O}_S. \tag{3.11}$$

(P3) (Holonomicity) *For any local sections V_1, V_2, V_3 of $\mathcal{T}_S$,*

$$\mathcal{K}^F_\Omega\left(\nabla^\Omega_{V_1}\nabla^\Omega_{V_2}\zeta, \nabla^\Omega_{V_3}\zeta\right) \in t^{-3}\mathcal{O}_S \oplus t^{-2}\mathcal{O}_S; \tag{3.12}$$

$$\mathcal{K}^F_\Omega\left(\nabla^\Omega_{t\partial_t}\nabla^\Omega_{V_1}\zeta, \nabla^\Omega_{V_2}\zeta\right) \in t^{-3}\mathcal{O}_S \oplus t^{-2}\mathcal{O}_S. \tag{3.13}$$

(P4) (Homogeneity) *Recall the Euler vector field E is defined in* (3.3)*. There is a constant $r \in \mathbb{C}$ such that*

$$\left(\nabla^\Omega_{t\partial_t} + \nabla^\Omega_E\right)\zeta = r\zeta.$$

Remark 3.22. *The hypothesis* (3.13) *follows from the combination of* (P2)*,* (3.12) *and* (P4)*.*

The properties of a primitive form give rise to a set of flat coordinates and an associated Frobenius manifold structure on the universal unfolding S, which was originally called by the third author the *flat structure* [46]. This is exposed in detail via the modern point of view in [48]. An exposition for Calabi-Yau manifolds parallel with our current setting is given in [35] . As we will see in the next section, primitive forms can be constructed by using good opposite filtrations $\mathcal{L}$ of $\mathcal{H}^{f,\Omega}$. This leads to a concrete way to compute the flat coordinates and the potential function of the associated Frobenius manifold structure [32].

3.5. **Intrinsic properties.** So far we have fixed a relative holomorphic volume form $\Omega := \Omega_{Z/S}$ (see formula (3.4)) in order to define a primitive form. In this subsection, we will discuss properties of higher residue pairings and primitive forms associated with different choices $\frac{1}{\lambda}\Omega$, where $\lambda \in H^0(Z, \mathcal{O}_Z^*)$ is a nowhere vanishing holomorphic function on Z. We will use the subscript $\frac{1}{\lambda}\Omega$ to indicate such a choice. For instance, we will denote $Q_{F,\frac{1}{\lambda}\Omega}$ for the differential and $\mathcal{H}^{F,\frac{1}{\lambda}\Omega}$ to specify the dependence on the choice. We will denote by $[x]_{\frac{1}{\lambda}\Omega}$ a local section of $\mathcal{H}^{F,\frac{1}{\lambda}\Omega}$, represented by a $\mathrm{PV}(Z/S)$-valued Laurent series x in t. In this subsection, we will also identify our primitive forms with the primitive forms in the original approach via differential forms.

Recall that we have isomorphisms of cochain complexes

$$\hat{\Gamma}^{\pm}_{\frac{1}{\lambda}\Omega} : \left(\mathrm{PV}(Z/S)((t)), Q_{F,\frac{1}{\lambda}\Omega}\right) \xrightarrow{\cong} \left(\mathcal{A}(Z/S)((t)), d_F^{\pm}\right).$$

Here we will use $\hat{\Gamma}^{+}_{\frac{1}{\lambda}\Omega}$, while we remark that the other isomorphism $\hat{\Gamma}^{-}_{\frac{1}{\lambda}\Omega}$ also works.

Let us still denote by $\hat{\Gamma}^{+}_{\frac{1}{\lambda}\Omega}$ the composition

$$\hat{\Gamma}^{+}_{\frac{1}{\lambda}\Omega} : \mathcal{H}^{F,\frac{1}{\lambda}\Omega} \xrightarrow{\cong} R^\bullet p_*\left(\mathrm{PV}(Z/S)((t)), Q_{F,\frac{1}{\lambda}\Omega}\right) \xrightarrow{\cong} R^\bullet p_*\left(\mathcal{A}(Z/S)((t)), d_F^{+}\right)$$

of isomorphisms of sheaves of $\mathcal{O}_S$-modules.

Lemma 3.23. *For any $x \in \mathrm{PV}(Z/S)((t))$ and $\lambda \in H^0(Z, \mathcal{O}_Z^*)$, we have*

$$(1)\ \hat{\Gamma}^+_{\frac{1}{\lambda}\Omega}(\lambda x) = \hat{\Gamma}^+_{\Omega}(x); \quad (2)\ Q_{F,\frac{1}{\lambda}\Omega}(\lambda x) = \lambda Q_{F,\Omega}(x).$$

Proof. Statement (1) is obvious. (2) follows from the observation $\partial_{\frac{1}{\lambda}\Omega}(\lambda x) = \lambda \partial_\Omega(x)$. □

As a direct consequence, we have

Corollary 3.24. *For every $\lambda \in H^0(Z, \mathcal{O}_Z^*)$, the map $\psi_\lambda : \mathcal{H}^{F,\Omega} \to \mathcal{H}^{F,\frac{1}{\lambda}\Omega}$ by $[x]_\Omega \mapsto [\lambda x]_{\frac{1}{\lambda}\Omega}$ is well defined, and defines an isomorphism of sheaves of $\mathcal{O}_S$-modules. Furthermore, we have $\hat{\Gamma}^+_{\frac{1}{\lambda}\Omega} \circ \psi_\lambda = \hat{\Gamma}^+_\Omega$.*

We refer to the next proposition as the *intrinsic properties* among the higher residue pairings and the associated Gauss-Manin connection.

Proposition 3.25. *Let $\lambda \in H^0(Z, \mathcal{O}_Z^*)$. For any local sections $[x]_\Omega, [y]_\Omega \in \mathcal{H}^{F,\Omega}$,*

$$\mathcal{K}^F_{\frac{1}{\lambda}\Omega}\left([\lambda x]_{\frac{1}{\lambda}\Omega}, [\lambda y]_{\frac{1}{\lambda}\Omega}\right) = \mathcal{K}^F_\Omega\left([x]_\Omega, [y]_\Omega\right).$$

Furthermore, we have

$$\nabla^{\frac{1}{\lambda}\Omega}_V [\lambda x]_{\frac{1}{\lambda}\Omega} = [\lambda \nabla^\Omega_V x]_{\frac{1}{\lambda}\Omega} \quad \textit{and} \quad \nabla^{\frac{1}{\lambda}\Omega}_{t\partial_t} [\lambda x]_{\frac{1}{\lambda}\Omega} = [\lambda \nabla^\Omega_{t\partial_t} x]_{\frac{1}{\lambda}\Omega},$$

where V is a local section of the holomorphic tangent sheaf $\mathcal{T}_S$.

Proof. Recall that we have defined the operator V_F and T^t_ρ, and ι_t^{-1} can be realized by T^t_ρ. Note

$$Q_{\frac{1}{\lambda}\Omega}(\lambda x) = \lambda Q_\Omega(x), \quad V_F(\lambda x) = \lambda V_F(x), \quad \{F, \lambda x\} = \lambda \{F, x\}.$$

It follows that $T^t_{\rho,\frac{1}{\lambda}\Omega}(\lambda x) = \lambda T^t_{\rho,\Omega}(x)$. Hence,

$$\begin{aligned}
\mathcal{K}^F_{\frac{1}{\lambda}\Omega}\left([\lambda x]_{\frac{1}{\lambda}\Omega}, [\lambda y]_{\frac{1}{\lambda}\Omega}\right) &= \int_{Z/S} \left(\left(T^t_{\rho,\frac{1}{\lambda}\Omega}(\lambda x) \wedge \overline{T^t_{\rho,\frac{1}{\lambda}\Omega}(\lambda y)}\right) \vdash \frac{1}{\lambda}\Omega\right) \wedge \frac{1}{\lambda}\Omega \\
&= \int_{Z/S} \left(\left(T^t_{\rho,\Omega}(x) \wedge \overline{T^t_{\rho,\Omega}(y)}\right) \vdash \Omega\right) \wedge \Omega \\
&= \mathcal{K}^F_\Omega\left([x]_\Omega, [y]_\Omega\right).
\end{aligned}$$

The compatible relations on the Gauss-Manin connections $\nabla^{\frac{1}{\lambda}\Omega}$ follow from that fact that they are all induced from that on differential forms via the isomorphism $\hat{\Gamma}^+_{\frac{1}{\lambda}\Omega}$. □

As a direct consequence of the above proposition, we have

Proposition 3.26. *A section $\zeta = [x]_\Omega \in \Gamma(S, \mathcal{H}^{F,\Omega}_{(0)})$ is a primitive form with respect to the choice Ω if and only if the section $[\lambda x]_{\frac{1}{\lambda}\Omega} \in \mathcal{H}^{F,\frac{1}{\lambda}\Omega}_{(0)}$ is a primitive form with respect to the choice $\frac{1}{\lambda}\Omega$.*

In order to identify the primitive forms in our setting with the original ones, we let $t = \delta_w^{-1}$ following the notation convention in [48], and recall some facts therein. The natural embedding of $(\Omega^*_{Z/S}((t)), d_F^-) \hookrightarrow (\mathcal{A}(Z/S)((t)), d_F^-)$ is a quasi-isomorphism (by Lemma 2.2 together with $\hat{\Gamma}^-_{\Omega_X}$). This gives a canonical isomorphism

$$\mathcal{H}_F := R^\bullet p_*(\Omega^*_{Z/S}((t)), d_F^-) \xrightarrow{\cong} R^\bullet p_*(\mathcal{A}(Z/S)((t)), d_F^-).$$

The de Rham cohomology group $\mathcal{H}_F$ is equipped a filtration $\{\mathcal{H}_F^{(-k)}\}_{k\in\mathbb{Z}}$ given in Definition 3.9. There is also the Gauss-Manin connection $\nabla^{\mathrm{GM}} : \mathcal{H}_F \to \mathcal{H}_F$ defined the same formula as in (3.6), which is extended along δ_W by

$$\nabla^{\mathrm{GM}}_{\delta_w\partial_{\delta_w}}[\alpha] := [(\delta_w\partial_{\delta_w} + \delta_w F)\alpha].$$

Theorem 3.27. (1) *The extended Gauss-Manin connections ∇^Ω and ∇^{GM} are compatible in the sense:*

$$\hat{\Gamma}^-_\Omega \circ \nabla^\Omega_{t\partial_t} = -\nabla^{\mathrm{GM}}_{\delta_w\partial_{\delta_w}} \circ \hat{\Gamma}^-_\Omega, \quad \hat{\Gamma}^-_\Omega \circ \nabla^\Omega_V = \nabla^{\mathrm{GM}}_V \circ \hat{\Gamma}^-_\Omega, \text{ for any local holomorphic vector filed } V.$$

(2) *The next pairing $\mathcal{K}_F$ coincides with the third author's higher resider pairing on $\mathcal{H}_F$ in* [45].

$$\mathcal{K}_F : \mathcal{H}_F \otimes \mathcal{H}_F \to \mathcal{O}_S((t))$$
$$\mathcal{K}_F([\alpha_1], [\alpha_2]) := t^n \mathcal{K}^F_\Omega\big((\hat{\Gamma}^-_\Omega)^{-1}([\alpha_1]), (\hat{\Gamma}^-_\Omega)^{-1}([\alpha_2])\big)$$

(3) *A section $[x] \in \Gamma(S, \mathcal{H}^{F,\Omega}_{(0)})$ is a primitive form with respect to $\{\mathcal{H}^{F,\Omega}_{(0)}, \mathcal{K}^F_\Omega(\text{-},\text{-}), \nabla^\Omega\}$ if and only if the section $[x\Omega] \in \Gamma(S, \mathcal{H}_F^{(0)})$ is a primitive form with respect to $\{\mathcal{H}_F^{(0)}, \mathcal{K}_F(\text{-},\text{-}), \nabla^{\mathrm{GM}}\}$.*

Proof. (1) The relation $t = \delta_w^{-1}$ implies $t\partial_t = -\delta_w\partial_{\delta_w}$. The first identify follows from direct calculations. The second identity also follows from the definitions, by observing that each section $[x]$ of $\mathcal{H}^{F,\Omega}$ is represented by an element x of $\mathcal{O}_Z((t))$.

(2) It follows directly from Proposition 3.20 that the pairing $\mathcal{K}_F(\text{-},\text{-})$ satisfies all the properties of Theorem 5.1 of [48]. Therefore it coincides with the third author's higher residue pairing on $\mathcal{H}_F$ by the uniqueness as stated in the same theorem.

(3) The statement also follows by directly checking according to the definitions. □

Primitive forms are invariant under the stably equivalence. Precisely, for any $m \in \mathbb{Z}_{>0}$, we define a holomorphic function f_m on $X_m := X \times \mathbb{C}^m$, by $f_m := f + \sum_{i=1}^m z_{n+i}^2$. Let π_m denote the natural projection from $Z_m := Z \times \mathbb{C}^m$ to Z. Then we obtain a frame (Z_m, S, p_m, F_m) associated to (X_m, f_m), where $p_m := p \circ \pi_m$ is a Stein map from Z_m to S, and $F_m := F + \sum_{i=1}^m z_{n+i}^2$ is the universal unfolding of f_m. We let $\Omega_{Z_m/S} := \Omega_{Z/S} \wedge dz_{n+1} \wedge \cdots \wedge dz_{n+m}$. Because of the coincidence of primitive forms in our setting with the original sense by using differential forms in Theorem 3.27, we have

Proposition 3.28 (see section 6 of [43])**.** *A section $\zeta \in \Gamma(S, \mathcal{H}^{F,\Omega}_{(0)})$ is a primitive form with respect to the choice $\Omega_{Z/S}$ if and only if $\pi_m^*(\zeta)$ is a primitive form in $\Gamma(S, \mathcal{H}^{F_m}_{(0)})$ with respect to the choice $\Omega_{Z_m/S}$.*

Furthermore if m is even, then there is an isomorphism of $\mathcal{O}_S$-modules,

$$\rho_m : \mathcal{H}^{F,\Omega}_{(\ell)} \xrightarrow{\cong} \mathcal{H}^{F_m}_{(\ell)},\ \forall \ell \in \mathbb{Z}_{\leq 0},$$

which is equivariant with respect to the extended Gauss-Manin connection.

Remark 3.29. *In the original approach to primitive forms by differential forms, there is a degree shifting on the left hand side of the isomorphism ρ_m. More precisely, the degree on the left should be shifted to $\ell - \frac{m}{2}$, and the isomorphism is given by $\left(\nabla^{\mathrm{GM}}_{\partial_t}\right)^{\frac{m}{2}} dz_{n+1} \wedge \cdots \wedge dz_{n+m}$ with the Gauss-Manin connection $\nabla^{\mathrm{GM}}_{\partial_t}$ for differential forms.*

4. General constructions of primitive forms

In this section, we will give an explicit description of primitive forms in the germ of the universal unfolding of (X, f). The main idea is based on the third author's good sections [44] and Barannikov's formula [4] of semi-infinite period maps. See also [35] on its connection with Kodaira-Spencer gauge theory. The existence of a primitive form as a germ was already proved in [50] and [16] in the corresponding situations.

We will first construct primitive forms in the formal neighborhood of the reference point in the universal unfolding. The use of formal setting not only simplifies the presentation of the primitive forms, but also gives an explicit formula (Theorem 4.15) which is closely related to the oscillatory integral. Via this new approach, we find an explicit algorithm to compute the Taylor series expansion of the primitive forms up to arbitrary finite order. This is applied in the next section to recover known examples of primitive forms and new results for exceptional unimodular singularities. The analytic nature of the formal primitive forms follows by comparing the formal construction with the original analytic construction [46].

Let (Z, S, p, F) be a frame (see Definition 3.1) associated to (X, f). In addition, a projection $\pi_X : Z \to X$ has been chosen, so that we obtain an embedding $\pi_X \times p : Z \hookrightarrow X \times S$ (where Z is made smaller if necessary). We will choose the relative holomorphic volume form

$$\Omega_{Z/S} = \pi_X^* \Omega_X$$

where Ω_X is a fixed holomorphic volume form on X. By Proposition 3.14, the (non-extended) Gauss-Manin connection ∇^{Ω} can be simplified as equation (3.8), namely

$$\nabla^{\Omega}_V [s] = \left[\partial_V s + \frac{\partial_V F}{t} s\right],$$

where we have abused the notation V with its lifting $\hat{V}$ (see (3.7)) on the right hand side.

We will let

$$F_0 := \pi_X^* f$$

denote the trivial unfolding.

4.1. **Formal geometry.**

4.1.1. *Formal set-up.* We fix some notions to be used in our formal set-up first. Let $\mathfrak{m}$ denote the maximal ideal of the local ring $\mathcal{O}_{S,0}$ of the reference point $0 \in S$. Let $R_N := \mathcal{O}_{S,0}/\mathfrak{m}^N$ where N is a positive integer. We consider the inverse system

$$\mathcal{R} := \{R_N \mid N \in \mathbb{Z}_{>0}\},$$

where, for $N \geq N'$, the morphism $\varphi_{N,N'} : R_N \to R_{N'}$ is given by the natural projection. Clearly, the associated system $\{\mathrm{Spec}(R_N)\}_N$ of $\mathcal{R}$ form a basis of the formal neighborhood of the reference point $0 \in S$. Here on after we simply denote the subscript $N \in \mathbb{Z}_{>0}$ as N without confusion.

Definition 4.1. *By a $\mathcal{R}$-module $\mathcal{F}$, we mean a set $\{\mathcal{F}_N\}_N$, each $\mathcal{F}_N$ being a R_N-module, together with morphisms $\Phi_{N+1,N} : \mathcal{F}_{N+1} \to \mathcal{F}_N$ of R_{N+1}-modules (by viewing $\mathcal{F}_N$ as a R_{N+1}-module via $\varphi_{N+1,N}$) that induce isomorphisms of R_N-modules: $\mathcal{F}_{N+1} \otimes_{R_{N+1}} R_N \xrightarrow{\cong} \mathcal{F}_N$. Direct product for $\mathcal{R}$-modules is defined in the similar fashion.*

Let $\Gamma(\mathcal{R}, \mathcal{F})$ denote the space of sections of $\mathcal{F}$ on $\mathcal{R}$, where by a section s we mean a set $\{s_N\}_N$ with each s_N an element in $\mathcal{F}_N$, satisfying the compatibility condition:

$$s_N = \Phi_{N+1,N}(s_{N+1}), \quad \forall\, N.$$

The space $\Gamma(\mathcal{R}, \mathcal{F})$ is naturally an $\hat{\mathcal{O}}_{S,0}$-module, where $\hat{\mathcal{O}}_{S,0}$ denotes the formal completion of $\mathcal{O}_{S,0}$ at the reference point 0 with respect to the maximal ideal $\mathfrak{m}$. In particular, it is also an $\mathcal{O}_{S,0}$-module.

Definition 4.2. *Let $\mathcal{E}$ be an $\mathcal{O}_S$-module on S. We define the induced $\mathcal{R}$–module $\check{\mathcal{E}} := \{\check{\mathcal{E}}_N\}_N$ by setting*

$$\check{\mathcal{E}}_N := \mathcal{E} \otimes_{\mathcal{O}_{S,0}} R_N,$$

via the natural projection $\varphi_{R_N} : \mathcal{O}_{S,0} \to R_N$.

Lemma/Definition 4.3. *Let $\mathcal{E}$ be an $\mathcal{O}_S$-module with a connection $\nabla : \mathcal{E} \to \Omega^1_S \otimes \mathcal{E}$, then there is an induced map*

$$\check{\nabla} : \Gamma(\mathcal{R}, \check{\mathcal{E}}) \to \Omega^1_S \otimes_{\mathcal{O}_S} \Gamma(\mathcal{R}, \check{\mathcal{E}})$$

satisfying the Leibniz rule:

$$\check{\nabla}(as) = da \otimes s + a \otimes \check{\nabla} s, \quad \textit{for any } a \in \mathcal{O}_S \ \textit{ and } \ s \in \Gamma(\mathcal{R}, \check{\mathcal{E}}).$$

Whenever there is no confusion, we will simply denote $\check{\nabla}$ as ∇ by abuse of notations.

Proof. Let $s = \{s_N\}_N$ be a section in $\Gamma(\mathcal{R}, \check{\mathcal{E}})$, and V be a local section of $\mathcal{T}_S$. Let $\tilde{s}_{N+1} \in \mathcal{E}$ be a lifting of s_{N+1}, and we define

$$(\check{\nabla}_V s)_N := \varphi_{R_N}(\nabla_V \tilde{s}_{N+1}).$$

It is easy to check that $\check{\nabla}_V s$ does not depend on the choice of the lifting, and defines an element in $\Gamma(\mathcal{R}, \check{\mathcal{E}})$. The Leibniz rule follows easily. □

4.1.2. *Formal primitive forms.* Recall that given the frame (Z, S, p, F), we have the free $\mathcal{O}_S[[t]]$-module $\mathcal{H}^{F,\Omega}_{(0)}$ and the free $\mathcal{O}_S((t))$-module $\mathcal{H}^{F,\Omega}$ with the extended Gauss-Manin connection ∇^Ω. As $\mathcal{O}_S$-modules, they induce the corresponding $\mathcal{R}$-modules $\check{\mathcal{H}}^{F,\Omega}_{(0)}$ and $\check{\mathcal{H}}^{F,\Omega}$ respectively. The space $\Gamma(\mathcal{R}, \check{\mathcal{H}}^{F,\Omega}_{(0)})$ (resp. $\Gamma(\mathcal{R}, \check{\mathcal{H}}^{F,\Omega})$) inherits the structure of $\mathbb{C}[[t]]$-module (resp. $\mathbb{C}((t))$-module). By Lemma 4.3, $\Gamma(\mathcal{R}, \check{\mathcal{H}}^{F,\Omega})$ is equipped with the induced non-extended Gauss-Manin connection, which is easily seen to be extended along the $t\partial_t$ direction in the similar way. Such extended flat connection on $\Gamma(\mathcal{R}, \check{\mathcal{H}}^{F,\Omega})$ is again denoted as ∇^Ω without confusion.

Remark 4.4. *Given a section $\{s_N\}_N$ of $\check{\mathcal{E}}$, we can take the inverse limit $\hat{s} := \varprojlim_N s_N$ in $\varprojlim_N \mathcal{E}/\mathfrak{m}^N\mathcal{E}$. When $\mathcal{E}$ is given by the $\mathcal{O}_S$-module $\mathcal{H}^{F,\Omega}_{(0)}$, the inverse limit $\hat{s}$ is still a formal power series in the descendent variable t. However, when $\mathcal{E} = \mathcal{H}^{F,\Omega}$, the power of t in the limit $\hat{s}$ may go to $-\infty$.*

The higher residue pairing defines a natural pairing for each N, by tensoring with R_N,

$$\hat{\mathcal{K}}^{F,N}_\Omega : \left(\mathcal{H}^{F,\Omega}_{(0)} \otimes_{\mathcal{O}_S} R_N\right) \otimes_{R_N} \left(\mathcal{H}^{F,\Omega}_{(0)} \otimes_{\mathcal{O}_S} R_N\right) \to R_N[[t]]$$

Lemma/Definition 4.5. *There is an induced formal higher residue pairing*

$$\hat{\mathcal{K}}^{F}_\Omega : \Gamma(\mathcal{R}, \check{\mathcal{H}}^{F,\Omega}_{(0)}) \otimes_{\hat{\mathcal{O}}_{S,0}} \Gamma(\mathcal{R}, \check{\mathcal{H}}^{F,\Omega}_{(0)}) \to \hat{\mathcal{O}}_{S,0}[[t]]$$

satisfying similar properties as in Proposition 3.20.

Proof. Let $s = \{s_N\}_N, s' = \{s'_N\}_N \in \Gamma(\mathcal{R}, \check{\mathcal{H}}^{F,\Omega}_{(0)})$. The pairing $\hat{\mathcal{K}}^{F}_\Omega(s, s')$ is defined by

$$\hat{\mathcal{K}}^{F}_\Omega(s, s') := \varprojlim_N \hat{\mathcal{K}}^{F,N}_\Omega(s_N, s'_N).$$

□

Now we formulate primitive forms in the formal neighborhood as well.

Definition 4.6. *A section $\zeta \in \Gamma(\mathcal{R}, \check{\mathcal{H}}^{F,\Omega}_{(0)})$ is called a* formal primitive form *if it satisfies the following conditions:*

(P1)$^\vee$ *The section ζ induces an $\hat{\mathcal{O}}_{S,0}$-module isomorphism*

$$t\nabla^\Omega\zeta : \hat{\mathcal{O}}_{S,0} \otimes_{\mathcal{O}_S} \mathcal{T}_S \to \Gamma(\mathcal{R}, \check{\mathcal{H}}^{F,\Omega}_{(0)})/t\Gamma(\mathcal{R}, \check{\mathcal{H}}^{F,\Omega}_{(0)}); \quad V \mapsto t\nabla^\Omega_V\zeta.$$

(P2)$^\vee$ *For any local holomorphic vector fields V_1, V_2 on S,*

$$\hat{\mathcal{K}}^{F}_\Omega\left(\nabla^\Omega_{V_1}\zeta, \nabla^\Omega_{V_2}\zeta\right) \in t^{-2}\hat{\mathcal{O}}_{S,0}.$$

(P3)$^\vee$ *For any local holomorphic vector fields V_1, V_2, V_3 on S,*

$$\hat{\mathcal{K}}^{F}_\Omega\left(\nabla^\Omega_{V_1}\nabla^\Omega_{V_2}\zeta, \nabla^\Omega_{V_3}\zeta\right) \in t^{-3}\hat{\mathcal{O}}_{S,0} \oplus t^{-2}\hat{\mathcal{O}}_{S,0};$$
$$\hat{\mathcal{K}}^{F}_\Omega\left(\nabla^\Omega_{t\partial_t}\nabla^\Omega_{V_1}\zeta, \nabla^\Omega_{V_2}\zeta\right) \in t^{-3}\hat{\mathcal{O}}_{S,0} \oplus t^{-2}\hat{\mathcal{O}}_{S,0}.$$

(P4)$^\vee$ *Recall the Euler vector field is given in* (3.3). *There is a constant* $r \in \mathbb{C}$ *such that*

$$\left(\nabla^{\Omega}_{t\partial_t} + \nabla^{\Omega}_{E}\right)\zeta = r\zeta.$$

There is a natural map by restricting to the formal neighborhood

$$\{\text{primitive forms in the germ of 0 around } S\} \to \{\text{formal primitive forms around } 0\}$$

We will see later that this map is in fact a bijection. This reduces the construction of primitive forms to the formal neighborhood.

4.1.3. *Formal splitting.*

Lemma/Definition 4.7. *The map*

$$e^{(f-F)/t} : \mathcal{H}^{f,\Omega} \to \Gamma(\mathcal{R}, \check{\mathcal{H}}^{F,\Omega}), \tag{4.1}$$

given by

$$[s] \mapsto e^{(f-F)/t}[s] := \{[\check{s}_N]\}_N \quad \textit{with} \quad \check{s}_N := \varphi_{R_N}(e^{(F_0-F)/t}\pi_X^*(s)), \tag{4.2}$$

is well-defined, independent of choices of the representatives s. *Furthermore,* $e^{(f-F)/t}[s]$ *is flat with respect to the non-extended Gauss-Manin connection* ∇^{Ω}, *which will be called the formal flat extension of* $[s]$.

The above map $e^{(f-F)/t}$ *induces an isomorphism of* R_N*-modules*

$$e^{(f-F)/t}|_N : \mathcal{H}^{f,\Omega} \otimes_{\mathbb{C}} R_N \to \mathcal{H}^{F,\Omega} \otimes_{\mathcal{O}_S} R_N, \quad \textit{defined by } [s] \otimes r \mapsto [r\check{s}_N], \tag{4.3}$$

whose inverse is

$$e^{(F-f)/t}|_N : \mathcal{H}^{F,\Omega} \otimes_{\mathcal{O}_S} R_N \to \mathcal{H}^{f,\Omega} \otimes_{\mathbb{C}} R_N, \quad \textit{defined by } [b] \otimes 1 \mapsto [\varphi_{R_N}(e^{(F-F_0)/t}b)]. \tag{4.4}$$

We will skip the subscript "$|_N$*" whenever there is no confusion.*

Proof. Let $s = \alpha + Q_f(\beta)$ for some $\alpha \in \Gamma(X, \mathcal{O}_X)((t))$ and $\beta \in \mathrm{PV}(X)((t))$. The map in (4.1) is in fact the composition of the following maps:

$$\mathcal{H}^{f,\Omega} \to \mathcal{H}^{f,\Omega} \otimes_{\mathbb{C}} \mathcal{O}_S \xrightarrow{\cong} \mathcal{H}^{F_0,\Omega} \xrightarrow{e^{(F_0-F)/t}} \Gamma(\mathcal{R}, \check{\mathcal{H}}^{F,\Omega}),$$

where the first map is given by $[s] \mapsto [s] \otimes 1$, the second map is an isomorphism of $\mathcal{O}_S$-modules defined by $[s] \otimes g \mapsto [g\pi_X^*(s)]$, and the third map is defined by using (4.2). By direct calculations, we have

$$\varphi_{R_N}\big(e^{(F_0-F)/t}\pi_X^*\left(Q_f(\beta)\right)\big) = \varphi_{R_N}\big(e^{(F_0-F)/t}Q_{F_0}(\pi_X^*(\beta))\big) = \varphi_{R_N}\big(Q_F\big(e^{(F_0-F)/t}\pi_X^*\beta\big)\big).$$

Since $F_0 - F \in \mathcal{O}_X \otimes \mathfrak{m}$, the series expansion of $\varphi_{R_N}(e^{(F_0-F)/t}) = \sum_{k=0}^{N} \frac{1}{k!}\frac{(F_0-F)^k}{t^k} \in R_N[t, t^{-1}]$, whose pole in t is of finite order. Therefore $[\check{s}_N]$ is a well-defined element in $\mathcal{H}^{F,\Omega} \otimes_{\mathcal{O}_S} R_N$, independent of choices of representatives s. It is easy to see that $[\check{s}] := \{[\check{s}_N]\}_N$ satisfies the compatibility condition, hence lies in $\Gamma(\mathcal{R}, \check{\mathcal{H}}^{F,\Omega})$. Moreover,

$$\nabla^{\Omega}_V[\check{s}_N] = \nabla^{\Omega}_V[\varphi_{R_N}\big(\big(e^{(F_0-F)/t}\pi_X^*(s)\big)\big)] = [\varphi_{R_N}\big(\big(\partial_V + \frac{\partial_V F}{t}\big)\big(e^{(F_0-F)/t}\pi_X^*(s)\big)\big)] = 0.$$

Hence, the extension $[\check{s}]$ of $[s]$ is flat.

The map in (4.3) is given by the composition: $\mathcal{H}^{f,\Omega} \otimes_{\mathbb{C}} R_N \overset{\cong}{\to} \mathcal{H}^{F_0,\Omega} \otimes_{\mathcal{O}_S} R_N \to \mathcal{H}^{F,\Omega} \otimes_{\mathcal{O}_S} R_N$, where the first map is the canonical isomorphism of R_N-modules. The second map is defined by sending $[a] \otimes r$ to $[r\varphi_{R_N}(e^{(F_0-F)/t}a)]$, which is also an isomorphism of R_N-modules since its inverse is constructed by multiplying $e^{(F-F_0)/t}$ (whose details are left to the readers). □

Remark 4.8. *Geometrically, the map $e^{(f-F)/t}$ can be considered as the local trivialization of $\check{\mathcal{H}}^{F,\Omega}$ with respect to the Gauss-Manin connection.*

Let $\mathcal{L} \subset \mathcal{H}^{f,\Omega}$ be an opposite filtration (see Defintion 2.16). Recall that

$$B := t\mathcal{L} \cap \mathcal{H}^{f,\Omega}_{(0)} \cong \mathcal{H}^{f,\Omega}_{(0)}/t\mathcal{H}^{f,\Omega}_{(0)}$$

is isomorphic to $\mathrm{Jac}(f)$, and we have

$$\mathcal{H}^{f,\Omega} = \mathcal{H}^{f,\Omega}_{(0)} \oplus \mathcal{L}$$

with

$$\mathcal{H}^{f,\Omega}_{(0)} = B[[t]], \quad \mathcal{L} = t^{-1}B[t^{-1}] \quad \text{and} \quad \mathcal{H}^{f,\Omega} = B((t)).$$

Lemma/Definition 4.9. *Let $\mathcal{L} \subset \mathcal{H}^{f,\Omega}$ be an opposite filtration. We define an associated $\mathcal{R}$-module $\mathcal{L}_{\mathcal{R}}$ by*

$$\mathcal{L}_{\mathcal{R}} := \{\mathcal{L}_{\mathcal{R}}(R_N)\}_N \quad \textit{with} \quad \mathcal{L}_{\mathcal{R}}(R_N) := e^{(f-F)/t}(\mathcal{L} \otimes_{\mathbb{C}} R_N),$$

using (4.3). *The splitting $\mathcal{H}^{f,\Omega} = \mathcal{H}^{f,\Omega}_{(0)} \oplus \mathcal{L}$ induces a splitting of $\check{\mathcal{H}}^{F,\Omega}(R_N)$,*

$$\text{(a)} \quad \check{\mathcal{H}}^{F,\Omega}(R_N) = \check{\mathcal{H}}^{F,\Omega}_{(0)}(R_N) \oplus \mathcal{L}_{\mathcal{R}}(R_N)$$

for each R_N, where

$$\check{\mathcal{H}}^{F,\Omega}(R_N) := \mathcal{H}^{F,\Omega} \otimes_{\mathcal{O}_S} R_N \quad \textit{and} \quad \check{\mathcal{H}}^{F,\Omega}_{(0)}(R_N) := \mathcal{H}^{F,\Omega}_{(0)} \otimes_{\mathcal{O}_S} R_N.$$

The $\mathcal{R}$-submodule $\mathcal{L}_{\mathcal{R}}$ of $\check{\mathcal{H}}^{F,\Omega}$ further satisfies the following:

$$\text{(b)} \ \ t^{-1}\mathcal{L}_{\mathcal{R}}(R_N) \subset \mathcal{L}_{\mathcal{R}}(R_N); \qquad \text{(c)} \ \ \hat{\mathcal{K}}^{F,N}_{\Omega}(\mathcal{L}_{\mathcal{R}}(R_N), \mathcal{L}_{\mathcal{R}}(R_N)) \subset t^{-2}R_N[t^{-1}].$$

$\mathcal{L}_{\mathcal{R}}$ will be called the formal opposite filtration *associated to the opposite filtration $\mathcal{L}$.*

The splitting (a) *gives rise to a decomposition of $\mathcal{R}$-modules*

$$\check{\mathcal{H}}^{F,\Omega} = \check{\mathcal{H}}^{F,\Omega}_{(0)} \oplus \mathcal{L}_{\mathcal{R}}.$$

Consequently, every section s in $\Gamma(\mathcal{R}, \check{\mathcal{H}}^{F,\Omega})$ admits a unique splitting:

$$s = s_+ \oplus s_-$$

with $s_+ \in \Gamma(\mathcal{R}, \check{\mathcal{H}}^{F,\Omega}_{(0)})$ and $s_- \in \Gamma(\mathcal{R}, \mathcal{L}_{\mathcal{R}})$.

Proof. To show (a), let $\{\phi_j\}_j$, $\phi_j \in B$, represent a basis of $\mathrm{Jac}(f)$. Let $U \subset S$ be an open neighborhood of the reference point $0 \in S$, such that $\{\pi_X^*\phi_j\}_j$ remain representatives of a basis of $p_*\mathcal{O}_{\mathcal{C}(F)}$. Let $B_F := \mathrm{Span}_{\mathcal{O}_S|_U}\{\pi_X^*\phi_j\}_j \subset \mathcal{H}^{F,\Omega}|_U$, which is isomorphic to $B \otimes_{\mathbb{C}} \mathcal{O}_S|_U$. It follows that there are natural identifications

$$\check{\mathcal{H}}^{F,\Omega}_{(0)}(R_N) = B_F[[t]] \otimes_{\mathcal{O}_S} R_N, \quad \check{\mathcal{H}}^{F,\Omega}(R_N) = B_F((t)) \otimes_{\mathcal{O}_S} R_N$$

and $e^{(f-F)/t}(\mathcal{L} \otimes_{\mathbb{C}} R_N) = e^{(f-F)/t}(t^{-1}B[t^{-1}] \otimes_{\mathbb{C}} R_N)$. We only need to show that the composition π_- in the diagram

$$\begin{array}{ccc} e^{(f-F)/t}(\mathcal{L} \otimes_{\mathbb{C}} R_N) & \longrightarrow & \check{\mathcal{H}}^{F,\Omega}(R_N) \\ & \searrow^{\pi_-} & \downarrow \\ & & t^{-1}B_F[t^{-1}] \otimes_{\mathcal{O}_S} R_N \end{array}$$

is an isomorphism. Here we have to transform elements of the form $e^{(f-F)/t}b, b \in B$ into elements of the form $B_F((t)) \otimes_{\mathbb{C}} R_N$ up to Q_F-exact term. Clearly, π_- is $R_N[t^{-1}]$-linear, and it is an isomorphism modulo the maximal ideal of R_N. It follows that π_- is an isomorphism. Thus (a) holds.

Property (b) follows by construction.

To show (c), we only need to prove for the pairing on $\varphi_{R_N}(e^{(f-F)/t}(\mathcal{L} \otimes_{\mathbb{C}} 1))$ since $\hat{\mathcal{K}}^{F,N}_\Omega$ is R_N-linear. Let $s, s' \in \mathcal{L} \subset \mathcal{H}^{f,\Omega}$, and $\check{s} = \{\check{s}_N\}, \check{s}' = \{\check{s}'_N\}$ be the formal flat extensions of s, s' respectively. Let T be the $\mathcal{O}_S$-module $\mathcal{O}_S((t))$ with the natural flat connection. Then

$$\{\hat{\mathcal{K}}^{F,N}_\Omega(\check{s}_N, \check{s}'_N)\}_N \in \Gamma(\mathcal{R}, \check{T})$$

Let V be a holomorphic vector field on S. ∂_V is defined on $\Gamma(\mathcal{R}, \check{T})$ by Lemma 4.3, and we will denote by

$$\{\partial_V \hat{\mathcal{K}}^{F,N}_\Omega(\check{s}, \check{s}')\}_N := \partial_V\{\hat{\mathcal{K}}^{F,N}_\Omega(\check{s}_N, \check{s}'_N)\}_N$$

Since $\check{s}, \check{s}'$ are flat, we have

$$\partial_V \hat{\mathcal{K}}^{F,N}_\Omega(\check{s}, \check{s}') = \hat{\mathcal{K}}^{F,N}_\Omega(\nabla^\Omega_V \check{s}, \check{s}') + \hat{\mathcal{K}}^{F,N}_\Omega(\check{s}, \nabla^\Omega_V \check{s}') = 0$$

for any V and N. Hence, $\hat{\mathcal{K}}^{F,N}_\Omega(\check{s}_N, \check{s}'_N) \in \mathbb{C}((t)) \subset R_N((t))$ for all N. This implies $\hat{\mathcal{K}}^{F,N}_\Omega(\check{s}_N, \check{s}'_N) = \hat{\mathcal{K}}^{\Omega}_{F,1}(\check{s}_1, \check{s}'_1) \in t^{-2}\mathbb{C}[t^{-1}]$ under the natural identification of $\mathbb{C} \subset R_N$ with $\mathbb{C} = R_1$. Thus (c) holds.

The splittings (a) satisfy the compatibility condition, so that they give rise to a splitting of $\check{\mathcal{H}}^{F,\Omega}$.

Given a section $s = \{s_N\}_N$ in $\Gamma(\mathcal{R}, \check{\mathcal{H}}^{F,\Omega})$, we have a splitting $s_N = s_+(R_N) \oplus s_-(R_N)$ in $\check{\mathcal{H}}^{F,\Omega}_{(0)}(R_N) \oplus \mathcal{L}_{\mathcal{R}}(R_N)$ for every $s_N \in \check{\mathcal{H}}^{F,\Omega}(R_N)$. Clearly, both $s_+ := \{s_+(R_N)\}_N$ and $s_- := \{s_-(R_N)\}_N$ satisfy the compatibility condition, and hence we obtain a splitting of sections $s = s_+ \oplus s_-$. □

Remark 4.10. *The R_N-components of the formal flat extensions $e^{(f-F)/t}\mathcal{L}$ generate a R_N-submodule of $\check{\mathcal{H}}^{F,\Omega}(R_N)$ which coincides with $\mathcal{L}_{\mathcal{R}}(R_N)$. Hence, we do not distinguish them and simply denote both by $e^{(f-F)/t}\mathcal{L}\otimes_{\mathbb{C}} R_N$.*

Lemma 4.11. *The non-extended Gauss-Manin connection ∇^{Ω} preserves $\mathcal{L}_{\mathcal{R}}$, i.e.*

$$\nabla^{\Omega} : \Gamma(\mathcal{R},\mathcal{L}_{\mathcal{R}}) \to \Omega^1_S \otimes_{\mathcal{O}_S} \Gamma(\mathcal{R},\mathcal{L}_{\mathcal{R}}).$$

Proof. Let $s = \{s_N\}_N$ be a section in $\Gamma(\mathcal{R},\mathcal{L}_{\mathcal{R}})$, and V be a local section of $\mathcal{T}_S$. Each s_N is an element in $e^{(f-F)/t}\mathcal{L}\otimes_{\mathbb{C}} R_N$. Therefore we can write $s_{N+1} = \sum_i \varphi_{R_{N+1}}(e^{(f-F)/t})a_i \otimes_{\mathbb{C}} b_i$, where $a_i \in \mathcal{L}$ and $b_i \in R_{N+1}$ for all i. Since $\varphi_{R_N}(e^{(f-F)/t}a_i)$ is a flat extension of a_i by Lemma 4.7, it follows that

$$(\nabla^{\Omega}_V s)(R_N) = \sum_i \varphi_{R_N}(e^{(f-F)/t}a_i)\otimes_{\mathbb{C}} \varphi_{R_N}(\partial_V \tilde{b}_i) \in \mathcal{L}_{\mathcal{R}}(R_N),$$

where $\tilde{b}_i \in \mathcal{O}_S$ is a lifting of b_i for each i. □

4.2. **Construction of formal primitive form.**

Definition 4.12. *An opposite filtration $\mathcal{L}$ of $\mathcal{H}^{f,\Omega}$ is called* good *if $\mathcal{L}$ is preserved by $\nabla^{\Omega}_{t\partial_t} = t\partial_t - \frac{f}{t}$.*

Here $\nabla^{\Omega}_{t\partial_t}$ is the restriction of the extended Gauss-Manin connection to the central fiber, which is a well-defined operator on $\mathcal{H}^{f,\Omega}$.

Definition 4.13. *An element $\zeta_0 \in \mathcal{H}^{f,\Omega}_{(0)}$ is called* primitive *with respect to a good opposite filtration $\mathcal{L}$ if*

(1) (primitivity) *the projection of ζ_0 generates* $\mathrm{Jac}(f) = \mathcal{H}^{f,\Omega}_{(0)}/t\mathcal{H}^{f,\Omega}_{(0)}$ *as* $\mathrm{Jac}(f)$*-module;*
(2) (homogeneity) $\nabla^{\Omega}_{t\partial_t}\zeta_0 - r\zeta_0 \in \mathcal{L}$ *for some constant $r \in \mathbb{C}$.*

We will describe all good opposite filtrations and primitive elements for weighted homogeneous polynomials in next section.

Remark 4.14. *The notion of a* good section $v : \mathrm{Jac}(f) \to \mathcal{H}^{f,\Omega}_{(0)}$ *in* [46] *coincides with the above notion of a good opposite filtration $\mathcal{L}$, under the identification $v \mapsto \mathcal{L} := t^{-1}v(\mathrm{Jac}(f))[t^{-1}]$ (recall Lemma 2.16). The notion of a primitive element was also described in* [46] *without introducing the terminology, where it is generally expected that there exists a unique primitive element with respect to a given good opposite filtration, up to a nonzero complex number.*

Given an element ζ_0 in $\mathcal{H}^{f,\Omega}_{(0)}$, we obtain a flat section $e^{(f-F)/t}\zeta_0$ in $\Gamma(\mathcal{R},\check{\mathcal{H}}^{F,\Omega})$ by Lemma 4.7. From this we obtain a unique splitting:

$$e^{(f-F)/t}\zeta_0 = \zeta_+ + \zeta_- \tag{4.5}$$

with $\zeta_+ \in \Gamma(\mathcal{R},\check{\mathcal{H}}^{F,\Omega}_{(0)})$ and $\zeta_- \in \Gamma(\mathcal{R},\mathcal{L}_{\mathcal{R}})$ by Lemma 4.9 .

Theorem 4.15. *Let $\mathcal{L}$ be a good opposite filtration of $\mathcal{H}^{f,\Omega}$ and ζ_0 be a primitive element of $\mathcal{H}^{f,\Omega}_{(0)}$ with respect to $\mathcal{L}$. Then the section $\zeta_+ \in \Gamma(\mathcal{R}, \check{\mathcal{H}}^{F,\Omega}_{(0)})$, given by the splitting* (4.5)*, is a formal primitive form. The converse is also true, i.e., a formal primitive form determines a unique good opposite filtration $\mathcal{L}$ together with a unique primitive element ζ_0 with respect to $\mathcal{L}$.*

Proof. We first prove that ζ_+ is a formal primitive form for the given $(\mathcal{L}, \zeta_0)$.

We check the four conditions for defining a formal primitive form in Definition 4.6. For simplicity, we will refer to the corresponding formulas in Definition 3.21, and simply use the notations ∇^Ω and $\mathcal{K}^F_\Omega$ instead of $\check{\nabla}^\Omega$ and $\hat{\mathcal{K}}^F_\Omega$, respectively. Let $R \in \mathcal{R}$.

(P1)$^\vee$ We need to show that

$$t\nabla^\Omega \zeta_+(R) : \mathcal{T}_S \otimes_{\mathcal{O}_S} R \to \check{\mathcal{H}}^{F,\Omega}_{(0)}(R)/t\check{\mathcal{H}}^{F,\Omega}_{(0)}(R)$$

is an isomorphism of free R-modules. For any local section V of $\mathcal{T}_S$, by definition we have $[t\nabla^\Omega_V \zeta_+(R)] = [\varphi_R((\partial_V F)\tilde{\zeta}_+(R))] \in \check{\mathcal{H}}^{F,\Omega}_{(0)}(R)/t\check{\mathcal{H}}^{F,\Omega}_{(0)}(R)$, where $\tilde{\zeta}_+(R) \in \mathcal{H}^{F,\Omega}_{(0)}$ is a lifting of $\zeta_+(R)$. That is, we have

$$t\nabla^\Omega \zeta_+(R) : \mathcal{T}_S \otimes_{\mathcal{O}_S} R \to p_*\mathcal{O}_{\mathcal{C}(F)}\tilde{\zeta}_+(R) \otimes_{\mathcal{O}_S} R \subset \check{\mathcal{H}}^{F,\Omega}_{(0)}(R)/t\check{\mathcal{H}}^{F,\Omega}_{(0)}(R) \cong p_*\mathcal{O}_{\mathcal{C}(F)} \otimes_{\mathcal{O}_S} R.$$

Note that when $R = \mathcal{O}_{S,0}/\mathfrak{m} \cong \mathbb{C}$, $\zeta_+(\mathbb{C}) = \zeta_0$ and the inclusion $\subset$ is in fact an isomorphism due to the primitivity of ζ_0 (see Definition 4.13). In this case the arrow also becomes an isomorphism, since the Kodaira-Spencer map (3.1) is an isomorphism. That is, $t\nabla^\Omega \zeta_+(\mathbb{C})$ is an isomorphism. Hence, $t\nabla^\Omega \zeta_+(R)$ is an isomorphism of R-modules by Nakayama Lemma.

(P2)$^\vee$ Let V_1, V_2 be local sections of $\mathcal{T}_S$. Since $e^{(f-F)/t}\zeta_0$ is flat, we have

$$t\nabla^\Omega_{V_1} \zeta_+(R) = -t\nabla^\Omega_{V_1} \zeta_-(R).$$

Since $t\nabla^\Omega_{V_1} \zeta_+(R) \in \check{\mathcal{H}}^{F,\Omega}_{(0)}(R)$, and $t\nabla^\Omega_{V_1} \zeta_-(R) \in t\mathcal{L}_\mathcal{R}(R)$ by Lemma 4.11, we find

$$t\nabla^\Omega_{V_1} \zeta_+(R) \in \check{\mathcal{H}}^{F,\Omega}_{(0)}(R) \cap t\mathcal{L}_\mathcal{R}(R).$$

By Lemma 4.9, we have

$$\mathcal{K}^F_\Omega(\text{-},\text{-}) : \mathcal{L}_\mathcal{R}(R) \otimes \mathcal{L}_\mathcal{R}(R) \to t^{-2}R[t^{-1}].$$

Hence,

$$\mathcal{K}^F_\Omega(\text{-},\text{-}) : \big(\check{\mathcal{H}}^{F,\Omega}_{(0)}(R) \cap t\mathcal{L}_\mathcal{R}(R)\big) \otimes \big(\check{\mathcal{H}}^{F,\Omega}_{(0)}(R) \cap t\mathcal{L}_\mathcal{R}(R)\big) \to R[[t]] \cap R[t^{-1}] = R,$$

where the value does not depend on t. In particular, the property in (P2)$^\vee$ holds:

$$\mathcal{K}^F_\Omega\big(\nabla^\Omega_{V_1}\zeta_+(R), \nabla^\Omega_{V_2}\zeta_+(R)\big) \in t^{-2}R.$$

(P3)$^\vee$ Let V_1, V_2 be local sections of $\mathcal{T}_S$. The identity $\nabla^\Omega_{V_1}\nabla^\Omega_{V_2}\zeta_+(R) = -\nabla^\Omega_{V_1}\nabla^\Omega_{V_2}\zeta_-(R)$ implies

$$\nabla^\Omega_{V_1}\nabla^\Omega_{V_2}\zeta_+(R) \in t^{-2}\check{\mathcal{H}}^{F,\Omega}_{(0)}(R) \cap \mathcal{L}_\mathcal{R}(R) = \big(t^{-2}\check{\mathcal{H}}^{F,\Omega}_{(0)}(R) \cap t^{-1}\mathcal{L}_\mathcal{R}(R)\big) \oplus \big(t^{-1}\check{\mathcal{H}}^{F,\Omega}_{(0)}(R) \cap \mathcal{L}_\mathcal{R}(R)\big).$$

This implies the first property in (P3)$^\vee$:

$$\mathcal{K}^F_\Omega\big(\nabla^\Omega_{V_1}\nabla^\Omega_{V_2}\zeta_+(R),\nabla^\Omega_{V_3}\zeta_+(R)\big)\in t^{-3}R\oplus t^{-2}R.$$

For any $[s]\in\mathcal{H}^{f,\Omega}$, it is easy to see that

$$\nabla^\Omega_{t\partial_t}[\varphi_R(e^{(F_0-F)/t}\pi_X^*(s))]=[\varphi_R(e^{(F_0-F)/t}\pi_X^*\big((t\partial_t-\frac{f}{t})s\big))], \tag{4.6}$$

the right hand side of which equals $[\varphi_R(e^{(F_0-F)/t}\pi_X^*(\nabla^\Omega_{t\partial_t}s))]$ by definition. In particular if $[s]\in\mathcal{L}\subset\mathcal{H}^{f,\Omega}$, then $\nabla^\Omega_{t\partial_t}[s]\in\mathcal{L}$ (since $\mathcal{L}$ is a good opposite filtration). Consequently,

$$\nabla^\Omega_{t\partial_t}[\varphi_R(e^{(F_0-F)/t}\pi_X^*(s))]\in\mathcal{L}_{\mathcal{R}}(R).$$

That is, $\nabla^\Omega_{t\partial_t}$ preserves $\mathcal{L}_{\mathcal{R}}(R)$. On the other hand $\nabla^\Omega_{t\partial_t}:\check{\mathcal{H}}^{F,\Omega}_{(0)}(R)\to t^{-1}\check{\mathcal{H}}^{F,\Omega}_{(0)}(R)$, it follows that

$$\nabla^\Omega_{t\partial_t}\nabla^\Omega_{V_1}\zeta_+(R)=-\nabla^\Omega_{t\partial_t}\nabla^\Omega_{V_1}\zeta_-(R)\in t^{-2}\check{\mathcal{H}}^{F,\Omega}_{(0)}(R)\cap\mathcal{L}_{\mathcal{R}}(R).$$

By the same arguments as above, the second property in (P3)$^\vee$ holds.

(P4)$^\vee$ First, let us observe that the operator $\nabla^\Omega_{t\partial_t}+\nabla^\Omega_E$ preserves $\mathcal{H}^{F,\Omega}_{(0)}$. Indeed, by the definition of the Euler vector field E, there exist $g_i\in\Gamma(Z,\mathcal{O}_Z)$, $i=1,\cdots,n$, such that $F-\partial_EF=\sum_i g_i\partial_{z^i}F$, where we remind that a projection $\pi_X:Z\to X$ is fixed. Hence, as operators on $\mathcal{H}^{F,\Omega}$, we have

$$\begin{aligned}\big(\nabla^\Omega_{t\partial_t}+\nabla^\Omega_E\big)[s]&=[\big(t\partial_t-\frac{F}{t}+\partial_E+\frac{\partial_EF}{t}\big)(s)]\\&=[\big(t\partial_t+\partial_E-t^{-1}\sum_i g_i\partial_{z^i}F\big)(s)]=[\big(t\partial_t+\partial_E+\sum_i\lambda\frac{\partial}{\partial z_i}\frac{g_i}{\lambda}+\sum_i g_i\frac{\partial}{\partial z_i}\big)(s)],\end{aligned}$$

where $[s]$ denotes a local section of $\mathcal{H}^{F,\Omega}$, and which clearly preserves $\mathcal{H}^{F,\Omega}_{(0)}$. Therefore,

$$\big(\nabla^\Omega_{t\partial_t}+\nabla^\Omega_E\big)\check{\mathcal{H}}^{F,\Omega}_{(0)}(R)\subset\check{\mathcal{H}}^{F,\Omega}_{(0)}(R).$$

Similarly

$$(\nabla^\Omega_{t\partial_t}+\nabla^\Omega_E)\mathcal{L}_{\mathcal{R}}(R)\subset\mathcal{L}_{\mathcal{R}}(R)$$

by (4.6) and Lemma 4.11. We find the decomposition

$$\big((\nabla^\Omega_{t\partial_t}+\nabla^\Omega_E)e^{(f-F)/t}\zeta_0\big)(R)=\Big(\nabla^\Omega_{t\partial_t}+\nabla^\Omega_E\Big)\zeta_+(R)+\Big(\nabla^\Omega_{t\partial_t}+\nabla^\Omega_E\Big)\zeta_-(R)\in\check{\mathcal{H}}^{F,\Omega}_{(0)}(R)\oplus\mathcal{L}_{\mathcal{R}}(R).$$

On the other hand,

$$\big(\nabla^\Omega_{t\partial_t}e^{(f-F)/t}\zeta_0\big)(R)-r\big(e^{(f-F)/t}\zeta_0\big)(R)\in\mathcal{L}_{\mathcal{R}}(R)$$

which follows from (4.6) and the homogeneity of ζ_0. It leads to another decomposition

$$\big((\nabla^\Omega_{t\partial_t}+\nabla^\Omega_E)e^{(f-F)/t}\zeta_0\big)(R)=\big(\nabla^\Omega_{t\partial_t}e^{(f-F)/t}\zeta_0\big)(R)\in r(e^{(f-F)/t}\zeta_0)(R)+\mathcal{L}_{\mathcal{R}}(R)=r\zeta_+(R)+\mathcal{L}_{\mathcal{R}}(R).$$

By the uniqueness of the splitting of $\big((\nabla^\Omega_{t\partial_t}+\nabla^\Omega_E)e^{(f-F)/t}\zeta_0\big)(R)$, we find

$$\Big(\nabla^\Omega_{t\partial_t}+\nabla^\Omega_E\Big)\zeta_+(R)=r\zeta_+(R).$$

In addition, it is easy to see the all $(P2)^\vee$, $(P3)^\vee$ and $(P4)^\vee$ above satisfy the compatibility condition with respect to the inverse system $\mathcal{R}$. Hence, ζ_+ is a formal primitive form.

Conversely, let ζ_+ be a formal primitive form. Let $B(R)$ be the image of the map

$$t\nabla^\Omega\zeta_+(R) : \mathcal{T}_S \otimes_{\mathcal{O}_S} R \to \check{\mathcal{H}}^{F,\Omega}_{(0)}(R), \quad V \to t\nabla^\Omega_V\zeta_+(R),$$

which is a R-submodule of $\check{\mathcal{H}}^{F,\Omega}_{(0)}(R)$. Then we have $\check{\mathcal{H}}^{F,\Omega}(R) = B(R)((t))$ and $\check{\mathcal{H}}^{F,\Omega}_{(0)}(R) = B(R)[[t]]$. Define $\mathcal{L}_{\mathcal{R}}(R) := t^{-1}B(R)[t^{-1}] \subset \check{\mathcal{H}}^{F,\Omega}(R)$. It is easy to check (by reversing the above argument) that $\mathcal{L}_{\mathcal{R}}(R)$ defines a sub $\mathcal{R}$-module of $\check{\mathcal{H}}^{F,\Omega}$ which is preserved by the Gauss-Manin connection. Then $\mathcal{L} := \mathcal{L}_{\mathcal{R}}(\mathbb{C})$ defines a good opposite filtration of $\mathcal{H}^{f,\Omega}$ (by using $(P2)^\vee$, $(P3)^\vee$), and we obtain a primitive element $\zeta_0 := \zeta_+(\mathbb{C})$ with respect to such $\mathcal{L}$ (by using $(P1)^\vee$, $(P4)^\vee$). □

Corollary 4.16. *Let $\mathcal{L}$ be a good opposite filtration and ζ_0 be a primitive element with respect to $\mathcal{L}$. Then there is a unique section η in $\Gamma(\mathcal{R}, \check{\mathcal{H}}^{F,\Omega}_{(0)})$ satisfying*

$$e^{(F-f)/t}\eta(R) \in \zeta_0 \otimes_{\mathbb{C}} 1 + \mathcal{L} \otimes_{\mathbb{C}} R$$

for all $R \in \mathcal{R}$. Furthermore, such η is precisely given by the formal primitive form ζ_+ defined in (4.5).

The above corollary provides a convenient way to compute the primitive form in practice. In particular, this leads to our algorithm in Section 5.4 and is used in Section 6 to compute primitive forms for explicit examples.

4.3. **Analyticity.** We have constructed the formal primitive form in the formal neighborhood of the universal unfolding with respect to the choice of a good opposite filtration together with a primitive element. In fact, more is true: the formal primitive form as constructed in Theorem 4.15 extends to certain analytic neighborhood. This follows from the comparison between the formal construction and the original analytic construction by the third author, as described below.

Consider the natural map

$$p_*\Omega^n_{Z/S}/dF \wedge d\left(p_*\Omega^{n-2}_{Z/S}\right) \to \mathcal{H}^{F,\Omega}_{(0)}, \quad [\xi] \to [\frac{\xi}{\Omega_{Z/S}}].$$

There is a canonical filtration on the left hand side defined by the power of the operator ∇_δ^{-1} in [46], which coincides with the filtration $\mathcal{H}^{F,\Omega}_{(-k)}$ on the right hand side under the above map. Actually, there are also the Gauss-Manin connection and the higher residue pairing defined on the left hand side, with which the above map is equivariant. Such map becomes an isomorphism if we take the formal completion of the left hand side with respect to the above filtration [45].

The space $p_*\Omega^n_{Z/S}/dF \wedge d\left(p_*\Omega^{n-2}_{Z/S}\right)$ is sometimes called the *Brieskorn lattice*, which has a purely analytic nature. The original primitive form is formulated as an analytic object inside this space satisfying the same properties, where the notion of good opposite filtrations $\mathcal{L}$ which we used here is identified with the notion of *good sections* v (see Remark 4.14) and the primitive element was similarly described. It was shown in [46] on the correspondence between primitive

forms and good sections, and the analyticity of the primitive form was proved via solving a version of Riemann-Hilbert-Birkhoff problem. The construction we provide in the present paper is the formulation of this correspondence in the formal setting. In particular, we have the maps

$$\begin{array}{ccc} \{\text{analytic primitive forms}\} & \longrightarrow & \{\text{formal primitive forms}\} \\ \downarrow & & \downarrow \\ \{\text{good sections}\} & = & \{\text{good opposite filtrations}\} \end{array}$$

Here the vertical maps are fibered by the primitive elements, and the top row is defined by restriction to the formal neighborhood. It follows that the top row is in fact an bijection. As a consequence, we have

Theorem 4.17. *Let $\mathcal{L}$ be a good opposite filtration and ζ_0 be a primitive element with respect to $\mathcal{L}$. Then the inverse limit of the formal primitive form ζ_+ defined in* (4.5) *is the Taylor series expansion of an analytic primitive form.*

The reason we use the formal setting is that the explicit formula $e^{(f-F)/t}$ allows us to compute the Taylor series expansion of the primitive form up to arbitrary finite order. In the next section we will study the moduli space of primitive forms for all all weighted homogeneous polynomial, and provide an explicit computation algorithm for the primitive forms.

5. Primitive forms for weighted homogeneous polynomials

In this section, we will present applications of Theorem 4.15 when f is a weighted homogeneous polynomial. We will describe the moduli space of all primitive forms. In addition, we will provide a concrete algorithm to compute the Taylor series expansions of primitive forms up to an arbitrary order.

5.1. **Weighted-homogeneity.** A polynomial $f : (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}, 0)$ is called a *weighted homogeneous polynomial* (of total degree 1) with weights $(q_1, \cdots, q_n)$, where each q_i is a rational number with $0 < q_i \leq \frac{1}{2}$, if $f(\lambda^{q_1} z_1, \cdots, \lambda^{q_n} z_n) = \lambda f(z_1, \cdots, z_n)$ holds for all $\lambda \in \mathbb{R}_{>0}$. The rational numbers $q_1, \cdots, q_n$ are called *weight degrees*. It defines the weight degree for polynomials by assigning $\deg z_i = q_i$. In particular, $\deg f = 1$.

Remark 5.1. *Since $\sum_{i=1}^n q_i z_i \partial_{z_i} f = f$, f belongs to its Jacobian ideal $(\partial_{z_1} f, \cdots, \partial_{z_n} f)$. In fact, more is true: a holomorphic function-germ $g : (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}, 0)$, with $\mathbf{0}$ being an isolated critical point, is analytically equivalent to a weighted homogeneous polynomial if and only if g belongs to its Jacobian ideal* [41].

Throughout this section, we will fix one such f. Let

$$s(f) := \sum_{i=1}^{n} (1 - 2q_i)\,, \tag{5.1}$$

which was introduced in [42] to classify weighted homogenous polynomials f satisfying $0 < s(f) \leq 1$. The invariant $s(f)$ sometimes borrows the name *central charge* since it coincides with the notion in the superconformal field theory associated to the Landau-Ginzburg model defined by f.

Example 5.2 (Calabi-Yau hypersurface)**.** *Every weighted homogenous polynomial $f(z_1, \cdots, z_n)$ with $\sum_{i=1}^n q_i = 1$ defines a Calabi-Yau hypersurface $f = 0$ in the weighted projective space $\mathbb{P}^{q_1,\cdots,q_n}$, whose dimension is equal to $s(f) = n - 2$.*

Let $\{\phi_1, \cdots, \phi_\mu\}$ be a set of weighted homogeneous polynomials that represent a basis of $\mathrm{Jac}(f)$ with $\deg \phi_1 \leq \deg \phi_2 \leq \cdots \leq \deg \phi_\mu$. As a well-known fact, we have $0 = \deg \phi_1 < \deg \phi_i < \deg \phi_\mu = s(f)$ for any $1 < i < \mu$, and

$$\deg \phi_i + \deg \phi_j = s(f) \quad \text{whenever} \quad i + j = \mu + 1.$$

Take $Z \subset X \times S \subset \mathbb{C}^n \times \mathbb{C}^\mu$ with $S \subset \mathbb{C}^\mu$ a (small) Stein domain containing the origin, such that the holomorphic function $F : Z \to \mathbb{C}$, defined by

$$F(\mathbf{z}, \mathbf{u}) = f(\mathbf{z}) + \sum_{j=1}^{\mu} u_j \phi_j(\mathbf{z}), \tag{5.2}$$

is a universal unfolding. Then (Z, S, p, F) is a frame (see Definition 3.1). We will extend the weight degree to deformation parameters

$$\deg u_j := 1 - \deg \phi_j \tag{5.3}$$

for each j. F becomes a weight-homogeneous polynomial in $(\mathbf{z}, \mathbf{u})$,

$$\left(\sum_{i=1}^{n} q_i z_i \partial_{z_i} + \sum_{j=1}^{\mu} (\deg u_j) u_j \partial_{u_j} \right) F = F.$$

As a consequence, the Euler vector field (defined by (3.3)) is given by

$$E = \sum_{j=1}^{\mu} (\deg u_j) u_j \partial_{u_j}. \tag{5.4}$$

We will discuss primitive forms for weighted homogeneous polynomials f with respect to (Z, S, p, F) and the choice

$$\Omega_{Z/S} := dz_1 \wedge \cdots \wedge dz_n.$$

Now we briefly review a partial classification of weighted homogeneous polynomials (of total degree 1). Two function-germs $g_1 : (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}, \mathbf{0})$ and $g_2 : (\mathbb{C}^m, \mathbf{0}) \to (\mathbb{C}, \mathbf{0})$ are called *stably equivalent* if $g_1(z_1, \cdots, z_n) + z_{n+1}^2 + \cdots + z_k^2$ and $g_2(\tilde{z}_1, \cdots, \tilde{z}_m) + \tilde{z}_{m+1}^2 + \cdots + \tilde{z}_k^2$ define the same function-germ $(\mathbb{C}^k, \mathbf{0}) \to (\mathbb{C}, \mathbf{0})$ for some $k \geq \max\{m, n\}$ up to a local analytic coordinate transformation. By Proposition 3.28, it is sufficient to deal with f up to stable equivalence. The classification of f, which was started by V.I. Arnold, is still open in general. However, the case of $s(f) \leq 1$ was already completely classified. For $s(f) < 1$, there are in total 5 lists of f, which

are called *simple singularities* [1] or *ADE-singularities*; for $s(f) = 1$, there are in total 3 lists, which are called *simple elliptic singularities* in [42]. Among those weighted homogeneous polynomials satisfying $1 < s(f) < 2$, there are 14 polynomials (in three variables) named *exceptional unimodular singularities* of type $E_{12}, E_{13}, E_{14}, Z_{11}, Z_{12}, Z_{13}, W_{12}, W_{13}, Q_{10}, Q_{11}, Q_{12}, S_{11}, S_{12}$ and U_{12} respectively [1]. The degrees of the weighted homogeneous basis for these 14 polynomials are given by

$$\deg \phi_i = \frac{m_i - a - b - c}{h} + 1 \tag{5.5}$$

with a, b, c, h and the *exponent* m_i read off directly from Table 3 of [47].

5.2. **A grading operator.** We can adapt the weight degrees to polyvector fields.

Definition 5.3. *Define an operator* $\mathbb{E}_f : \mathrm{PV}(X)((t)) \to \mathrm{PV}(X)((t))$, *by*

$$\mathbb{E}_f := t\partial_t + \sum_{i=1}^n q_i\big(z_i\partial_{z_i} - \bar{z}_i\partial_{\bar{z}_i}\big) + \sum_{i=1}^n \big((1-q_i)\partial_i \wedge \frac{\partial}{\partial \partial_i} - q_i d\bar{z}_i \frac{\partial}{\partial d\bar{z}_i}\big),$$

where the operator $\frac{\partial}{\partial d\bar{z}_i}$ *is defined similarly to* $\frac{\partial}{\partial \partial_i}$.

Clearly, $\mathbb{E}_f$ perserves $\mathrm{PV}_c(X)((t))$.

We may formally treat t (resp. $\bar{z}_i, \partial_i, d\bar{z}_i$) as of weight degree 1 (resp. $-q_i$, $1-q_i$, $-q_i$). Then the differential Q_f is an operator of weight degree 0 on $\mathrm{PV}(X)((t))$. The next lemma becomes a trivial consequence of this observation.

Lemma 5.4. *The operator* $\mathbb{E}_f$ *commutes with the coboundary operator* Q_f *(recall* (2.6)*).*

It follows that $\mathbb{E}_f$ is well defined on the cohomology $\mathcal{H}^{f,\Omega}$, and it induces a $\mathbb{Q}$-grading on $\mathcal{H}^{f,\Omega}_{(0)}$ and $\mathcal{H}^{f,\Omega}$. Classes in $\mathcal{H}^{f,\Omega}$ are represented by elements in $\Gamma(X, \mathcal{O}_X)[[t]]$, for which

$$\mathbb{E}_f(ht^k) = \big(t\partial_t + \sum_i q_i z_i \partial_{z_i}\big)(ht^k), \quad \text{where } h \in \Gamma(X, \mathcal{O}_X).$$

The $\mathbb{Q}$-grading is just the weight degree of the representatives. In the following discussions, we will always assume the extended weight degree assignments

$$\deg z_i = q_i, \quad \deg u_j = 1 - \deg \phi_j, \quad \deg t = 1, \quad 1 \le i \le n, 1 \le j \le \mu.$$

Recall that the higher residue $\widehat{\mathrm{Res}}_f$ is defined by (2.14).

Proposition 5.5. *The higher residue map* $\widehat{\mathrm{Res}}_f : \mathcal{H}^{f,\Omega}_{(0)} \to \mathbb{C}[[t]]$ *is homogeneous of weight degree* $-s(f)$*: if* $\deg \alpha = m$, *then* $\deg(\widehat{\mathrm{Res}}_f(\alpha)) = m - s(f)$.

Proof. This follows from a direct calculation on integration by part and we leave it to the reader. □

The next corollary is a direct consequence of the above proposition

Corollary 5.6. *The higher residue pairing* $\mathcal{K}_\Omega^f(\text{-},\text{-}) : \mathcal{H}_{(0)}^{f,\Omega} \otimes \mathcal{H}_{(0)}^{f,\Omega} \to \mathbb{C}[[t]]$ *is of weight degree* $-s(f)$.

The higher residue map $\widehat{\text{Res}}^f$ induces a map on the associated graded vector space of the Hodge filtration, the restriction of which to $\mathcal{H}_{(0)}^{f,\Omega}/\mathcal{H}_{(-1)}^{f,\Omega} = \text{Jac}(f)$ coincides with the usual residue map by Proposition 2.5. As a direct consequence of Proposition 5.5, we reprove the next well-known fact.

Corollary 5.7. *The residue* $\text{Res} : \text{Jac}(f) \to \mathbb{C}$ *is only non-zero at the weighted homogeneous component of (top) weight degree* $s(f)$ *in* $\text{Jac}(f)$.

The grading operator $\mathbb{E}_f$ on $\text{PV}(X)((t))$ is naturally extended to the universal unfolding

$$\mathbb{E}_F := \mathbb{E}_f + \sum_{j=1}^{\mu} (\deg u_j)\big(u_j\partial_{u_j} - \bar{u}_j\partial_{\bar{u}_j} - d\bar{u}_j\frac{\partial}{\partial d\bar{u}_j}\big),$$

which acts on $\text{PV}(Z/S)((t))$ and commutes with Q_F. Hence, $\mathbb{E}_F$ acts on $\mathcal{H}_{(0)}^{F,\Omega}$, and induces a $\mathbb{Q}$-grading on it. As a family version of Proposition 5.5, we have

Proposition 5.8. *The higher residue map* $\widehat{\text{Res}}^F : \mathcal{H}_{(0)}^{F,\Omega} \to \mathcal{O}_S[[t]]$ *is of weight degree* $-s(f)$.

Proof. This follows from the homogeneity of the choice of the holomorphic form $\Omega_{Z/S} = dz_1 \wedge \cdots \wedge dz_n$ and a similar calculation on integration by part. □

Corollary 5.9. *The higher residue pairing* $\mathcal{K}_\Omega^F(\text{-},\text{-}) : \mathcal{H}_{(0)}^{F,\Omega} \otimes \mathcal{H}_{(0)}^{F,\Omega} \to \mathcal{O}_S[[t]]$ *is of weight degree* $-s(f)$.

Proposition 5.10. *A section* $[\eta] \in \Gamma(S, \mathcal{H}_{(0)}^{F,\Omega})$ *satisfies property* (P4) *of Definition 3.21 if and only if* $[\eta]$ *is weighted homogeneous with respect to the* $\mathbb{Q}$*-grading induced by* $\mathbb{E}_F$.

Proof. Write $\eta = \sum_{j=0}^{\infty} a_j t^j, a_j \in \Gamma(Z, \mathcal{O}_Z)$. Modulo the coboundary operator Q_F, we have

$$-\left(\sum_i q_i z_i \partial_{z_i} F\right)\eta \equiv t\sum_i q_i z_i \partial_{z_i}\eta + t\left(\sum_i q_i\right)\eta \mod \text{Im}(Q_F).$$

Hence,

$$\begin{aligned}
&(\nabla_{t\partial_t}^\Omega + \nabla_E^\Omega)[\eta] = [(t\frac{\partial}{\partial t} - \frac{F}{t} + \partial_E + \frac{\partial_E F}{t})\eta] \\
=&[(t\frac{\partial}{\partial t} + \partial_E - \frac{1}{t}(\sum_{i=1}^n q_i z_i \partial_{z_i} F))\eta] = [(t\frac{\partial}{\partial t} + \partial_E + \sum_i q_i z_i \partial_{z_i} + \sum_i q_i)\eta] \\
=&[\sum_q (t\frac{\partial}{\partial t} + \partial_E + \sum_i q_i z_i \partial_{z_i})\eta + (\sum_i q_i)\eta] = [\mathbb{E}_F\eta + (\sum_i q_i)\eta].
\end{aligned}$$

Hence, property (P4) holds if and only if $\mathbb{E}_F[\eta] = (r - \sum_i q_i)[\eta]$, i.e., $[\eta]$ is weighted homogeneous of degree $r - \sum_i q_i$. □

5.3. **Moduli space of primitive forms.** As we have shown in the previous section, a primitive form in $\Gamma(S, \mathcal{H}^{F,\Omega}_{(0)})$ is equivalent to a good opposite filtration $\mathcal{L}$ of $\mathcal{H}^{f,\Omega}$ together with a primitive element with respect to $\mathcal{L}$.

Proposition 5.11. *Let $\mathcal{L} \subset \mathcal{H}^{f,\Omega}$ be an opposite filtration. The following are equivalent:*

$$(1)\ \mathcal{L} \text{ is good}; \quad (2)\ \mathbb{E}_f \text{ preserves } \mathcal{L}; \quad (3)\ \mathbb{E}_f \text{ preserves } \mathcal{H}^{f,\Omega}_{(0)} \cap t\mathcal{L}.$$

Furthermore, $1 \in \mathcal{H}^{f,\Omega}_{(0)}$ is the unique primitive element with respect to a given good opposite filtration, up to a scalar multiple by nonzero complex numbers.

Proof. Denote $r := \sum_i q_i$. For any $[\eta] \in \mathcal{H}^{f,\Omega}_{(0)}$, we have

$$\nabla^{\Omega}_{t\partial_t}[\eta] = [(t\partial_t - \frac{f}{t})\eta] = [(t\partial_t - \sum_i \frac{q_i z_i \partial_{z_i} f}{t})\eta] = [(r + t\partial_t + \sum_i q_i z_i \partial_{z_i})\eta] = \big(r + \mathbb{E}_f\big)\,[\eta].$$

Hence, $\mathcal{L}$ is good if and only if $\mathbb{E}_f\mathcal{L} \subset \mathcal{L}$, which holds if and only if $\mathbb{E}_f$ preserves $B := \mathcal{H}^{f,\Omega}_{(0)} \cap t\mathcal{L}$ (since $\mathcal{L} = t^{-1}B[t^{-1}]$ and $\mathbb{E}_f\mathcal{H}^{f,\Omega}_{(0)} \subset \mathcal{H}^{f,\Omega}_{(0)}$).

Since $\zeta_0 \in \mathcal{H}^{f,\Omega}_{(0)}$ is primitive only if ζ_0 is a weighted homogenous element (of degree zero) such that $\mathrm{Jac}(f)\zeta_0 = \mathrm{Jac}(f)$, the latter statement follows. □

We now construct all good opposite filtrations for any given weighted homogeneous singularity. The method is a generalization of the classical construction of the flat coordinates for the case of finite reflection group quotient [49], by including the parameter t. Let us fix one set $\{\phi_1, \cdots, \phi_\mu\} \subset \mathbb{C}[\mathbf{z}]$ of weighted homogeneous polynomials such that

(1) $\{\phi_1, \cdots, \phi_\mu\}$ represent a basis of $\mathrm{Jac}(f)$.
(2) $\deg \phi_1 < \deg \phi_2 \leq \cdots \leq \deg \phi_{\mu-1} < \deg \phi_\mu$, and $\phi_1 = 1$.
(3) The residue pairing (ϕ_i, ϕ_j) is zero unless $i + j = \mu + 1$, which implies that each $(\phi_i, \phi_{\mu+1-i})$ is nonzero due to the non-degeneracy of the residue pairing.

Let

$$r(i,j) := \deg \phi_i - \deg \phi_j.$$

Whenever referring to an element $\mathbf{c} = \big(c_{ij}\big)_{1\leq i,j\leq \mu} \in \mathbb{C}^{\mu^2}$ below, we always require $c_{ij} = 0$ unless $r(i,j) \in \mathbb{Z}_{>0}$. For such $\mathbf{c}$, we set

$$B(\mathbf{c}) := \mathrm{Span}_{\mathbb{C}}\{\Phi_1(\mathbf{c}), \cdots, \Phi_\mu(\mathbf{c})\} \subset \mathcal{H}^{f,\Omega}_{(0)}$$

with

$$\Phi_i = \Phi_i(\mathbf{c}) := \phi_i + \sum_{j=1}^{i-1} c_{ij}\phi_j t^{r(i,j)}.$$

Let

$$Y := \{\mathbf{c} \in \mathbb{C}^{\mu^2} \mid \mathcal{K}^f_{\Omega}\big(B(\mathbf{c}), B(\mathbf{c})\big) \subset \mathbb{C}, c_{ij} = 0 \text{ if } r(i,j) \notin \mathbb{Z}_{>0}\} \subset \mathbb{C}^{\mu^2}.$$

Proposition 5.12. *There is a bijection map*

$$Y \xrightarrow{\cong} \{\text{good opposite filtrations of } \mathcal{H}^{f,\Omega}\}$$

defined by $\mathbf{c} \in Y \mapsto \mathcal{L}(\mathbf{c}) := t^{-1}B(\mathbf{c})[t^{-1}]$.

Proof. For any $\mathbf{c} \in Y$, we have $\mathcal{K}^f_\Omega(B(\mathbf{c}), B(\mathbf{c})) \subset \mathbb{C}$. Moreover, it follows by construction that $\mathcal{H}^{f,\Omega}_{(0)} = B(\mathbf{c})[[t]]$, $\mathcal{H}^{f,\Omega} = B(\mathbf{c})((t)) = \mathcal{H}^{f,\Omega}_{(0)} \oplus \mathcal{L}(\mathbf{c})$ and $t^{-1}\mathcal{L}(\mathbf{c}) \subset \mathcal{L}(\mathbf{c})$. Hence, $\mathcal{L}(c)$ is an opposite filtration (recall Definition 2.16). Clearly, $B(\mathbf{c})$ is preserved by $\mathbb{E}_f$, since $\{\Phi_j\}$ are weighted homogeneous with respect to $\mathbb{E}_f$. Therefore $\mathcal{L}(\mathbf{c})$ is good by Proposition 5.11, and the given map is defined. The injectivity of such map follows from the linear independence of $\{\phi_1, \cdots, \phi_\mu\}$ in $\mathcal{H}^{f,\Omega}$.

Now we assume $\mathcal{L}$ to be a good opposite filtration of $\mathcal{H}^{f,\Omega}$ and let $B := \mathcal{H}^{f,\Omega}_{(0)} \cap t\mathcal{L}$. Then we have $\mathcal{K}^f_\Omega(B, B) \subset \mathbb{C}$ by definition, and $\mathbb{E}_f$ preserves B by Proposition 5.11. It follows that $B = B(\mathbf{c})$ for some $\mathbf{c} \in Y$. That is, the given map is surjective. □

Let $\mathbb{Z}^{\text{odd}}_{>0}$ be the set of positive odd integers, and set

$$D := \sharp\{(i,j) \mid r(i,j) \in \mathbb{Z}_{>0}, i+j < \mu+1\} + \sharp\{(i,j) \mid r(i,j) \in \mathbb{Z}^{\text{odd}}_{>0}, i+j = \mu+1\}.$$

Proposition 5.13. *Y is an algebraic subvariety of $\mathbb{C}^{\mu^2}$ defined by quadratic equations, which is biregular to $\mathbb{C}^D$. In particular if $D = 0$, then $Y = \{\mathbf{0}\}$ consists of the origin of $\mathbb{C}^{\mu^2}$ only.*

Proof. For $1 \le i, j \le \mu$, we let $a_{ij}(t) := \mathcal{K}^f_\Omega(\phi_i, \phi_j)$. We also let $d_{ij}(t) := c_{ij}t^{r(i,j)}$ if $r(i,j) \in \mathbb{Z}_{>0}$, or 1 if $i = j$, or 0 otherwise. It follows from Corollary 5.6 that every $a_{ij}(t)$ is a monomial in $\mathbb{C}[t]$ of degree $\deg\phi_i + \deg\phi_j - s(f)$. Our assumption on $\{\phi_i\}$ implies that $a_{ij}(t) = 0$ if $i + j < \mu + 1$, and $a_{ij}(t)$ is a nonzero constant if $i + j = \mu + 1$. Since $\Phi_i = \sum_{k=1}^{i} d_{ik}(t)\phi_k$, the pairing

$$\mathcal{K}^f_\Omega(\Phi_i, \Phi_j) = \begin{cases} 0, & \text{if } i+j < \mu+1 \\ = a_{ij} \in \mathbb{C}^*, & \text{if } i+j = \mu+1 \end{cases}.$$

By the skew-symmetric property of the higher residue pairing, $\mathbf{c} \in Y$ if and only if all the monomials $\mathcal{K}^f_\Omega(\Phi_i, \Phi_{j'})$ are constant functions for those entries (i, j') with $i \le j'$ and $i + j' > \mu + 1$. Write $j' = \mu + 1 - j$. The condition is equivalent to that $\mathcal{K}^f_\Omega(\Phi_i, \Phi_{\mu+1-j})$ is a constant for all i, j with $i - j > 0$ and $i \le \mu + 1 - j$. Namely the monomial $K_{ij} := \mathcal{K}^f_\Omega(\Phi_i, \Phi_{\mu+1-j})$ in t of degree $r(i,j)$ satisfies

$$K_{ij} = d_{i,j}(t)a_{j,\mu+1-j} + a_{i,\mu+1-i}d_{\mu+1-j,\mu+1-i}(-t) + \sum_{\substack{j<k\le i \\ \mu+1-i<l\le\mu+1-j}} d_{ik}(t)a_{kl}(t)d_{\mu+1-j,l}(-t) \in \mathbb{C}.$$

Let us refer to $k - l$ as the step of a coordinate c_{kl}. Then K_{ij} contains precisely two (possibly the same) coordinates c_{ij}, $c_{\mu+1-j,\mu+1-i}$ of step $i - j$, and other coordinates of step strictly less than $i - j$. We analyze the constraints $K_{ij} \in \mathbb{C}$ according to $r(i,j) = r(\mu+1-j, \mu+1-i)$ as follows.

Assume $r(i,j) \notin \mathbb{Z}$ or $r(i,j) = 0$. Then $K_{ij} \in \mathbb{C}$ is a trivial condition.

Assume $r(i,j) \in \mathbb{Z}_{>0}$ and $i < \mu + 1 - j$, then the parameter $c_{\mu+1-j,\mu+1-i}$ (which is of step $i-j$ with $(\mu+1-j) > \mu+1-(\mu+1-i)$) is uniquely determined by the free parameter $c_{ij} \in \mathbb{C}$ and those $c_{k',l'}$ with smaller steps.

Assume $r(i,j) \in \mathbb{Z}_{>0}$ and $i = \mu + 1 - j$. Then $\mathcal{K}^f_\Omega(\Phi_i, \Phi_{\mu+1-j}) = \mathcal{K}^f_\Omega(\Phi_i, \Phi_i)$ is an even function in t by the skew-symmetric property of the higher residue pairing. Thus if $r(i,j)$ is odd, the pairing must vanish. In this case, LHS $\in \mathbb{C}$ is a trivial condition with a free parameter c_{ij} of the largest step. If $r(i,j)$ is even, then c_{ij} is uniquely determined by the summation in LHS, which involves coordinates of step strictly less than $i-j$ only.

In summary, there are precisely D free parameters. In particular if $D = 0$, then $c_{ij} = 0$ unless $r(i,j)$ is a positive even integer and $i = \mu + 1 - j$. However, such c_{ij} is equal to a summation of quadratic terms in coordinates of step strictly less than $i-j$, which are equal to zero. Hence, we have $c_{ij} = 0$ for all i, j. That is, $\mathbf{c} = \mathbf{0}$ if $D = 0$. Hence, the statement follows. □

We say two primitive forms ζ_1, ζ_2 of $\mathcal{H}^{F,\Omega}_{(0)}$ are *equivalent*, denoted as $\zeta_1 \sim \zeta_2$, if there exists $\lambda \in \mathbb{C}^*$ and open neighborhood $U \subset S$ of 0 such that $\zeta_1|_U = \lambda\zeta_2|_U \in \Gamma(U, \mathcal{H}^{F,\Omega})$. Let

$$\mathcal{M} := \{\text{primtive forms of } \mathcal{H}^{F,\Omega}_{(0)}\}/\sim \ .$$

Theorem 5.14. *There is a bijection map $Y \xrightarrow{\cong} \mathcal{M}$, inducing a bijection map $\mathcal{M} \xrightarrow{\cong} \mathbb{C}^D$.*

Proof. By using Theorem 4.15, Theorem 4.17 and Proposition 5.11, we obtain a bijection between $\mathcal{M}$ and the set of good opposite filtrations of $\mathcal{H}^{f,\Omega}$. The first statement follows from Proposition 5.12, and the second statement follows from Proposition 5.13. □

Corollary 5.15. *When $D = 0$, the representatives of any two weighted homogeneous bases of* $\mathrm{Jac}(f)$ *span the same vector subspace B in $\mathcal{H}^{f,\Omega}$. This defines the unique good opposite filtration of $\mathcal{H}^{f,\Omega}$ by*

$$\mathcal{L}_{\mathrm{can}} = t^{-1}B[t^{-1}].$$

If f is given by one of the exceptional unimodular singularities, then there exists a unique primitive form ζ_+, up to a scalar of nonzero complex numbers. Furthermore, the Taylor expansion of ζ_+ up to order $N-1$ is given by the projection of the element $[\varphi_{R_N}(e^{\frac{F_0-F}{t}})]$ in $\mathcal{H}^{F,\Omega} \otimes_{\mathcal{O}_S} R_N = (\mathcal{H}^{F,\Omega}_{(0)} \otimes_{\mathcal{O}_S} R_N) \oplus \left(e^{\frac{f-F}{t}}\mathcal{L}_{\mathrm{can}} \otimes_{\mathbb{C}} R_N\right)$ to $\mathcal{H}^{F,\Omega}_{(0)} \otimes_{\mathcal{O}_S} R_N$, where $R_N = \mathcal{O}_{S,0}/\mathfrak{m}^N$, and $\varphi_{R_N} : \mathcal{O}_{S,0} \to R_N$ is the natural projection.

Proof. By Proposition 5.12, there is a unique good opposite filtration $\mathcal{L}(\mathbf{0}) = \mathbf{0} = t^{-1}B(\mathbf{0})[t^{-1}]$ with respect to the choice $B = \mathrm{Span}_{\mathbb{C}}\{\phi_1, \cdots, \phi_\mu\}$. Let B' be spanned by $\{\phi'_1, \cdots, \phi'_\mu\} \subset \Gamma(X, \mathcal{O}_X)$, which represent another basis of $\mathrm{Jac}(f)$ with $\deg \phi'_1 \leq \cdots \deg \phi'_\mu = s(f)$. It follows

from Proposition 5.12 again that $\mathcal{L}'(\mathbf{0}) = t^{-1}B'(\mathbf{0})[t^{-1}]$ is the unique good opposite filtration. Hence, $\mathcal{L}'(\mathbf{0}) = \mathcal{L}(\mathbf{0})$ by the uniqueness, implying $B' = B$.

Now we consider the case when f is given by one of the exceptional unimodular singularities. All the degrees $\deg \phi_i$ can be read off directly from Table 3 of [47] by using equality (5.5). It follows that $\deg \phi_i + \deg \phi_j - s(f)$ is not an integer unless it vanishes, by direct calculations. Hence, we have $D = 0$, and therefore $Y = \{\mathbf{0}\}$. Then the statement follows immediately from Theorem 5.14, Theorem 4.15 and Theorem 4.17. □

5.4. **The perturbative formula of a primitive form.** Let $\{\phi_1, \cdots, \phi_\mu\} \subset \mathbb{C}[\mathbf{z}]$ be a set of weighted homogeneous polynomials as in the previous subsection, which represent a basis of $\mathrm{Jac}(f)$. As we have seen, a point $\mathbf{c} \in Y$ corresponds to a good opposite filtration $\mathcal{L}(\mathbf{c})$ of $\mathcal{H}^{f,\Omega}$, and hence a unique primitive form ζ_+ of $\mathcal{H}^{F,\Omega}_{(0)}$ with respect to $\mathcal{L}(\mathbf{c})$ together with the primitive element 1. In this subsection, we provide an explicit algorithm to compute ζ_+, up to an arbitrary finite order N.

Let $R = \mathbb{C}[\mathbf{u}]/\mathfrak{m}^{N+1}$. Consider the natural projection $\pi_X : Z \subset X \times S \to X$. The map

$$e^{\frac{F-f}{t}} : \mathcal{H}^{F,\Omega}_{(0)} \otimes_{\mathcal{O}_S} R \longrightarrow \mathcal{H}^{f,\Omega} \otimes_{\mathbb{C}} R = B((t)) \otimes_{\mathbb{C}} R$$

(which is the inverse of (4.3) in Lemma 4.7) defines a $\mu \times \mu$ matrix $A(\mathbf{u}, t, t^{-1})$ valued in R, by

$$e^{\frac{F-F_0}{t}} \begin{pmatrix} \pi_X^*\Phi_1 \\ \vdots \\ \pi_X^*\Phi_\mu \end{pmatrix} = A(\mathbf{u}, t, t^{-1}) \begin{pmatrix} \Phi_1 \\ \vdots \\ \Phi_\mu \end{pmatrix}.$$

Here we treat $\Phi_i = \Phi_i(\mathbf{c})$ and $\pi_X^*\Phi_i$ as elements in $\mathcal{H}^{f,\Omega} \otimes_{\mathbb{C}} R$ and $\mathcal{H}^{F,\Omega} \otimes_{\mathcal{O}_S} R$, respectively. Let

$$a := \max\{[N(s(f)-1) + s(f)], [s(f)]\}$$

where by $[r]$ we mean the largest integer that is less than or equal to r. It follows from degree reason that

$$A(\mathbf{u}, t, t^{-1}) = \sum_{k=-\infty}^{\infty} t^k A^{(k)}(\mathbf{u}) = \sum_{k=-N}^{a} t^k A^{(k)}(\mathbf{u})$$

where every $A^{(k)}$ ($k \neq 0$) is a $\mu \times \mu$ matrix with entries in $\mathfrak{m}/\mathfrak{m}^{N+1}$; so is $A_{(0)} - \mathrm{Id}$. As a matrix valued in R, $A^{(k)}$ is zero whenever $k > a$ or $k < -N$.

We consider the following $(a+1)\mu \times (a+1)\mu$ matrix

$$\Psi := \begin{pmatrix} A_{(0)} - \mathrm{Id} & A^{(1)} & \cdots & A^{(a)} \\ A^{(-1)} & A_{(0)} - \mathrm{Id} & \cdots & A^{(a-1)} \\ \cdots & \cdots & \cdots & \cdots \\ A^{(-a)} & A^{(-a+1)} & \cdots & A_{(0)} - \mathrm{Id} \end{pmatrix}$$

and the $1 \times (a+1)\mu$ matrix

$$\mathbf{e} := (1, 0, \cdots, 0).$$

Theorem 5.16. *The first $N+1$ terms of the Taylor series expansion of the primitive form ζ_+ around the reference point 0 is given by*

$$\zeta_+(R) = \sum_{i=0}^{a} t^i \vec{g}^{(i)} \begin{pmatrix} \Phi_1 \\ \vdots \\ \Phi_\mu \end{pmatrix}$$

where

$$(\vec{g}_{(0)}, \vec{g}^{(1)}, \cdots, \vec{g}^{(a)}) = \mathbf{e} \sum_{k=0}^{N} (-1)^k \Psi^k.$$

Proof. Let $\mathcal{L} = t^{-1}B(\mathbf{c})[t^{-1}]$. By Corollary 4.16, the hypothesis

$$\varphi_R(e^{\frac{F-f}{t}}\eta) \in 1 + \mathcal{L} \otimes_{\mathbb{C}} R$$

admits a unique solution given by the primitive form $\eta = \zeta_+(R)$ constructed with respect to $\mathcal{L}$. Since $\{\pi_X^* \Phi_i\}_i$ are a basis of $\mathcal{H}^{F,\Omega} \otimes_{\mathcal{O}_S} R$, we can write

$$\zeta_+(R) = \sum_{i=1}^{\mu} g_i \Phi_i, \text{ where } g_i = g_i(\mathbf{u}, t) \in \mathbb{C}[[t]] \otimes_{\mathbb{C}} R.$$

Since $\zeta_+(R)$ is of weight degree zero,

$$(g_1, \cdots, g_\mu) =: \vec{g} = \sum_{k=0}^{a} \vec{g}^{(k)} t^k,$$

where each $\vec{g}^{(i)}$ is a $1 \times \mu$ matrix with entries in R. Therefore, we have

$$(g_1, g_2, \cdots, g_\mu) \sum_{k=-N}^{a} t^k A^{(k)} = (1 + h_1, h_2, \cdots, h_\mu)$$

for some $h_j = h_j(\mathbf{u}, t) \in t^{-1}\mathbb{C}[t^{-1}] \otimes_{\mathbb{C}} R$, $j = 1, 2, \cdots, \mu$. Equivalently, we have the system of equations:

$$(\vec{g}_{(0)}, \vec{g}^{(1)}, \cdots, \vec{g}^{(a)}) \begin{pmatrix} A_{(0)} & A^{(1)} & \cdots & A^{(a)} \\ A^{(-1)} & A_{(0)} & \cdots & A^{(a-1)} \\ \cdots & \cdots & \cdots & \cdots \\ A^{(-a)} & A^{(-a+1)} & \cdots & A_{(0)} \end{pmatrix} = \mathbf{e}$$

Since

$$1 + \Psi = \begin{pmatrix} A_{(0)} & A^{(1)} & \cdots & A^{(a)} \\ A^{(-1)} & A_{(0)} & \cdots & A^{(a-1)} \\ \cdots & \cdots & \cdots & \cdots \\ A^{(-a)} & A^{(-a+1)} & \cdots & A_{(0)} \end{pmatrix}$$

and Ψ is a matrix with entries in $\mathfrak{m}/\mathfrak{m}^{N+1}$, we have

$$(1+\Psi)^{-1} = \sum_{k=0}^{N} (-\Psi)^k.$$

Thus the statement follows. □

6. Examples

In this section, we provide some concrete examples of primitive forms. Whenever a good opposite filtration $\mathcal{L}$ is specified below, we always consider the unique primitive form associated to $\mathcal{L}$ together with the primitive element 1.

6.1. **ADE-singularities.** Let f be a weighted homogeneous polynomial of type ADE-singularity. That is, $s(f) < 1$. It is easy to see that $D = 0$, $Y = \{0\}$. As a consequence, $\mathcal{L}_{\text{can}}$ is the only good opposite filtration that is given by Y, which consists elements of negative weight degrees. Here we only refer to the degree of the variables z_i, t. Therefore $\deg\left((F - F_0)/t\right) < 0$. It follows that

$$e^{(F-F_0)/t} \in 1 + \mathcal{L}_{\text{can}} \otimes_{\mathbb{C}} R$$

The next proposition [46] follows from Corollary 4.16.

Proposition 6.1. *The element $1 \in \Gamma(S, \mathcal{H}^{F,\Omega}_{(0)})$ is the unique primitive form up to a nonzero scalar.*

6.2. **Simple elliptic singularity.** Let $f = \frac{1}{3}\left(z_1^3 + z_2^3 + z_3^3\right)$. Then f is a weighted homogeneous polynomial with $s(f) = 1$. It is called a simple elliptic singularity of type $E_6^{(1,1)}$. Take a (ordered) monomial basis by

$$\{\phi_1, \cdots, \phi_8\} = \{1, z_1, z_2, z_3, z_1z_2, z_2z_3, z_3z_1, z_1z_2z_3\},$$

and consider the universal unfolding

$$F = \frac{1}{3}z_1^3 + \frac{1}{3}z_2^3 + \frac{1}{3}z_3^3 + \sigma\phi_8 + \sum_{i=1}^{7} u_i\phi_i,$$

where $(u_1, \cdots, u_7, \sigma)$ are the local coordinates of S. We take $\Omega_{Z/S} = dz_1 \wedge dz_2 \wedge dz_3$.

By definition, $D = 1$, and hence $Y \cong \mathbb{C}$. There is only one free parameter in $\mathbf{c} = (c_{ij})$, say c. We have

$$\mathcal{L}(c) = t^{-1}B(c)[t^{-1}] \quad \text{where} \quad B(c) = \text{Span}_{\mathbb{C}}\{\phi_1, \cdots, \phi_7, \phi_8 + ct\} \subset \mathcal{H}^{f,\Omega}_{(0)}.$$

Every u_j in F belongs to the maximal ideal of R. By Corollary 4.16, the associated primitive form $\zeta_+(R) \in \mathcal{H}^{F,\Omega}(R)$ is determined by the relation

$$e^{(F-f)/t}\zeta_+(R) \in 1 + \mathcal{L}(c) \otimes_{\mathbb{C}} R.$$

Lemma 6.2. *As elements in $\mathcal{H}^{f,\Omega}$, the class $e^{\sigma\phi_8/t}$ is well-defined ($\phi_8 = z_1z_2z_3$), and it is given by*

$$e^{\sigma\phi_8/t} = g(\sigma) + t^{-1}h(\sigma)\phi_8,$$

where $g(\sigma)$ and $h(\sigma)$ are respectively given by

$$g(\sigma) = 1 + \sum_{r=1}^{\infty} \frac{(-1)^r\sigma^{3r}\prod_{j=1}^{r}(3j-2)^3}{(3r)!}, \quad h(\sigma) = \sigma + \sum_{r=1}^{\infty} \frac{(-1)^r\sigma^{3r+1}\prod_{j=1}^{r}(3j-1)^3}{(3r+1)!}.$$

Furthermore, $g(\sigma), h(\sigma)$ are the fundamental solutions to the following Picard-Fuchs equation:

$$\big((1+\sigma^3)\partial_\sigma^2 + 3\sigma^2\partial_\sigma + \sigma\big)\nu(\sigma) = 0. \tag{6.1}$$

Proof. Note $Q_f(z_2^2z_3^2\partial_1) = (\partial_{z_1}f)z_2^2z_3^2 = z_1^2z_2^2z_3^2 = \phi_8^2$. That is, $[\phi_8^2] = [0]$ in $\mathcal{H}^{f,\Omega}$. Similarly, for $k \geq 3$, it follows from direct calculations that

$$[\phi_8^k] = [z_1^{k-2}z_2^kz_3^k\partial_{z_1}f] = [-t(k-2)z_1^{k-3}z_2^kz_3^k] = \cdots = [-t^3(k-2)^3\phi_8^{k-3}]$$

Hence, in $\mathcal{H}^{f,\Omega}$ we have

$$e^{\sigma\phi_8/t} = \sum_{k=0}^{\infty}\frac{\sigma^k\phi_8^k}{k!t^k} = \sum_{r=0}^{\infty}\frac{\sigma^{3r}\phi_8^{3r}}{(3r)!t^{3r}} + \sum_{r=0}^{\infty}\frac{\sigma^{3r+1}\phi_8^{3r+1}}{(3r+1)!t^{3r+1}} = g(\sigma) + t^{-1}h(\sigma)\phi_8.$$

It is easy to check that g, h satisfy the differential equation (6.1). Since g, h are linearly independent, they are the fundamental solutions. In particular, they are convergent. □

As a consequence, in $\mathcal{H}^{f,\Omega}$, the class

$$e^{(F-F_0)/t} - (g(\sigma) + t^{-1}h(\sigma)\phi_8) = e^{\frac{\sigma\phi_8}{t}}(e^{(F-F_0-\sigma\phi_8)/t} - 1)$$

consists of elements of negative degree (viewed as elements in $\mathcal{H}^{f,\Omega}$), and hence lies in $\mathcal{L}(c)\otimes_{\mathbb{C}} R$. Therefore $e^{(F-F_0)/t} - (g(\sigma) - c\cdot h(\sigma)) \in \mathcal{L}(c)\otimes_{\mathbb{C}} R$. Then for $\zeta_+(R) := \frac{1}{g(\sigma)-c\cdot h(\sigma)} \in \mathcal{H}^{F,\Omega}_{(0)}\otimes_{\mathcal{O}_S} R$, we have

$$e^{(F-f)/t}\zeta_+(R) \in 1 + \mathcal{L}(c)\otimes_{\mathbb{C}} R.$$

It follows that

$$\zeta_+(R) = \frac{1}{g(\sigma) - c\cdot h(\sigma)}$$

is a primitive form by Corollary 4.16.

The above hypergeometric equation (6.1) is precisely the Picard-Fuchs equation for period integrals on elliptic curves. Consider the elliptic curve $E_\sigma \subset \mathbb{P}^2$ defined by the equation $f + \sigma z_1z_2z_3 = 0$, with holomorphic top form

$$\Omega_{E_\sigma} = \mathrm{Res}\,\frac{z_1dz_2dz_3 - z_2dz_1dz_3 + z_3dz_1dz_2}{f + \sigma z_1z_2z_3}$$

where $\mathrm{Res} : \Omega^2_{\mathbb{P}^2}(E_\sigma) \to \Omega^1_{E_\sigma}$ denotes the residue map. Let Γ be a topological 1-cycle in E_σ, then the period integral $\int_\Gamma \Omega_{E_\sigma}$ satisfies the Picard-Fuchs equation (6.1) In particular, an arbitrarily

fixed basis $\{\Gamma_1, \Gamma_2\}$ of $H_1(E_\sigma, \mathbb{Z})$ gives gives the fundamental solutions $\int_{\Gamma_1} \Omega_{E_\sigma}, \int_{\Gamma_2} \Omega_{E_\sigma}$ to the Picard-Fuchs equation. Hence, we reprove the next proposition [46].

Proposition 6.3. *$\zeta \in \Gamma(S, \mathcal{H}^{F,\Omega}_{(0)})$ is a primitive form if and only if*

$$\zeta = \frac{1}{a\int_{\Gamma_1} \Omega_{E_\sigma} + b\int_{\Gamma_2} \Omega_{E_\sigma}}$$

for some $(a, b) \in \mathbb{C}^* \times \mathbb{C}$.

6.3. **E_{12}-singularity.** The exceptional singularity of type E_{12} is given by the weighted homogeneous polynomial:

$$f(x, y) = x^3 + y^7.$$

with $s(f) = \frac{22}{21}$. Take the (ordered) set

$$\{\phi_1, \cdots, \phi_{12}\} := \{1, y, y^2, x, y^3, xy, y^4, xy^2, y^5, xy^3, xy^4, xy^5\} \subset \mathbb{C}[x, y],$$

which represent a basis of $\mathrm{Jac}(f)$. Consider the universal unfolding

$$F(x, y, \mathbf{u}) = x^3 + y^7 + \sum_{i=1}^{12} u_i \phi_i$$

and take $\Omega_{Z/S} = dx \wedge dy$. In this case, $Y = \{\mathbf{0}\}$ by Theorem 5.14. That is, there is a unique primitive form up to a nonzero scalar. Hence, $\Phi_j = \phi_j$ for all $1 \leq j \leq 12$. As an example of Theorem 5.16, we compute $\zeta_+(R)$ with $R = \mathbb{C}[x, y]/\mathfrak{m}^{11}$ where $\mathfrak{m} = (x, y)$ is the maximal ideal at the origin.

By direct calculations (with a computer), we have

$$\begin{aligned}
\zeta_+(R) = 1 &+ \frac{4}{3\cdot 7^2}u_{11}u_{12}^2 - \frac{64}{3\cdot 7^4}u_{11}^2u_{12}^4 - \frac{76}{3^2\cdot 7^4}u_{10}u_{12}^5 + \frac{937}{3^3\cdot 7^5}u_9u_{12}^6 + \frac{218072}{3^4\cdot 5\cdot 7^6}u_{11}^3u_{12}^6 \\
&+ \frac{1272169}{3^4\cdot 5\cdot 7^7}u_{10}u_{11}u_{12}^7 + \frac{28751}{3^4\cdot 7^7}u_8u_{12}^8 - \frac{1212158}{3^4\cdot 7^8}u_9u_{11}u_{12}^8 - \frac{38380}{3^3\cdot 7^8}u_7u_{12}^9 \\
&+ \Big(\frac{1}{7^2}u_{12}^3 - \frac{101}{5\cdot 7^4}u_{11}u_{12}^5 + \frac{1588303}{3^4\cdot 5\cdot 7^7}u_{11}^2u_{12}^7 + \frac{378083}{3^4\cdot 5\cdot 7^7}u_{10}u_{12}^8 - \frac{108144}{3\cdot 7^8}u_9u_{12}^9\Big)x \\
&+ \Big(\frac{1447}{3^3\cdot 7^6}u_{12}^7 - \frac{71290}{3^3\cdot 7^8}u_{11}u_{12}^9\Big)y - \frac{45434}{3^4\cdot 7^8}u_{12}^{10}xy \\
&- \Big(\frac{53}{3^2\cdot 7^4}u_{12}^6 - \frac{46244}{3^3\cdot 7^7}u_{11}u_{12}^8\Big)x^2 + \frac{22054}{3^4\cdot 7^7}u_{12}^9x^3
\end{aligned}$$

Remark 6.4. *For exceptional unimodular singularities, the first few orders of the primitive forms are sufficient to establish the LG-LG mirror symmetry between their Saito-Givental theory (LG B-model) and their mirror FJRW theory (LG A-model)* [32].

6.4. **Mirror of $\mathbb{P}^1$.** Consider the mirror Landau-Ginzburg model (X, f) of the complex projective line $\mathbb{P}^1$. Here $X = \mathbb{C}^*$ and the superpotential $f : X \to \mathbb{C}$ is given by $f(z) = z + \frac{q}{z}$, where q is a nonzero constant in $\mathbb{C}^*$. The quantum cohomology ring $QH^*(\mathbb{P}^1, \mathbb{C})$ of $\mathbb{P}^1$ is isomorphic to the Jacobian ring $\mathrm{Jac}(f) = \mathbb{C}[z, z^{-1}]/\langle z\frac{\partial f}{\partial z}\rangle$ of f. We will consider the Milnor fibration (Z, S, p, F), where $Z = \mathbb{C}^* \times \mathbb{C}^2$, $S = \mathbb{C}^2$ and the universal unfolding is given by

$$F(z, u_0, u_1) = u_0 + z + \frac{q\exp(u_1)}{z}.$$

Consider primitive forms with respect to the choice $\Omega_{Z/S} := \dfrac{dz}{z}$. Then Theorem 3.7 of [54] (where Z and S are chosen smaller) can be restated as follows. Here we provide an alternative proof, by using Theorem 4.15.

Remark 6.5. *The pair $((\mathbb{C}^*)^n, f)$ with $f(\mathbf{z}) := z_1 + \cdots + z_n + \frac{q}{z_1 \cdots z_n}$ is a mirror Landau-Ginzburg model of $\mathbb{P}^n$. In this case, it has been shown in* [3] *that the identity section* 1 *is a primitive form with respect to the relative holomorphic volume form $\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$.*

Proposition 6.6. *The identity section $1 \in \Gamma(S, \mathcal{H}^{F,\Omega}_{(0)})$ is a primitive form.*

Proof. To prove the statement, we define a subspace $\mathcal{L}$ of $\mathcal{H}^{f,\Omega}$, given by

$$\mathcal{L} := t^{-1}B[t^{-1}] \quad \text{where} \quad B := \mathrm{Span}_{\mathbb{C}}\left\{1, \frac{q}{z}\right\} \subset \mathcal{H}^{f,\Omega}.$$

We first show that $\mathcal{L}$ is an opposite filtration. Note that $\{1, \frac{q}{z}\}$ represent a basis of $\mathrm{Jac}(f)$. Clearly, we have $t^{-1}\mathcal{L} \subset \mathcal{L}$ and $\mathcal{H}^{f,\Omega} = \mathcal{H}^{f,\Omega}_{(0)} \oplus \mathcal{L}$. Assume the next claim on the higher residue pairings on $B \subset \mathcal{H}^{f,\Omega}$ first.

Claim: $\mathcal{K}^f_\Omega(1,1) = 0, \quad \mathcal{K}^f_\Omega(\frac{q}{z}, \frac{q}{z}) = 0, \quad \mathcal{K}^f_\Omega(1, \frac{q}{z}) = -1.$

Then It follows that $\mathcal{K}^f_\Omega(B, B) = \mathbb{C}$, and hence $\mathcal{L}$ is an opposite filtration by definition.

Since $Q_f(z\partial_z) = z\partial_z f = z - \frac{q}{z}$, we have $[z] = [\frac{q}{z}]$ in $\mathcal{H}^{f,\Omega}_{(0)}$. Hence,

$$\nabla^\Omega_{t\partial_t} 1 - 0 \cdot 1 = \nabla^\Omega_{t\partial_t} 1 = [-\frac{f}{t}] = -[\frac{z + \frac{q}{z}}{t}] = -2[\frac{q}{z}]t^{-1} \in \mathcal{L}.$$

Since $Q_f(\partial_z) = \partial_z f - \frac{t}{z} = 1 - \frac{q}{z^2} - \frac{t}{z}$, we have $[-\frac{q}{z^2}] = [\frac{t}{z} - 1]$ in $\mathcal{H}^{f,\Omega}_{(0)}$. Hence,

$$\nabla^\Omega_{t\partial_t}[\frac{q}{z}] = [-\frac{z + \frac{q}{z}}{t} \cdot \frac{q}{z}] = [-\frac{q}{t} + \frac{1}{t}(\frac{qt}{z} - q)] = [-\frac{2q}{t} + \frac{q}{z}] \in t^{-1}B + B.$$

It follows that $\nabla^\Omega_{t\partial_t}\mathcal{L} \subset \mathcal{L}$. Hence, the opposite filtration $\mathcal{L}$ is good and 1 is a primitive element with respect to $\mathcal{L}$.

Note $\frac{F - F_0}{t} = t^{-1}\big(u_0 + \frac{q}{z}(e^{u_1} - 1)\big) \in \mathcal{L} \otimes_{\mathbb{C}} R$. For $k \geq 2$, in $\mathcal{H}^{f,\Omega}_{(0)}$, we have

$$\frac{q}{z^k} = \frac{1}{z^{k-2}} - \frac{1}{z^{k-2}}\partial_z f = \frac{1}{z^{k-2}} + t \cdot z\partial_z \frac{1}{z^{k-2} \cdot z} = \frac{1}{z^{k-2}} - (k-1)t\frac{1}{z^{k-1}}.$$

It follows that $1-e^{\frac{F-F_0}{t}} \in \mathcal{L}\otimes_{\mathbb{C}} R$. Hence,

$$e^{\frac{F_0-F}{t}} = 1+(e^{\frac{F_0-F}{t}}-1) \in (\mathcal{H}^{F,\Omega}_{(0)}\otimes_{\mathcal{O}_S} R)\oplus \big(e^{\frac{f-F}{t}}\mathcal{L}\otimes_{\mathbb{C}} R\big).$$

Therefore, the identity section 1 is a primitive form by Theorem 4.15.

It remains to prove the above claim as follows.

We simply denote $f' := \partial_z f$. It is easy to compute that

$$V_f = \frac{1}{f'}\partial_z\wedge, \quad \text{and} \quad \big[Q, V_f\big]\, g = t\frac{1}{f'}z\frac{\partial}{\partial z}(\frac{g}{z}) \quad \text{for any } g\in\Gamma(X,\mathcal{O}_X).$$

Denote $B^{\pm}_{\varepsilon} := \{z\in X \mid |z-(\pm\sqrt{q})|=\varepsilon\}$ and $C^{\pm}_{\varepsilon} := \partial B^{\pm}_{\varepsilon}$, where $0<\varepsilon<<1$. Take a smooth cut-off function ρ on X with $\rho|_{B^{\pm}_{\varepsilon}}\equiv 1$ and $\rho|_{X\setminus(B^{+}_{2\varepsilon}\cup B^{-}_{2\varepsilon})}\equiv 0$. Similar to Example 2.15, we have

$$\begin{aligned}
\mathcal{K}^f_\Omega(\frac{1}{z^i},\frac{1}{z^j}) &= -\sum_{r\geq 0} t^r \oint_{C^+_\varepsilon\cup C^-_\varepsilon} \frac{1}{z^i}\frac{dz}{z^2\cdot f'}\left(-z\frac{\partial}{\partial z}\frac{1}{zf'}\right)^r\left(\frac{1}{z^j}\right)\\
&= \sum_{r\geq 0} t^r \operatorname{Res}_{z\in\{0,\infty\}} \frac{1}{z^i}\frac{dz}{z^2\cdot f'}\left(-z\frac{\partial}{\partial z}\frac{1}{zf'}\right)^r\left(\frac{1}{z^j}\right)\\
&= \sum_{r\geq 0} (-t)^r \operatorname{Res}_{z\in\{0,\infty\}} \frac{dz}{z}\frac{1}{z^i}\frac{1}{z-q/z}\left(z\frac{\partial}{\partial z}\frac{1}{z-q/z}\right)^r\frac{1}{z^j}
\end{aligned}$$

where we use the observation that poles of the integrant can only occur at $\pm\sqrt{q}$, 0 or ∞. By direct calculations, the leading term (up to a constant) of the Laurant series expansion of

$$A_r(z) := \frac{1}{z^i}\frac{1}{z-q/z}\left(z\frac{\partial}{\partial z}\frac{1}{z-q/z}\right)^r\frac{1}{z^j}.$$

around the origin $z=0$ is given by $z^{r+1-i-j}$. The leading term (up to a constant) of the Laurant series expansion of $A_r(z)$ around the origin $z=\infty$ is given by $(\frac{1}{z})^{k+1+i+j}$. It follows easily that

$$\mathcal{K}^f_\Omega(1,1)=0, \quad \mathcal{K}^f_\Omega\left(1,\frac{q}{z}\right) = q\operatorname{Res}_{z=0}\frac{dz}{z(z^2-q)} = -1,$$

$$\text{and} \quad \mathcal{K}^f_\Omega(\frac{1}{z},\frac{1}{z}) = (-t)\cdot\operatorname{Res}_{z=0}\frac{dz}{z(z^2-q)}z\frac{\partial}{\partial z}(\frac{1}{z^2-q}) = 0.$$

□

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