A PLÜCKER COORDINATE MIRROR FOR PARTIAL FLAG VARIETIES AND QUANTUM SCHUBERT CALCULUS

CHANGZHENG LI, KONSTANZE RIETSCH, MINGZHI YANG, AND CHI ZHANG

Abstract. We construct a Plücker coordinate superpotential $\mathcal{F}_-$ that is mirror to a partial flag variety $\mathbb{F}\ell(n_\bullet)$. Its Jacobi ring recovers the small quantum cohomology of $\mathbb{F}\ell(n_\bullet)$, and we prove a folklore conjecture in mirror symmetry. Namely, we show that the eigenvalues for the action of the first Chern class $c_1(\mathbb{F}\ell(n_\bullet))$ on quantum cohomology are equal to the critical values of $\mathcal{F}_-$. We achieve this by proving new identities in quantum Schubert calculus that are inspired by our formula for $\mathcal{F}_-$ and the mirror symmetry conjecture.

Contents

1. Introduction

Mirror symmetry is a fascinating phenomenon arising in string theory: two apparently completely different objects on A-model and B-model give rise to equivalent physics. Mathematical descriptions of mirror symmetry, in terms of equivalence of mathematical structures, were first made for pairs of Calabi-Yau manifolds in early 1990s (see e.g. [HKetal]). The (closed string) mirror symmetry was extended to Fano manifolds X on the topological A-model soon after by Givental [Giv95, Giv98] and Eguchi-Hori-Xiong [EHX97]. In this case, the topological B-model is given by a Landau-Ginzburg model $(\check{X}, W)$, consisting of a non-compact Kähler manifold $\check{X}$ and a holomorphic function $W : \check{X} \to \mathbb{C}$ called the superpotential. Mirror symmetry predicts equivalences between both sides on various levels. For instance on one level, the (small) quantum cohomology ring $QH^*(X)$ should be isomorphic to the Jacobi ring $Jac(W)$ of W.

Studying mirror symmetry for X a priori requires a good construction of the mirror superpotential W. However, this is only known for certain Fano manifolds, with toric Fano manifolds and complete intersections inside toric manifolds being typical examples, following work of Givental [Giv95, Giv98] and Hori-Vafa [HV00]. In this article, we will focus on the case when $X = \mathbb{F}\ell(n_\bullet)$ is a partial flag variety parameterizing flags of quotient vector subspaces of $\mathbb{C}^n$. Special cases include complex Grassmannians $Gr(k, n)$ and complete flag variety $\mathbb{F}\ell_n$. Candidate Landau-Ginzburg models for $Gr(k, n)$ and $\mathbb{F}\ell_n$ were constructed by Eguchi-Hori-Xiong [EHX97] and Givental [Giv97] respectively. They were later generalized to $\mathbb{F}\ell(n_\bullet)$ by Batyrev-Ciocan-Fontanine-Kim-van Straten [BCFKS00]. See also [NNU10] for a construction using holomorphic disk counts. Here different approaches turned out to result in identical versions of the superpotential, namely arriving at a particular Laurent polynomial W_{tor} defined on a complex torus of dimension $\dim \mathbb{F}\ell(n_\bullet)$. It turned out that there is a toric degeneration of $\mathbb{F}\ell(n_\bullet)$ with the central fiber a singular toric variety X_0, and the superpotential W_{tor} coincides with the superpotential mirror to X_0 as constructed by Givental and Hori-Vafa. This superpotential, however, does not contain enough information to be an honest mirror superpotential for $\mathbb{F}\ell(n_\bullet)$ sometimes. For instance for $Gr(2, 4)$ the correct mirror superpotential would be expected to have $6 = \dim H^*(Gr(2, 4))$ critical points, but W_{tor} only has 4.

In [Rie08], the second-named author wrote down a Lie theoretical superpotential, namely a function $\mathcal{F}_{\text{Lie}} : Z_P \to \mathbb{C}$ defined on a subvariety Z_P of B_-. Here G is a connected complex reductive Lie group, and P is a parabolic subgroup of G containing a Borel subgroup B_-. This function had appeared separately earlier in a different context, as part of a theory of geometric crystals [BK07]. It is shown in [Rie08] that the fiberwise critical locus of $\mathcal{F}_{\text{Lie}}$ is isomorphic to (an open dense part of) the so-called Peterson variety stratum Y_P in the flag variety G/B_-. This relates the superpotential $\mathcal{F}_{\text{Lie}}$ to quantum cohomology via the remarkable isomorphism of Dale Peterson's, described in his unpublished lecture notes [Peterson], between $\mathbb{C}[Y_P]$ and the small quantum cohomology ring $QH^*(G^\vee/P^\vee)$ of the Langlands dual flag variety. A proof of Peterson's isomorphism for the type A case, that is for $G^\vee/P^\vee = \mathbb{F}\ell(n_\bullet)$, was given in [Rie03]. Some other cases were covered in [Cheo09, LS10], and the general case was proved in a recent preprint [Chow22]. The combination of both isomorphisms leads to mirror symmetry for flag varieties on the level of small quantum cohomology. Namely, the ring $QH^*(G^\vee/P^\vee)$, with inverse quantum parameters adjoined, is isomorphic to the (fiberwise) Jacobi ring of $\mathcal{F}_{\text{Lie}}$.

This is not the end of the story, but only the end of the beginning. The function $\mathcal{F}_{\text{Lie}}$ is defined quite indirectly, and it is desirable to find a compact expression in terms of coordinates on the mirror space Z_P. In the special case of $\mathbb{F}\ell(n_\bullet) = Gr(n-k, n)$, a natural isomorphic interpretation of the mirror space was given in [MR20]. There, Z_P was identified with a trivial family over $\mathbb{C}^*$ with fiber a particular open log Calabi-Yau subvariety in the Langlands dual Grassmannian $Gr(k, n)$. Moreover, [MR20] gave a very compact and clean expression for $\mathcal{F}_{\text{Lie}}$ using the Plücker coordinates of $Gr(k, n)$. This also led to an improved mirror symmetry result on the higher level of D-modules.

One generalization of this $Gr(n-k,n)$ construction is to cominuscule Grassmannians of other types. The fiber of the mirror space is then inside the Langlands dual minuscule Grassmannian, which has (generalized) Plücker coordinates, due to its embedding into the projective space of a minuscule representation. Corresponding coordinate presentation of $\mathcal{F}_{\mathrm{Lie}}$ have been individually obtained for quadrics [PRW16], Lagrangian Grassmannians [PR13], the Cayley plane and the Freudenthal variety [SW23].

The generalization of $Gr(n-k,n)$ of interest to us here is the partial flag variety $X = \mathbb{F}\ell(n_\bullet)$. As the first main result of this paper, we provide a Plücker coordinate formula version $\mathcal{F}_-$ of the superpotential $\mathcal{F}_{\mathrm{Lie}}$ for this case. To construct the domain we consider the Langlands dual partial flag variety $F\ell_{n_\bullet} = F\ell_{n_1,\cdots,n_r;n} = P\backslash G$ that parameterizes flags of vector subspaces V_{n_j} in the dual vector space of $\mathbb{C}^n$. Let $(P\backslash G)^\circ$ denote the complement of the Knutson-Lam-Speyer anti-canonical divisor $-K_{F\ell_{n_\bullet}}$ [KLS14], which consists of $(n-1+r)$ irreducible components (see Proposition 3.12).

Theorem 1.1. *There is an isomorphism*

$$\psi_- : Z_P \longrightarrow (P\backslash G)^\circ \times \prod_{i\in I^P} \mathbb{C}_q^*, \qquad \text{where } I^P := \{n_1, \cdots, n_r\}.$$

The superpotential $\mathcal{F}_- := \mathcal{F}_{\mathrm{Lie}} \circ \psi_-^{-1} : (P\backslash G)^\circ \times \prod_{i\in I^P} \mathbb{C}_q^* \to \mathbb{C}$ *is of the form*

$$\mathcal{F}_-(Pz, \mathbf{q}) = \sum_{i\in I^P} q_i v_{i,i+1} + \sum_{i=1}^{n-1} u_{i,i+1},$$

consisting of $(n-1+r)$ *summands, and these satisfy*

(1) *the* $v_{i,i+1}$ *are all of the form* $\frac{p_{J'}}{p_J}$ *for some Plücker coordinates;*
(2) *the* $u_{i,i+1}$ *are of the form* $\frac{p_{J'}}{p_J}$ *if* $i \in I^P$, *or if* $1 \le i \le n_1$ *or* $n_r \le i \le n-1$. *Otherwise, if* $n_j < i < n_{j+1}$ *for some* $j \in \{1, \cdots, r-1\}$, *then* $u_{i,i+1}$ *is of the form* $\frac{f_1}{f_2}$ *with each* f_i *a quadratic polynomial in the Plücker coordinates;*
(3) *all* $v_{i,i+1}$ *and* $u_{i,i+1}$ *have pole of order 1 along a (unique) irreducible component of* $-K_{F\ell_{n_\bullet}}$.

In fact, the denominators in the summands of $\mathcal{F}_-$ are precisely the defining equations of the irreducible components of $-K_{F\ell_{n_\bullet}}$ [LSZ23].

The isomorphism Ψ_- will be constructed explicitly in Definition 3.6. Explicit expressions for $v_{i,i+1}$ and $u_{i,i+1}$ will be given in **Theorem 3.18**. Here we provide an example to give a first impression.

Example 1.2. *For* $F\ell_{n_\bullet} = F\ell_{2,4;7} \hookrightarrow \mathbb{P}(\bigwedge^2 \mathbb{C}^7) \times \mathbb{P}(\bigwedge^4 \mathbb{C}^7)$, *we have*

$$\mathcal{F}_- = q_2\frac{p_{46}}{p_{67}} + q_4\frac{p_{1467}}{p_{4567}} + \frac{p_{27}}{p_{17}} + \frac{p_{24}p_{1567} - p_{14}p_{2567} + p_{12}p_{4567}}{p_{23}p_{1567} - p_{13}p_{2567} + p_{12}p_{3567}} + \frac{p_{2346}}{p_{2345}} + \frac{p_{3457}}{p_{3456}} + \frac{p_{13}}{p_{12}} + \frac{p_{1235}}{p_{1234}}.$$

Remark 1.3. *A straightforward generalization of the superpotential in* [MR20] *leads to another superpotential* $\mathcal{F}_+$ *defined on* $(P_+\backslash G)^\circ \times \prod_{i\in I^P} \mathbb{C}_q^*$. *The Plücker coordinate expression of* $\mathcal{F}_+$, *however, appears to be complicated and does not have the similar good properties (especially property (3) above). We refer to Examples 3.2 and 3.19 for a comparison of* $\mathcal{F}_+$ *and* $\mathcal{F}_-$ *in the case of the complete flag variety* $F\ell_3$.

Remark 1.4. *In* [GS18], *Gu and Sharpe proposed a construction of 'non-abelian' mirrors, examples of which included* $\mathbb{F}\ell(n_\bullet)$. *Their approach is closely related to* [HV00] *but involves more variables than the dimension of* $\mathbb{F}\ell(n_\bullet)$ *(see also* [GK20]*). Their mirror diverges already* *in the Grassmannian case from the mirror constructions* [EHX97, Rie08, MR20] *that are related to ours here, see* [GS18, Section 4.9].

Another mirror construction inspired by viewing flag varieties as non-abelian GIT quotients was given by Kalashnikov [Kala22]. *Namely, Kalashnikov proposed a generalization of the superpotential from* [MR20] *for* $Gr(n-k,n)$ *to partial flag varieties* $\mathbb{F}\ell(n_\bullet)$ *in the form of a rational function on a product of Grassmannians, expressed explicitly in terms of Plücker coordinates, which recovers the aforementioned Laurent polynomial* $W_{\rm tor}$ *in a cluster chart. Kalashnikov also described a relation (on the level of critical points) between her superpotential and Gu-Sharpe's superpotential in a special case.*

To compare the Kalashnikov formula with our $\mathcal{F}_-$*, consider the partial flag variety* $\mathbb{F}\ell(n_\bullet) = \mathbb{F}\ell(4;2,1)$. *Kalashnikov's superpotential is a rational function on* $Gr(2,4)\times Gr(1,2)\times(\mathbb{C}^*)^2$ *described in terms of Plücker coordinates* $[p_{ij};\hat{p}_k]$ *by*

$$W_{\rm Kal} = \frac{p_{13}}{p_{12}} + \frac{p_{24}+q_2}{p_{23}} + \frac{p_{24}}{p_{14}} + \frac{q_2 p_{13}\hat{p}_2}{p_{34}} + \frac{\hat{p}_2}{\hat{p}_1} + \frac{q_1}{\hat{p}_2}.$$

Our superpotential $\mathcal{F}_-$ *is a rational function on* $F\ell_{1,2;4}\times(\mathbb{C}^*)^2$*, given in terms of* $[p_k;p_{ij}]$ *by*

$$\mathcal{F}_- = q_1\frac{p_3}{p_4} + q_2\frac{p_{14}}{p_{34}} + \frac{p_2}{p_1} + \frac{p_{13}}{p_{12}} + \frac{p_{24}}{p_{23}}.$$

Apart from the formula looking more complicated, the superpotential $W_{\rm Kal}$ *turns out not to have the full set of critical points in this example. Namely, while* $\mathcal{F}_-$ *has* 12 *critical points along its* $q_1=q_2=1$ *fiber, in agreement with* $\dim H^*(\mathbb{F}\ell(4;2,1)) = 12$*, one of these critical points, the one with critical value* -3*, is missing for* $W_{\rm Kal}|_{\mathbf{q}=(1,1)}$.

Let us now recall that, on the A-side, the (small) quantum cohomology ring $QH^*(X) = (H^*(X,\mathbb{C})\otimes\mathbb{C}[\mathbf{q}],\cdot)$ of the Fano manifold X is a deformation of the classical cohomology ring $H^*(X,\mathbb{C})$ by incorporating genus zero, 3-point Gromov-Witten invariants. The quantum multiplication by the first Chern class of X induces a linear operator

$$\hat{c}_1(\mathbf{q}) : QH^*(X) \longrightarrow QH^*(X);\ \beta\mapsto c_1(X)\cdot\beta$$

depending on the values of the deformation parameters $\mathbf{q}=(q_i)_i$, also called quantum parameters. Here we treat the q_i as nonzero complex numbers, so that $QH^*(X) = H^*(X)$ as vector spaces. On the B-side, we consider the superpotential $W = W_{\mathbf{q}}$ with the quantum parameters fixed correspondingly. Now let us state a celebrated folklore conjecture in mirror symmetry.

Conjecture 1.5. *The eigenvalues of the first Chern class operator* $\hat{c}_1(\mathbf{q})$ *coincide with the critical values of the mirror superpotential* $W_{\mathbf{q}}$.

There has been very little progress on this conjecture in the past two decades. The case of toric Fano manifolds was first proved by Auroux [Aur07], which was also known to Kontsevich and Seidel. Recently, Yuan [Yuan21] proved that the critical values of the family Floer mirror Landau-Ginzburg superpotential are the eigenvalues of the first Chern class, under certain assumptions. The cases of complex Grassmannians and quadrics were proved implicitly in [MR20] and [Hu22] respectively.

As a central result of this paper, we prove a theorem that implies this conjecture for any partial flag variety $X=\mathbb{F}\ell(n_\bullet)$.

Let us write $\mathbf{q}=(q_{n_1},\cdots,q_{n_r})$ for the quantum parameters associated to $\mathbb{F}\ell(n_\bullet)$, and view them as coordinates on an algebraic torus that we denote by $\prod_{i\in I^P}\mathbb{C}^*_q$. Let us consider the (fiberwise) Jacobi ring,

$$Jac(\mathcal{F}_-) := \mathcal{O}\left((P\backslash G)^\circ\times\prod_{i\in I^P}\mathbb{C}^*_q\right)/(\partial_{(P\backslash G)^\circ}\mathcal{F}_-), \tag{1.1}$$

where we are taking partial derivatives of $\mathcal{F}_-$ in the $(P\backslash G)^\circ$ directions only. Using Theorem 1.1, and the isomorphism between fiberwise Jacobi ring of $\mathcal{F}_{\mathrm{Lie}}$ and quantum cohomology resulting from [Peterson, Rie03, Rie08], we obtain an isomorphism of rings

$$\Theta : Jac(\mathcal{F}_-) \xrightarrow{\sim} QH^*(X)[q_{n_1}^{-1}, \cdots, q_{n_r}^{-1}]. \tag{1.2}$$

See Section 4.2 for a more detailed description. We can now state our second main theorem.

Theorem 1.6. *For the class $[\mathcal{F}_-]$ of $\mathcal{F}_-$ in the Jacobi ring $Jac(\mathcal{F}_-)$, we have*

$$\Theta([\mathcal{F}_-]) = c_1(X),$$

where $c_1(X)$ is the first Chern class of $X = \mathbb{F}\ell(n_\bullet)$, as element of the small quantum cohomology ring.

The above theorem is stated again in an isomorphic form in **Theorem 4.14**, using a version $\mathcal{F}_R$ of the superpotential whose domain relates more directly to the Peterson variety. By interpreting the critical values of $\mathcal{F}_-$ as eigenvalues for the operator of multiplication by $[\mathcal{F}_-]$ on $Jac(\mathcal{F}_-)$, we obtain the following corollary.

Corollary 1.7. *Conjecture 1.5 holds for $X = \mathbb{F}\ell(n_\bullet)$ and the mirror superpotential $\mathcal{F}_-$.*

We note that isomorphically changing the domain of the superpotential does not affect the critical values. Therefore the same corollary holds for $\mathcal{F}_{\mathrm{Lie}}$, and $\mathcal{F}_R$. We also note that $\mathcal{F}_R$ and $\mathcal{F}_-$ look to be related by the chiral map in [GL22].

An exciting aspect of this part of our paper is the interaction between the Conjecture 1.5 in mirror symmetry and identities in quantum Schubert calculus. The quantum cohomology ring $QH^*(\mathbb{F}\ell(n_\bullet))$ has a $\mathbb{C}[\mathbf{q}]$-basis of Schubert classes σ_w, that is indexed by permutations in S_n with descents at most in n_j, for $j \in \{1, \cdots, r\}$. The study of the ring structure of $QH^*(\mathbb{F}\ell(n_\bullet))$ in terms of this basis, referred to as (type A) quantum Schubert calculus, is an area of great independent interest from the viewpoint of enumerative geometry. One of the most central problems is to find a manifestly positive formula for the Schubert structure constants in the quantum product of two Schubert classes. Another topic of interest is the study of identities among quantum products of Schubert classes. For example, the quantum Schubert polynomials [FGP97, C-Fon99] are expressions for general Schubert classes as polynomials in special Schubert classes. The quantum Giambelli formula [Bert97] for complex Grassmannians is another example. It turns out, that mirror symmetry also helps us find identities of this kind. Let us illustrate this from the perspective of the following natural question. Consider the isomorphism Θ from (1.2)

Question 1.8. *What are the preimages of the Schubert classes in $QH^*(\mathbb{F}\ell(n_\bullet))$ under Θ?*

Assuming the answer, one may expect to find relations in $QH^*(\mathbb{F}\ell(n_\bullet))$ simply by studying the mirror superpotential. Indeed, in the special case of $\mathbb{F}\ell(n_\bullet) = Gr(n-k, n)$, our $\mathcal{F}_-$ turns out to coincide with the superpotential $\mathcal{F}_+$ of [MR20] (see Example 3.20). Therefore, we have $\Theta^{-1}(\sigma_w) = [p_w]$, where p_w denotes the (suitably normalised) Plücker coordinate corresponding to the Grassmannian permutation w, as described in [MR20]. Each term in $\mathcal{F}_-$ turns out to reveal a quantum cohomology relation, see [MR20, Remark 6.2], recovering an instance of the known quantum Pieri-Chevalley formula in this case. For more general partial flag varieties the question above can be answered for certain Schubert classes using work of Peterson, see Section 4.1. This means that quantum Schubert calculus relations involving these classes can be viewed as relations in the Jacobi ring.

The approach of using the superpotential for understanding quantum cohomology was used also in [CK23]. Namely, one can consider partial derivatives of the superpotential which naturally represent the zero class in the Jacobi ring, and translate these into quantum cohomology relations via the mirror isomorphism. Following this approach, [CK23] obtained certain relations involving 'quantum hooks' via W_{Kal}.

Our final result is a set of quantum Schubert calculus identities related to our formula for $\mathcal{F}_-$. The proof of Theorem 1.6, turns out to involve showing each term in $\mathcal{F}_-$ corresponds to specific class in quantum cohomology. This requires certain relations to be proved in $QH^*(\mathbb{F}\ell(n_\bullet))$. For instance, the term $\frac{p_{24}p_{1567}-p_{14}p_{2567}+p_{12}p_{4567}}{p_{23}p_{1567}-p_{13}p_{2567}+p_{12}p_{3567}}$ in Example 1.2 relates to an identity

$$\sigma_{24}\cdot\sigma_{1567}-\sigma_{14}\cdot\sigma_{2567}+\sigma_{12}\cdot\sigma_{4567}=(\sigma_{23}\cdot\sigma_{1567}-\sigma_{13}\cdot\sigma_{2567}+\sigma_{12}\sigma_{3567})\cdot(\sigma_{13}+\sigma_{1235})$$

in quantum Schubert calculus. Note that we have simplified the notations, for instance by σ_{24} above we mean the Schubert class $\sigma_{2413567}$ indexed by the Grassmannian permutation 2413567 in one-line notation. The above identity is equivalent to the following simpler one by using quantum Chevalley-Monk formula [C-Fon99, Buch05, Miha07],

$$\sigma_{1526347}\cdot\sigma_{2314567}-\sigma_{2516347}\cdot\sigma_{1324567}+\sigma_{3516247}\cdot\sigma_{1234567}=0. \tag{1.3}$$

Our final theorem, that we prove concurrently with Theorem 1.6, gives new relations in quantum Schubert calculus of $QH^*(\mathbb{F}\ell(n_\bullet))$, including identity (1.3) as one example.

Theorem 1.9. *In $QH^*(\mathbb{F}\ell(n_\bullet))$, there are quantum relations of the form*

$$\sum_J(-1)^{|J|}\sigma_{w_J}\sigma_{[1,n_j+d]\setminus J}=0.$$

We will postpone the explanations of the relevant notations to Section 5, and will restate the identity fully in **Theorem 5.3**. The proof of the above theorem goes via the complete flag variety $\mathbb{F}\ell_n$ using Peterson's remarkable extension property (see Proposition 5.23). The proof of Theorem 1.6 is closely linked to to the above result.

Remarks for further directions. Closed string mirror symmetry in full generality at genus zero predicts an isomorphism on the level of Frobenius manifolds. The notion of a *Frobenius manifold* was first introduced by B. Dubrovin in 1990s [Dubr96], while the first construction of a Frobenius manifold could date back to K. Saito [Sai83] in early 1980s in the name of *flat structures* using his primitive form theory. Mirror symmetry predicts that the Frobenius manifold associated to the Gromov-Witten theory of a Fano manifold X (the *big* quantum cohomology ring of X) should be isomorphic to the Frobenius manifold of the mirror Landau-Ginzburg model $(\check{X}, W)$ associated to an appropriate Saito's primitive form. This was indirectly proved for complex Grassmannians in [KS08, CFKS08] by a reduction to the case of projective spaces [Bara00]. The case of quadrics was recently proved in [Hu22], where the verification of Conjecture 1.5 is an important step. We expect that our Theorem 1.6 will play an important role in studying mirror symmetry $\mathbb{F}\ell(n_\bullet)$ on such level as well.

For the mirror symmetry on the intermediate level of D-modules, the Plücker coordinate versions of the superpotential of $\mathcal{F}_{\mathrm{Lie}}$ play a very important role in proving an explicit injective morphism of D-modules for complex Grassmannians and quadrics [MR20, PRW16]. An implicit isomorphism of D-modules for minuscule Grassmannians was proved in [LT23]. A proof for an equivariant D-module isomorphism for general $G^\vee/P^\vee$ was recently given in [Chow23]. However, the isomorphism therein seems not explicit enough either. Moreover, verification of the Gauss-Manin connection along z-direction was missing, which is an indispensable piece in the mirror symmetry on the level of Frobenius manifolds. We believe that our Theorem 1.6 will be helpful towards getting a better understanding of the D-module mirror symmetry for $\mathbb{F}\ell(n_\bullet)$.

Conjecture 1.5 also appeared in the context of Kontsevich's *homological mirror symmetry* [Kont95], which is one main approach to (open string) mirror symmetry (in addition to another main approach by Strominger-Yau-Zaslow [SYZ]). For $G^\vee/P^\vee$, homological mirror symmetry was so far only proved for complex Grassmannians $G(n-k,n)$ with n prime [Cast20], beyond the projective space case covered earlier [Abou09]. Here it is important to understand the superpotential $\mathcal{F}_-$ in a Floer theoretical way, which has only been achieved for very few cases including $Gr(2,n)$ [HKL23]. It will be desirable to understand the superpotential more deeply.

Another closely related direction is about the Gamma conjecture I and its underlying conjecture $\mathcal{O}$ proposed by Galkin-Golyshev-Iritani [GGI16]. Here conjecture $\mathcal{O}$ concerns the eigenvalues of the aforementioned linear operator $\hat{c}_1|_{\mathbf{q}=\mathbf{1}}$. For flag varieties $G^\vee/P^\vee$, conjecture $\mathcal{O}$ has already been proved in [CL17], while the Gamma conjecture I was only known for very few cases including complex Grassmannians and quadrics. One main approach to Gamma conjecture I in [GI19] relies on a B-side analogy of conjecture $\mathcal{O}$ and a conifold condition. Our Theorem 4.14, together with [CL17], ensures the B-side analogy of conjecture $\mathcal{O}$. Therefore it will play an important role in the study of the Gamma conjecture I for $\mathbb{F}\ell(n_\bullet)$ via this approach.

Finally, we would propose a deeper interaction between mirror symmetry and quantum Schubert calculus for $G^\vee/P^\vee$. Indeed, for the type C case, some conjectural quantum relations in the quantum cohomology of a Lagrangian Grassmannian were given in [PR13, Conjecture 4.1], inspired by Conjecture 1.5. Even in type A, we would ask which quantum relations arise from the natural partial derivatives of the mirror superpotential $\mathcal{F}_-$ via the mirror isomorphism. We also note that some new quantum relations in $QH^*(\mathbb{F}\ell(n_\bullet))$ related with a cluster algebra structure were discovered in [HZ23]. It will be interesting to investigate whether these relations could also be revealed using cluster charts in the domain of $\mathcal{F}_-$.

The paper is organized as follows. In Section 2, we introduce the basic notions. In Section 3, we construct two superpotentials $\mathcal{F}_\pm$, and provide the Plücker coordinate expression of $\mathcal{F}_-$. In Section 4, we prove Theorem 1.6 by assuming Lemma 4.19 first. Section 5 is devoted to a proof of Lemma 4.19 in terms of equivalent identities on quantum product of Schubert classes. Finally, in the Appendix we provide a description of $\mathcal{F}_-$ using Young diagrams.

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2. Preliminaries

We review some background in Lie theory (see e. g. [Borel] for details).

2.1. **Notation.** Let $G = GL_n(\mathbb{C})$. Let B_+ and B_- denote the upper-triangular and lower-triangular Borel subgroups of G with unipotent radicals U_+ and U_-, respectively. Then $T = B_- \cap B_+$ is the maximal torus of diagonal matrices in G.

Let $\mathfrak{b}_-, \mathfrak{b}_+, \mathfrak{u}_-, \mathfrak{u}_+, \mathfrak{h}$ be the Lie algebras of B_-, B_+, U_-, U_+ and T respectively. Let $\Delta = \{\alpha_1, \cdots, \alpha_{n-1}\}$ be the standard base of simple roots, and R (resp. R_+) be the set of (positive) roots. That is, we have the Cartan decomposition

$$\mathfrak{gl}(n,\mathbb{C}) = \mathfrak{h} \oplus \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha \quad \text{with} \quad \mathfrak{g}_{\alpha_i+\alpha_{i+1}+\cdots+\alpha_{j-1}} = \mathbb{C}E_{ij} \quad \forall 1 \le i < j \le n,$$

where $E_{i,j}$ is the matrix with entry 1 in row i and column j and zeros elsewhere. We will view elements of Δ as lying in the character group of T, so that

$$\alpha_i : T \to \mathbb{C}^* := \mathbb{C}\backslash\{0\}; \quad t = \mathrm{diag}(t_1, \cdots, t_n) \mapsto \alpha_i(t) = \frac{t_i}{t_{i+1}}.$$

For any positive integers k, m with $k < m$, we denote the integral interval $[k,m] := \{k, k+1, \cdots, m\}$, and simply denote $[m] := [1,m]$. Set $I = [n-1]$. The Weyl group W of $\mathfrak{gl}(n,\mathbb{C})$ is generated by simple reflections $s_i = s_{\alpha_i}$, $i \in I$. We will freely identify W with the Weyl group $N_G(T)/T$ of G as well as the symmetric group S_n, by using the isomorphisms

$$S_n \quad \xrightarrow{\cong} \quad W \quad \xrightarrow{\cong} \quad N_G(T)/T,$$

$$\text{where} \quad (i,i+1) \quad \mapsto \quad s_i \quad \mapsto \quad \dot{s}_i T \quad \text{ for } \quad \dot{s}_i = \exp(E_{i,i+1})\exp(-E_{i+1,i})\exp(E_{i,i+1}).$$

Moreover, we let $\ell : W \to \mathbb{Z}_{\geq 0}$ be the standard length function. For $w = s_{i_1} \cdots s_{i_m}$ with $m = \ell(w)$, the element $\dot{w} := \dot{s}_{i_1} \cdots \dot{s}_{i_m} \in N_G(T)$ is independent of the reduced expression chosen.

We let $P \supseteq B_-$ be a parabolic subgroup of G. Set $I_P = \{i \in I \mid \dot{s}_i \in P\}$ and $I^P = I \setminus I_P$. We write

$$I^P = \{n_1, \cdots, n_r\},$$

where $1 \leq n_1 < n_2 < \cdots < n_r \leq n-1$.

Let W_P be the Weyl subgroup of W associated to P, and W^P be the set of minimal length coset representatives in W/W_P. Precisely,

$$W_P = \langle s_i \mid i \in I_P \rangle, \quad W^P = \{u \in W \mid \ell(us_i) > \ell(u), \forall i \in I_P\}.$$

Denote by w_P (resp. w^P, w_0) the longest element in W_P (resp. W^P, W).

The Langlands dual group $G^\vee$ to G is again $GL(n,\mathbb{C})$, but plays a different role. Let $B_\pm^\vee, P_\pm^\vee, T^\vee, \Delta^\vee$ be the Langlands dual versions of $B_\pm, P_\pm, T, \Delta$, respectively. The base $\Delta^\vee$ for $G^\vee$ are canonically identified with the set $\{\alpha_1^\vee, \cdots, \alpha_{n-1}^\vee\}$ of simple coroots for G. In particular, we have $s_{\alpha_i} = s_{\alpha_i^\vee}$. The Weyl group for $G^\vee$ is again W, and $I_{P^\vee} = I_P$ for any parabolic subgroup P of G containing B_+ or B_-. The deeper relationship between the original group $GL_n(\mathbb{C})$ and its Langlands dual group is described by the geometric Satake correspondence [Lus83, Gin95, MV07].

2.2. **Langlands dual flag varieties.** A partial flag variety is a quotient of $GL(n,\mathbb{C})$ by a parabolic subgroup on the left or right. We can think of it as parameterizing flags of subspaces (of row vectors) in $Hom(\mathbb{C}^n,\mathbb{C})$ or flags of quotients of the space $\mathbb{C}^n$ (in column vectors), to be precisely described below. Since we will focus more on the B-side of mirror symmetry, we will use G there, and leave $G^\vee$ for the A-side of mirror symmetry.

On the B-side, recall that $P \supseteq B_-$ is the parabolic subgroup of G with $I^P = \{n_1, \cdots, n_r\}$. Denote $n_0 := 0$ and $n_{r+1} := n$, and set

$$a_j := n_j - n_{j-1}, \quad \forall j \in [r+1].$$

Then P consists of block-lower-triangular matrices in G with block-diagonal matrices of the form $\mathrm{diag}\{M_1, M_2, \cdots, M_{r+1}\}$, where each M_j is an $a_j \times a_j$ invertible matrix.

Consider the partial flag variety $F\ell_{n_\bullet} = F\ell_{n_1,\cdots,n_r;n}$ that parameterizes flag of vector subspaces V_{n_j} in $Hom(\mathbb{C}^n,\mathbb{C})$, namely

$$F\ell_{n_\bullet} = \{V_{n_1} \subset \cdots \subset V_{n_r} \subset Hom(\mathbb{C}^n,\mathbb{C}) \mid \dim V_{n_j} = n_j \,, 1 \leq j \leq r\}.$$

The Lie group G transitively acts on $F\ell_{n_\bullet}$ on the right, inducing an isomorphism

$$\tau_P : P\backslash G \xrightarrow{\cong} F\ell_{n_\bullet}.$$

The isomorphism τ_P sends Pb to the partial flag $V_\bullet$ such that V_{n_j} is spanned by the first n_j row vectors of the matrix b for all $j \in [r]$.

On the A-side, we consider the partial flag variety $\mathbb{F}\ell(n_\bullet) = \mathbb{F}\ell(n; n_r, \cdots, n_1)$ that parameterizes flag of quotients, namely

$$\mathbb{F}\ell(n_\bullet) = \{\mathbb{C}^n \twoheadrightarrow \Lambda_{n_r} \twoheadrightarrow \cdots \twoheadrightarrow \Lambda_{n_1} \to 0 \mid \dim \Lambda_{n_j} = n_j,\ \forall j \in [r]\}/\sim.$$

Here $\Lambda_\bullet \sim \Lambda'_\bullet$ if and only if there are isomorphisms $\Lambda_{n_j} \to \Lambda'_{n_j}$ making the diagram of the two flags of quotients commutative. There is a canonical isomorphism

$$G^\vee/P^\vee \xrightarrow{\cong} \mathbb{F}\ell(n_\bullet),$$

which sends $gP^\vee$ to the class of a flag $\Lambda_\bullet$ of quotients such that the kernel of $\mathbb{C}^n \twoheadrightarrow \Lambda_{n_j}$ is the vector subspace spanned by the last $n - n_j$ column vectors of the matrix g for all j.

We remark that there are many canonical isomorphisms between group quotients and different parameterizations of flag varieties floating around. Here we are taking the above isomorphisms, to fit the philosophy of Langlands dual as well as the word "mirror". It does not matter much if different isomorphisms are taken.

2.3. **Open Richardson varieties.** For $v, w \in W$, with $v \le w$ with respect to the Bruhat order, we have the *open Richardson subvarieties*,

$$\mathcal{R}^-_{v,w} := (B_-\backslash B_-\dot{v}^{-1}B_+) \cap (B_-\backslash B_-\dot{w}^{-1}B_-) \subset B_-\backslash G,$$
$$\mathcal{R}^+_{v,w} := (B_+\backslash B_+\dot{v}^{-1}B_-) \cap (B_+\backslash B_+\dot{w}^{-1}B_+) \subset B_+\backslash G.$$

These are smooth and irreducible of dimension $\ell(w) - \ell(v)$ [KL79].Note that the intersections above are empty if $v \not\le w$. The Zariski closures are called a (closed) Richardson varieties and denoted by $\overline{\mathcal{R}}^-_{v,w}$ and $\overline{\mathcal{R}}^+_{v,w}$, respectively. We will also consider the following (open) Richardson varieties in G/B_-.

$$\mathcal{R}^{R-}_{v,w} := (B_+\dot{v}B_-/B_-) \cap (B_-\dot{w}B_-/B_-) \subset G/B_-,$$
$$\mathcal{R}^{R+}_{v,w} := (B_-\dot{v}B_+/B_+) \cap (B_+\dot{w}B_+/B_+) \subset G/B_+..$$

Open Richardson varieties and their projections will be our main target spaces on the B-side.

We use the notation $(P\backslash G)^\circ$ for the top-dimensional projected open Richardson variety inside $P\backslash G$, namely $(P\backslash G)^\circ = \mathrm{pr}_P(\mathcal{R}^-_{\mathrm{id},w_0w_P})$ under the projection $\mathrm{pr}_P : B_-\backslash G \longrightarrow P\backslash G$.

2.3.1. *Schubert varieties.* On the A-side, we consider the Bruhat decompositions

$$X = G^\vee/P^\vee = \bigsqcup_{v\in W^P} B^\vee_+\dot{v}P^\vee/P^\vee = \bigsqcup_{w\in W^P} B^\vee_-\dot{w}P^\vee/P^\vee.$$

The Zariski closures of the (opposite) Schubert cells $B^\vee_+\dot{v}P^\vee/P^\vee$ and $B^\vee_-\dot{w}P^\vee/P^\vee$,

$$X^v = \overline{B^\vee_+\dot{v}P^\vee/P^\vee}, \quad X_w := \overline{B^\vee_-\dot{w}P^\vee/P^\vee}$$

are (opposite) Schubert varieties in X. They are of codimension $\ell(v)$ and dimension $\ell(w)$ respectively. It is well-known that the classical cohomology ring $H^*(X,\mathbb{Z})$ has a $\mathbb{Z}$-basis of Schubert classes σ_v:

$$H^*(X,\mathbb{Z}) = \bigoplus_{v\in W^P} \mathbb{Z}\sigma_v, \text{ where } \sigma_v := \mathrm{P.D.}[X^v] \in H^{2\ell(v)}(X,\mathbb{Z}),$$

and P.D.$[X^v]$ stands for the Poincaré dual of the fundamental homology class of X^v.

3. The Plücker coordinate superpotentials

In this section, we will construct two versions of a superpotential for $X = G^\vee/P^\vee$ defined on projected open Richardson varieties for G. The first superpotential $\mathcal{F}_+$ is a straightforward extension of a construction for complex Grassmannians given in [MR20]. It has a natural formula in terms of Plücker coordinates in the Grassmannian case, as was shown in [MR20], but this formula does not generalise well to more general partial flag varieties. The construction of the second superpotential $\mathcal{F}_-$, which has a natural Plücker coordinate presentation in general, is the main outcome of this section.

3.1. **The superpotential $\mathcal{F}_+$ generalising** [MR20]**.** Recall we have the following (open) Richardson variety in $B_+\backslash G$,

$$\mathcal{R}^+_{\mathrm{id},w_0w_P} := (B_+\backslash B_+B_-) \cap (B_+\backslash B_+\dot{w}_P^{-1}w_0^{-1}B_+) \subset B_+\backslash G$$

and the projection map $\mathrm{pr}_{P_+} : B_+\backslash G \to P_+\backslash G$, where P_+ is upper-triangular parabolic subgroup with $I^{P_+} = \{n_1, \cdots, n_r\}$. It is shown in [KLS14, Lemma 3.1] that $\mathrm{pr}_{P_+} : \mathcal{R}^+_{\mathrm{id},w_0w_P} \to (P_+\backslash G)^\circ = \mathrm{pr}_{P_+}(\mathcal{R}^+_{\mathrm{id},w_0w_P})$ is an isomorphism. Moreover, implicitly from [KLS14], the projected open Richardson variety $\mathrm{pr}_{P_+}(\mathcal{R}^+_{\mathrm{id},w_0w_P})$ is the complement of an anti-canonical divisor in $P_+\backslash G$.

As in [MR20], we use $GL(n,\mathbb{C})$ as the starting point instead of $PSL(n,\mathbb{C})$ used in [Rie08], and thus need to cut down to a codimension one subtorus in T. The torus T has an adjoint action by W. Consider the invariant subtorus

$$\mathcal{T}^{W_P} = \{t \in T \mid t_n = 1,\ \dot{w}t\dot{w}^{-1} = t, \forall w \in W_P\}.$$

We define a map

$$\begin{array}{rccc} \psi_+ : & B_- \cap U_+\mathcal{T}^{W_P}\dot{w}_P\dot{w}_0^{-1}U_+ & \longrightarrow & (P_+\backslash G)^\circ \times \prod_{i\in I^P} \mathbb{C}^*_q \\ & b_- = u_1 t\dot{w}_P\dot{w}_0^{-1}u_2 & \mapsto & (P_+b_-, \mathbf{q}(t)) \end{array}$$

where

$$\begin{array}{ccc} \mathcal{T}^{W_P} & \xrightarrow{\sim} & \prod_{i\in I^P} \mathbb{C}^*_q \\ t & \mapsto & \mathbf{q}(t) := (\alpha_{n_1}(t), \dots, \alpha_{n_r}(t)). \end{array} \tag{3.1}$$

It follows from [Rie08, Section 4] and [KLS14] that ψ_+ is an isomorphism.

Definition 3.1. *We define the superpotential $\mathcal{F}_+$ by*

$$\begin{array}{ccccc} \mathcal{F}_+ : (P_+\backslash G)^\circ \times \prod_{i\in I^P} \mathbb{C}^*_q & \xrightarrow{\psi_+^{-1}} & B_- \cap U_+\mathcal{T}^{W_P}\dot{w}_P\dot{w}_0^{-1}U_+ & \longrightarrow & \mathbb{C} \\ (P_+b_-, \mathbf{q}(t)) & \mapsto & b_- = u_1 t\dot{w}_P\dot{w}_0^{-1}u_2 & \mapsto & \sum_{i=1}^{n-1} e_i^*(u_1) + e_i^*(u_2). \end{array}$$

Where $e_k^* : U_+ \to \mathbb{C}$ is the map that sends $u = (u_{ij})$ in U_+ to its $(k, k+1)$-entry, namely $e_k^*(u) = u_{k,k+1}$. This is well-defined by [Rie08, Lemma 5.2].

This definition is a direct translation of the Lie-theoretic superpotential defined in [Rie08] via the isomorphism ψ_+. If P_+ is a maximal parabolic, then this definition agrees with the one used to give a Plücker coordinate superpotential for Grassmannians in [MR20]. In general, viewing $\mathcal{F}_+$, as a rational function on the partial flag variety $P_+\backslash G$ (depending additionally on parameters q_i), there will be a Plücker coordinate formula also in the partial flag setting. However, it turns out that this construction gives a superpotential that is not as well-suited for being expressed in terms of Plücker coordinates as we would like.

Example 3.2. *Consider the complete flag variety $G^\vee/B_-^\vee$ for $GL_3(\mathbb{C})$ and the associated superpotential $\mathcal{F}_+$. Fix a representative b_- of P_+b_-. For a subset I of $\{1,2,3\}$ let p_I denote the minor of b_- with column set I and row set the last $|I|$ many rows. Then*

$$\mathcal{F}_+(P_+b_-,(q_1,q_2)) = \frac{p_2}{p_1} + \frac{p_{13}}{p_{12}} + q_2\frac{p_1p_{13}}{p_3p_{12}} + q_1\frac{p_2p_{12}}{p_1p_{23}}.$$

This example will be useful for comparison between $\mathcal{F}_+$ and our alternative version of the superpotential that we construct in Section 3.3 (see Example 3.19).

3.2. **Superpotential $\mathcal{F}_{\mathrm{Lie}}$.** We now give the construction of the original Lie theoretic superpotential $\mathcal{F}_{\mathrm{Lie}}$ in a form that is suitable for our applications. The following definition is a slight change of conventions on [MR20, Definition 6.3] and [Rie08].

We recall the definition of the torus $\mathcal{T}^{W_P}$ and the isomorphism from (3.1), as well as the maps $e_k^*: U_+ \to \mathbb{C}$. We also recall that $P \supseteq B_-$ is the parabolic subgroup of G with $I^P = \{n_1,\cdots,n_r\}$.

Definition 3.3 (The Lie-theoretic superpotential)**.** *Let $Z_P := B_- \cap U_+\mathcal{T}^{W_P}\dot{w}_P^{-1}\dot{w}_0U_+$. Define the map $\mathcal{F}_{\mathrm{Lie}}: Z_P \longrightarrow \mathbb{C}$ by*

$$b_- = u_1t\dot{w}_P^{-1}\dot{w}_0u_2 \quad\mapsto\quad -\left(\sum_{i=1}^{n-1} e_i^*(u_1) + \sum_{i=1}^{n-1} e_i^*(u_2)\right).$$

It is an important fact that the map $\mathcal{F}_{\mathrm{Lie}}$ is well-defined even though u_1 and u_2 are not uniquely determined by b_-, see [Rie08, Lemma 5.2].

3.3. **The superpotential $\mathcal{F}_-$.** In this section we give a non-standard isomorphism from Z_P to the product of the projected open Richardson variety $(P\backslash G)^\circ = \mathrm{pr}_P(\mathcal{R}^-_{\mathrm{id},w_0w_P})$ with $\prod_{i\in I^P}\mathbb{C}_q^*$. The superpotential $\mathcal{F}_-$ will then be defined as a function on $(P\backslash G)^\circ \times \prod_{i\in I^P}\mathbb{C}_q^*$.

We start by considering the isomorphism

$$\begin{array}{rrcl}\gamma: & Z_P = B_-\cap U_+\mathcal{T}^{W_P}\dot{w}_P^{-1}\dot{w}_0U_+ & \longrightarrow & (B_-\cap U_+\dot{w}_P^{-1}\dot{w}_0U_+)\times\prod_{i\in I^P}\mathbb{C}_q^* \\ & b_- = u_1t\dot{w}_P^{-1}\dot{w}_0u_2 & \mapsto & (\hat{b} = t^{-1}b_-, \mathbf{q}(t)),\end{array}$$

where $\mathbf{q}(t)$ is as in (3.1). Here the first factor of the right-hand side may be considered as fiber of Z_P where t equals to the identity element. The $q_{n_i} = \alpha_{n_i}(t)$ can also be obtained directly using minors of b_-.

We translate the superpotential $\mathcal{F}_{\mathrm{Lie}}$ to a function on the right-hand side above, and write down a formula for it for future reference. Namely, we have $\hat{b} = \hat{v}\dot{w}_P^{-1}\dot{w}_0\,\hat{u}$ for $\hat{b} \in B_- \cap U_+\dot{w}_P^{-1}\dot{w}_0U_+$. Then

$$\mathcal{F}_{\mathrm{Lie}}\circ\gamma^{-1}(\hat{b},(q_i)_{i\in I^P}) = -\left(\sum_{i\in I_P}\hat{v}_{i,i+1} + \sum_{i\in I^P}q_i\hat{v}_{i,i+1} + \sum_{i=1}^{n-1}\hat{u}_{i,i+1}\right). \tag{3.2}$$

The main step in the construction now is to make a choice for $\hat{v}$ for which only the quantum terms in the formula above will appear, and the other $\hat{v}_{i,i+1}$ vanish. Recall that $\mathcal{R}^{R^+}_{\mathrm{id},w_Pw_0} = B_-B_+ \cap B_+w_Pw_0B_+/B_+$. We define the variety

$$\mathcal{Z} := U_+\dot{w}_P^{-1}\dot{w}_0 \cap \dot{w}_P^{-1}\dot{w}_0U_- \cap B_-B_+, \tag{3.3}$$

which will play a central role in our constructions.

Lemma 3.4. *Consider the intersection* $B_- \cap U_+\dot{w}_P^{-1}\dot{w}_0U_+$ *and the variety* $\mathcal{Z}$ *as defined above. We have the following isomorphisms*

$$\begin{array}{rccc} \zeta_1: & B_- \cap U_+\dot{w}_P^{-1}\dot{w}_0U_+ & \longrightarrow & \mathcal{R}^{R^+}_{\mathrm{id},w_Pw_0} \\ & \hat{b} & \mapsto & \hat{b}B_+, \end{array}$$

$$\begin{array}{rccc} \zeta_2: & \mathcal{Z} & \longrightarrow & \mathcal{R}^{R^+}_{\mathrm{id},w_Pw_0} \\ & z & \mapsto & zB_+. \end{array}$$

We consider the composition $\zeta = \zeta_2^{-1} \circ \zeta_1$,

$$\begin{array}{rccc} \zeta: & B_- \cap U_+\dot{w}_P^{-1}\dot{w}_0U_+ & \longrightarrow & \mathcal{Z} \\ & \hat{b} & \mapsto & z, \end{array}$$

where $z \in \mathcal{Z}$ *is the unique representative with* $zB_+ = \hat{b}B_+$.

Proof. The map ζ_1 is just the restriction to the fiber over $e \in T^{W_P}$ of the isomorphism $B_- \cap U_+T^{W_P}\dot{w}_P^{-1}\dot{w}_0U_+ \cong \mathcal{R}^{R^+}_{\mathrm{id},w_Pw_0} \times T^{W_P}$ from [Rie08, Section 4.1]. We now show that ζ_2 is an isomorphism.

Note that $\mathcal{R}^{R^+}_{\mathrm{id},w_Pw_0}$ is the open dense subset of the Bruhat cell $B_+w_Pw_0B_+/B_+$ obtained by intersecting with opposite big cell B_-B_+/B_+. We have the factorisation $U_+ = U_+^P U_{+,P}$, where

$$\begin{aligned} U_+^P &:= U_+ \cap \dot{w}_P^{-1}U_+\dot{w}_P, \\ U_{+,P} &:= U_+ \cap \dot{w}_P^{-1}U_-\dot{w}_P, \end{aligned} \tag{3.4}$$

and the map $u \mapsto u\dot{w}_P^{-1}\dot{w}_0B_+$ from U_+ restricts to an isomorphism $U_+^P \to B_+w_Pw_0B_+/B_+$. Equivalently, the projection map $g \mapsto gB_+$ restricts to an isomorphism

$$U_+^P\dot{w}_P^{-1}\dot{w}_0 \xrightarrow{\sim} B_+w_Pw_0B_+/B_+. \tag{3.5}$$

We now rewrite the definition of $\mathcal{Z}$ as follows,

$$\begin{aligned} \mathcal{Z} &= U_+\dot{w}_P^{-1}\dot{w}_0 \cap \dot{w}_P^{-1}\dot{w}_0U_- \cap B_-B_+ \\ &= \left(U_+ \cap \dot{w}_P^{-1}U_+\dot{w}_P\right)\dot{w}_P^{-1}\dot{w}_0 \cap B_-B_+ \\ &= U_+^P\dot{w}_P^{-1}\dot{w}_0 \cap B_-B_+. \end{aligned}$$

It follows that (3.5) restricts to an isomorphism $\mathcal{Z} \to \mathcal{R}^{R^+}_{\mathrm{id},w_Pw_0}$, and this is precisely the map ζ_2. □

Lemma 3.5. *Recall that* $(P\backslash G)^\circ = \mathrm{pr_P}(\mathcal{R}^-_{\mathrm{id},w_0w_P})$ *and let* $\mathcal{Z}$ *be as defined in* (3.3)*. We have an isomorphism*

$$\begin{array}{rccc} \pi: & \mathcal{Z} & \longrightarrow & (P\backslash G)^\circ, \\ & z & \mapsto & Pz. \end{array}$$

Proof. Note that

$$\mathcal{R}^-_{\mathrm{id},w_0w_P} = \mathcal{R}^-_{w_P,w_0}\dot{w}_0 = B_-\backslash(B_-\dot{w}_P^{-1}B_+ \cap B_-\dot{w}_0^{-1}B_-)\dot{w}_0.$$

Consider U_+^P and $U_{+,P}$ as defined in (3.4) and the factorisation $U_+ = U_{+,P}U_+^P$. The Bruhat cell $B_-\backslash B_-\dot{w}_P^{-1}B_+$ is isomorphic to U_+^P via the map $u \mapsto B_-\dot{w}_P^{-1}u$. It follows that

$$\begin{array}{ccc} \dot{w}_P^{-1}U_+^P \cap B_-\dot{w}_0^{-1}B_- & \to & B_-\backslash(B_-\dot{w}_P^{-1}B_+ \cap B_-\dot{w}_0^{-1}B_-) \\ \dot{w}_P^{-1}u & \mapsto & B_-\dot{w}_P^{-1}u. \end{array} \tag{3.6}$$

is an isomorphism. We can rewrite the definition of $\mathcal{Z}$ as follows,

$$\begin{aligned}\mathcal{Z} &= U_+\dot{w}_P^{-1}\dot{w}_0 \cap \dot{w}_P^{-1}\dot{w}_0 U_- \cap B_-B_+ \\ &= \dot{w}_P^{-1}\left(\dot{w}_P U_+ \dot{w}_P^{-1} \cap \dot{w}_0 U_- \dot{w}_0^{-1}\right)\dot{w}_0 \cap (B_-\dot{w}_0^{-1}B_-)\dot{w}_0. \\ &= \left(\dot{w}_P^{-1}U_+^P \cap B_-\dot{w}_0^{-1}B_-\right)\dot{w}_0.\end{aligned}$$

We now translate both sides of the isomorphism from (3.6) by $\dot{w}_0$ and obtain an isomorphism

$$\begin{array}{rccl}\pi' : & \mathcal{Z} & \to & \mathcal{R}^-_{\mathrm{id},w_0w_P}, \\ & z & \mapsto & B_-z.\end{array}$$

The composition of π' above with the isomorphism $\mathcal{R}^-_{\mathrm{id},w_0w_P} \xrightarrow{\sim} (P\backslash G)^\circ$ from [KLS14] is the map $\pi : \mathcal{Z} \to (P\backslash G)^\circ$, which proves that π is an isomorphism. □

Definition 3.6 (The superpotential $\mathcal{F}_-$)**.** *We denote by ψ_- the composition of isomorphisms, where $\mathcal{B} = B_- \cap U_+\dot{w}_P^{-1}\dot{w}_0 U_+$,*

$$\psi_- : Z_P \xrightarrow{\gamma} \mathcal{B} \times \prod_{i\in I^P}\mathbb{C}_q^* \xrightarrow{\zeta\times\mathrm{id}} \mathcal{Z} \times \prod_{i\in I^P}\mathbb{C}_q^* \xrightarrow{\pi\times\mathrm{id}} (P\backslash G)^\circ \times \prod_{i\in I^P}\mathbb{C}_q^*.$$

We now define the superpotential $\mathcal{F}_-$ by

$$\mathcal{F}_- := \mathcal{F}_{\mathrm{Lie}} \circ \psi_-^{-1} : (P\backslash G)^\circ \times \prod_{i\in I^P}\mathbb{C}_q^* \to \mathbb{C}.$$

3.4. **Notations for $\mathcal{F}_-$.** In summary, we have shown above that we may write any element of $(P\backslash G)^\circ$ uniquely as Pz for some $z \in \mathcal{Z}$. We can then write

$$z = v^{-1}\dot{w}_P^{-1}\dot{w}_0 \tag{3.7}$$

for a unique $v \in U_+$. Next let $\hat{b} := \zeta^{-1}(z) \in B_- \cap U_+\dot{w}_P^{-1}\dot{w}_0U_+$. We can write

$$\hat{b} = zu^{-1} = v^{-1}\dot{w}_P^{-1}\dot{w}_0u^{-1}, \tag{3.8}$$

for a unique $u \in U_+$. Finally, the superpotential $\mathcal{F}_-$ is computed by

$$\mathcal{F}_-(Pz,q) = \mathcal{F}_{\mathrm{Lie}} \circ \gamma^{-1}(\hat{b}, (q_i)_{i\in I^P}) = \sum_{i\in I_P} v_{i,i+1} + \sum_{i\in I^P} q_i v_{i,i+1} + \sum_{i=1}^{n-1} u_{i,i+1}, \tag{3.9}$$

using the description of $\mathcal{F}_{\mathrm{Lie}}$ in (3.2).

Definition 3.7. *Given $z \in \mathcal{Z}$ we will always use the notations above for the related elements $\hat{b}, v, u$ with $\hat{b} \in B_- \cap U_+\dot{w}_P^{-1}\dot{w}_0U_+$ and $u, v \in U^+$, satisfying*

$$\begin{aligned}\hat{b}B_+ &= zB_+, \\ z &= v^{-1}\dot{w}_P^{-1}\dot{w}_0, \\ \hat{b} &= zu^{-1} = v^{-1}\dot{w}_P^{-1}\dot{w}_0u^{-1}.\end{aligned}$$

We will also let $b := \hat{b}^{-1} = u\dot{w}_0^{-1}\dot{w}_Pv$.

We can immediately simplify the formula (3.9) using the following lemma.

Lemma 3.8. *Let $z \in \mathcal{Z}$. For $v \in U_+$ as in Definition 3.7, we have that $v \in \dot{w}_P^{-1}U_+\dot{w}_P$, and therefore the entry $v_{i,i+1} = 0$ for all $i \in I_P$.*

Remark 3.9. *This lemma says that $v \in U_+^P$, in the notation* (3.4) *from Lemma 3.4.*

Proof of Lemma 3.8. By (3.3) we have that $z \in \dot{w}_P^{-1}\dot{w}_0U_-$. The lemma follows from this and the fact that $v = \dot{w}_P^{-1}\dot{w}_0z^{-1}$. □

As a consequence of Lemma 3.8 we have the formula,

$$\mathcal{F}_-(Pz,\mathbf{q}) = \sum_{i\in I^P} q_i v_{i,i+1} + \sum_{i=1}^{n-1} u_{i,i+1}, \tag{3.10}$$

for $\mathcal{F}_-$ in terms of u, v.

We can now use the formula (3.10) as our starting point for studying $\mathcal{F}_-$. The rest of this section will be devoted to giving a compact description of $\mathcal{F}_-$ in terms of Plücker coordinates.

3.5. **The description of $\mathcal{Z}$.** We first give a concrete description of our variety

$$\mathcal{Z} = U_+\dot{w}_P^{-1}\dot{w}_0 \cap \dot{w}_P^{-1}\dot{w}_0 U_- \cap B_-B_+. \tag{3.11}$$

Recall that $a_j = n_j - n_{j-1}$ where $I^P = \{n_1, \ldots, n_r\}$.

Lemma 3.10. *Let I_{a_j} denote the $a_j \times a_j$ identity matrix. If $z \in \mathcal{Z}$ then z is of the following form,*

$$\begin{pmatrix} * & * & \cdots & * & * & I_{a_1} \\ * & * & \cdots & * & (-1)^{n_1} I_{a_2} & 0 \\ * & * & \iddots & \iddots & 0 & 0 \\ \vdots & \vdots & \iddots & \iddots & \vdots & \vdots \\ * & (-1)^{n_{r-1}} I_{a_r} & 0 & \cdots & 0 & 0 \\ (-1)^{n_r} I_{a_{r+1}} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{n\times n}. \tag{3.12}$$

We write $z = v^{-1}\dot{w}_P^{-1}\dot{w}_0$ as in Definition 3.7. Then the matrix $v \in U_+$ has its non-zero above-diagonal entries given by

$$v_{n_j,n_j+1} = (-1)^{n_j+1} z_{n_j, n-n_{j+1}+1}.$$

Proof. Given square matrices A_i, we let $\mathrm{diag}\{A_1, \cdots, A_m\}$ denote the block-diagonal matrix with diagonal blocks A_i. Similarly we write

$$\mathrm{antidiag}\{A_1, \cdots, A_m\} := \begin{pmatrix} & & & A_1 \\ & & A_2 & \\ & \iddots & & \\ A_m & & & \end{pmatrix}.$$

for the anti block-diagonal matrix with blocks A_i.

We have that $\mathcal{Z} \subset U_+\dot{w}_P^{-1}\dot{w}_0 \cap \dot{w}_P^{-1}\dot{w}_0 U_-$. It follows that $z \in \mathcal{Z}$ has anti-diagonal blocks according to $\dot{w}_P^{-1}\dot{w}_0$, and all other non-zero entries must lie above and to the left of these blocks.

By direct calculation we have $\dot{w}_0 = \mathrm{antidiag}\{1, -1, \cdots, (-1)^{n-1}\}$. Moreover, $\dot{w}_P$ is block diagonal with j-th diagonal block given by $\mathrm{antidiag}\{1, -1, \cdots, (-1)^{a_j-1}\}$. Therefore $\dot{w}_P^{-1} = \mathrm{diag}\{I_\pm^{(1)}, \cdots, I_\pm^{(r+1)}\}$ where $I_\pm^{(j)} := \mathrm{antidiag}\{(-1)^{a_j-1} \cdots, -1, 1\}$. It follows that $\dot{w}_P^{-1}\dot{w}_0$ is an anti-diagonal block matrix with top right-hand block given by I_{a_1}, and $(j+1)$-st block given by $(-1)^{n_j} I_{a_{j+1}}$,

$$\dot{w}_P^{-1}\dot{w}_0 = \mathrm{antidiag}\{I_{a_1}, (-1)^{n_1} I_{a_2}, \cdots, (-1)^{n_r} I_{a_{r+1}}\}. \tag{3.13}$$

Here the overall signs of the blocks follow from the fact that the n_{j+1}-st row of $\dot{w}_P^{-1}$ is the row vector $\delta_t^{n_j+1}$, and the $(n-n_j)$-th column of $\dot{w}_0$ is $(-1)^{n_j}\delta^t_{n_j+1}$.

This finishes the proof that the matrix z has the form indicated in (3.12). We can now check the formula for v_{n_j,n_j+1}. We apply v^{-1} to $\delta^t_{n_j+1}$, giving

$$v^{-1}\cdot \delta^t_{n_j+1} = z\dot{w}_0^{-1}\dot{w}_P \cdot \delta^t_{n_j+1} = (-1)^{n_j} z \cdot \delta^t_{n-n_{j+1}+1} \tag{3.14}$$

Here we used the inverse of (3.13),

$$\dot{w}_0^{-1}\dot{w}_P = \mathrm{antidiag}\{(-1)^{n_r} I_{a_{r+1}}, \cdots, (-1)^{n_1} I_{a_2}, I_{a_1}\}. \tag{3.15}$$

From (3.14) we get

$$(v^{-1})_{n_j, n_j+1} = (-1)^{n_j} z_{n_j, n-n_{j+1}+1}.$$

The formula now follows, since $(v^{-1})_{n_j,n_j+1} = -v_{n_j,n_j+1}$. □

Remark 3.11. *The complete description of $\mathcal{Z}$ is that it consists of those matrices of the form* (3.12) *for which the upper left-hand corner minors are all non-vanishing. This final condition encodes the intersection with B_-B_+ in* (3.11).

3.6. **A Plücker coordinate formula for $\mathcal{F}_-$.** For positive integers $j \le m$, we denote by $\binom{[m]}{j}$ the set of subsets J of $[m]$ of cardinality j. We denote the complement $J^c_{(m)} = [m] \setminus J$ simply as J^c whenever $J \subset [m]$ is well understood. We always write J, J^c as increasing sequences, and define $|J| := \sum_{i \in J} i$.

We consider the Plücker embedding

$$\begin{array}{rccc} \mathrm{Pl}: & P\backslash G & \longrightarrow & \mathbb{P}^{\binom{n}{n_1}-1} \times \cdots \times \mathbb{P}^{\binom{n}{n_r}-1} \\ & Pg & \mapsto & \left([p_{K_1}(g)]_{K_1 \in \binom{[n]}{n_1}}, \cdots, [p_{K_r}(g)]_{K_r \in \binom{[n]}{n_r}} \right) \end{array}$$

where the Plücker coordinate $p_{K_j}(g)$ of Pg is the determinant of the $n_j \times n_j$ sub-matrix of g with first n_j rows and the columns from K_j.

The next proposition is a combination of Propositions 2.2 and 3.9 in [LSZ23] with respect to the quotient $P\backslash G$, which was also implicitly contained in [KLS14].

Proposition 3.12. *The projected open Richardson variety $(P\backslash G)^\circ = \mathrm{pr}_P(\mathcal{R}^-_{\mathrm{id}, w_0 w_P})$ is the complement of the anti-canonical divisor $-K_{P\backslash G}$ in $P\backslash G$, where*

$$-K_{P\backslash G} \;=\; \sum_{i \in I} \mathrm{pr}_P(\overline{\mathcal{R}}^-_{s_i, w_0 w_P}) \;+\; \sum_{i \in I^P} \mathrm{pr}_P(\overline{\mathcal{R}}^-_{\mathrm{id}, w_0 s_i w_P}).$$

Definition 3.13. *For any homogeneous polynomial p in Plücker coordinates, we denote $\mathcal{V}(p) := \{Pg \in P\backslash G \mid p([p_{K_1}]_{K_1}, \cdots, [p_{K_r}]_{K_r})(Pq) = 0\}$ and define*

$$D_k := \begin{cases} \mathcal{V}(p_{[k]}), & \text{if } k \in \{n_1, \cdots, n_r\}, \\ \mathcal{V}(p_{[n]\setminus[k+1, n-n_1+k]}), & \text{if } 1 \le k < n_1, \\ \mathcal{V}(p_{[k-n_r+1, k]}), & \text{if } n_r < k \le n-1, \\ \mathcal{V}\Big(\sum\limits_{J \in \binom{[\min\{k, \hat{k}\}]}{k-n_j}} (-1)^{|J|} p_{[k] \smallsetminus J} \cdot p_{J \cup [\hat{k}+1, n]}\Big), & \text{if } n_j < k < n_{j+1} \text{ with } j \in [r-1] \\ & \text{where } \quad \hat{k} := n - n_{j+1} + k - n_j \\ \mathcal{V}(p_{[n-n_{k-n+1}, n]}), & \text{if } k \in \{n, \cdots, n-1+r\}. \end{cases},$$

The following proposition from [LSZ23], provides explicit equations for the irreducible components of the the anti-canonical divisor $-K_{P\backslash G}$ in terms of Plücker coordinates.

Proposition 3.14 ([LSZ23, Theorem 4.1])**.** *We have*

$$D_k = \begin{cases} \mathrm{pr}_P(\overline{\mathcal{R}}^-_{s_k, w_0 w_P}), & \text{if } 1 \le k \le n-1, \\ \mathrm{pr}_P(\overline{\mathcal{R}}^-_{\mathrm{id}, w_0 s_{n_{k-n+1}} w_P}) & \text{if } n \le k \le n-1+r, \end{cases}$$

and $\sum_{k=1}^{n-1+r} D_k$ is an anti-canonical divisor in $P\backslash G$, denoted as $-K_{P\backslash G}$.

We will give a Plücker coordinate expansion of $\mathcal{F}_-$ where each summand has a pole of order 1 along a (unique) irreducible component D_k of $-K_{P\backslash G}$, giving rise to a bijection between divisors D_k and summands of $\mathcal{F}_-$.

For any $n\times m$ matrix $A\in M_{n\times m}(\mathbb{C})$, we let $\Delta^J_K(A)$ denote the minor with row set J and column set K, whenever the sub-matrix is a square matrix. We will need the following generalization in [GAE02] of the famous Cramer's rule in linear algebra.

Lemma 3.15 (Generalized Cramer's rule)**.** *Let $A\in GL_n(\mathbb{C})$ and $X, Y\in M_{n\times m}(\mathbb{C})$ such that $AX=Y$. For any $J\in\binom{[n]}{l}$ and $K\in\binom{[m]}{l}$ where $l\le\min\{n,m\}$, we have*

$$\Delta^J_K(X)=\frac{\det(A_Y(J,K))}{\det A}$$

where $A_Y(J,K)$ denotes the matrix constructed from A by order-preserving replacing the column set J of A by the column set K of Y.

The special case of $X=A^{-1}$ in the generalized Cramer's rule yields Jacobi Theorem for the minors of an inverse matrix immediately as follows.

Corollary 3.16 (Jacobi Theorem)**.** *Let $A\in GL_n(\mathbb{C})$. For any $J,K\in\binom{[n]}{l}$ where $l\in[n]$, we write $J^c=[n]\setminus J$ and $K^c=[n]\setminus K$ in increasing sequences. We have*

$$\Delta^J_K(A^{-1})=\frac{(-1)^{|J|+|K|}}{\det A}\Delta^{K^c}_{J_c}(A).$$

Using Lemma 3.10, we have the next key proposition.

Proposition 3.17. *Let $z\in\mathcal{Z}$. We define b,u and v as in Definition 3.7, so that $b=u\dot{w}_0^{-1}\dot{w}_Pv=uz^{-1}$. Then*

$$u_{i,i+1}=\frac{\Delta^{[i]}_{[i-1]\cup\{i+1\}}(z)}{\Delta^{[i]}_{[i]}(z)},\qquad \textit{for any } i\in[n-1];$$

$$v_{i,i+1}=\begin{cases}\dfrac{\Delta^{[n_j]}_{\{n-n_{j+1}+1\}\cup[n-n_j+1,n]\setminus\{n-n_{j-1}\}}(z)}{\Delta^{[n_j]}_{[n-n_j+1,n]}(z)}, & \textit{if } i=n_j \textit{ with } j\in[r],\\ 0, & \textit{otherwise.}\end{cases}$$

Proof. Since $u\in U_+$ and $b\in B_-$, we have $\hat{u}:=u^{-1}\in U_+$ and $\hat{b}:=b^{-1}\in B_-$. For $m=n-i$, we let $\hat{u}^{(m)}$ (resp. $\hat{b}^{(m)}$) be the $n\times m$ matrix obtained by taking the last m columns of $\hat{u}$ (resp. $\hat{b}$). Since $b=uz^{-1}$, we have $z\hat{u}^{(m)}=\hat{b}^{(m)}$. By using Lemma 3.15 and noting $\hat{b}\in B_-$ and $\det z=1$, we have

$$\begin{aligned}\hat{u}_{n-m,n-m+1}&=\Delta^{\{n-m\}\cup[n-m+2,n]}_{[m]}(\hat{u}^{(m)})\\&=\det(z_{\hat{b}^{(m)}}(\{n-m\}\cup[n-m+2,n],[m]))\\&=-(\prod_{j=n-m+1}^{n}\hat{b}_{jj})\Delta^{[i]}_{[i-1]\cup\{i+1\}}(z),\end{aligned}$$

$$1=\Delta^{[n-m+1,n]}_{[m]}(\hat{u}^{(m)})=(\prod_{j=n-m+1}^{n}\hat{b}_{jj})\Delta^{[i]}_{[i]}(z).$$

Thus for $i \in [n-1]$, we have

$$u_{i,i+1} = -\hat{u}_{i,i+1} = -\frac{\hat{u}_{i,i+1}}{1} = \frac{\Delta^{[i]}_{[i-1]\cup\{i+1\}}(z)}{\Delta^{[i]}_{[i]}(z)}.$$

By Lemma 3.8, we have $v_{i,i+1} = 0$ if $i \neq \{n_1, \cdots, n_r\}$; We recall that by Lemma 3.10, z is of the form (3.12) and therefore for $j \in [r]$ we have

$$\begin{aligned} v_{n_j,n_j+1} &= (-1)^{n_j+1} z_{n_j, n-n_{j+1}+1} \\ &= (-1)^{n_j+1} \Delta^{\{n_j\}}_{\{n-n_{j+1}+1\}}(z) \\ &= (-1)^{n_j+1}(-1)^{a_j-1}(-1)^{n_{j-1}} \frac{\Delta^{[n_j]}_{\{n-n_{j+1}+1\}\cup[n-n_j+1,n]\setminus\{n-n_{j-1}\}}(z)}{\Delta^{[n_j]}_{[n-n_j+1,n]}(z)} \\ &= \frac{\Delta^{[n_j]}_{\{n-n_{j+1}+1\}\cup[n-n_j+1,n]\setminus\{n-n_{j-1}\}}(z)}{\Delta^{[n_j]}_{[n-n_j+1,n]}(z)}. \end{aligned}$$

□

Theorem 3.18. *Let $(q_{n_1}, \cdots, q_{n_r})$ denote the coordinates of $(\mathbb{C}^*)^r = \prod_{i\in I^P} \mathbb{C}^*_q$. The superpotential $\mathcal{F}_- : (P\backslash G)^\circ \times (\mathbb{C}^*)^r \longrightarrow \mathbb{C}$ is given by the explicit formula*

$$\begin{aligned} \mathcal{F}_- = &\sum_{i=1}^{n_1-1} \frac{p_{[i-1]\cup\{i+1\}\cup[n-n_1+i+1,n]}}{p_{[i]\cup[n-n_1+i+1,n]}} + \sum_{j=1}^{r-1}\sum_{i=n_j+1}^{n_{j+1}-1} S_i^{(j)} + \sum_{i=n_r+1}^{n-1} \frac{p_{[i-n_r+1,i+1]\setminus\{i\}}}{p_{[i-n_r+1,i]}} \\ &+ \sum_{j=1}^{r} \frac{p_{[n_j-1]\cup\{n_j+1\}}}{p_{[n_j]}} + \sum_{j=1}^{r} q_{n_j} \frac{p_{\{n-n_{j+1}+1\}\cup[n-n_j+1,n]\setminus\{n-n_{j-1}\}}}{p_{[n-n_j+1,n]}}, \end{aligned}$$

where

$$S_i^{(j)} = \frac{\sum_{J\in\binom{[\min\{i+1,\hat{i}\}]\setminus\{i\}}{i-n_j}} \epsilon(J)(-1)^{|J|} p_{[i-1]\cup\{i+1\}\setminus J} \cdot p_{J\cup[\hat{i}+1,n]}}{\sum_{J\in\binom{[\min\{i,\hat{i}\}]}{i-n_j}} (-1)^{|J|} p_{[i]\setminus J} \cdot p_{J\cup[\hat{i}+1,n]}}$$

with $\hat{i} = n - n_{j+1} + i - n_j$ and $\epsilon(J) = \begin{cases} 1, & \text{if } i+1 \notin J, \\ -1, & \text{if } i+1 \in J. \end{cases}$

Proof. We let $\hat{b}, b, u, v$ be defined as in Definition 3.7 for given $z \in \mathcal{Z}$. By Lemma 3.8, we have

$$\mathcal{F}_-(Pz, \mathbf{q}) = \sum_{i\in I^P} q_i v_{i,i+1} + \sum_{i=1}^{n-1} u_{i,i+1},$$

Since by Proposition 3.17 $u_{i,i+1}$ and $v_{i,i+1}$ are quotients of minors of z, it suffices to interpret those minors by the Plücker coordinates. Recall that for subsets $K \in \binom{[n]}{n_j}$, $p_K(z) = \Delta^{[n_j]}_K(z)$ is the determinant of the submatrix of z with first n_j rows and columns from K. Therefore if $i \in \{n_1, \cdots, n_r\}$, this is already done for both $u_{i,i+1}$ and $v_{i,i+1}$, as given in the last two sums of the expression of $\mathcal{F}_-$. Recall that z is of the form (3.12). For $i < n_1$ then both $\Delta^{[i]}_{[i-1]\cup\{i+1\}}(z) = \Delta^{[n_1]}_{[i-1]\cup\{i+1\}\cup[n-n_1+i+1,n]}(z)$ and $\Delta^{[i]}_{[i]}(z) = \Delta^{[n_1]}_{[i]\cup[n-n_1+i+1,n]}(z)$ hold. Then we have

$$u_{i,i+1} = \frac{p_{[i-1]\cup\{i+1\}\cup[n-n_1+i+1,n]}}{p_{[i]\cup[n-n_1+i+1,n]}}.$$

Next we consider the case $i > n_r$. Let $0_{\mu,\nu}$ denote the zero matrix with μ rows and ν columns. then the last $(i-n_r)$ rows for minors $\Delta^{[i]}_{[i-1]\cup\{i+1\}}(z)$ and $\Delta^{[i]}_{[i]}(z)$ are both given by $\Big((-1)^{n_r} I_{i-n_r}, 0_{i-n_r, n-i+n_r}\Big)_{(i-n_r)\times i}$. The Laplace expansion on the last $(i-n_r)$ rows leads to the third sum in the expression of $\mathcal{F}_-$ immediately.

It remains to discuss $u_{i,i+1}$ for the case $n_j < i < n_{j+1}$ for some $j \in [r-1]$. Denote $k = i - n_j$. The first i rows of z is given by

$$\begin{pmatrix} * & * & * & \cdots & * & * & I_{a_1} \\ * & * & * & \cdots & * & (-1)^{n_1} I_{a_2} & 0 \\ \vdots & \vdots & \vdots & \iddots & \iddots & 0 & 0 \\ * & * & * & (-1)^{n_{j-1}} I_{a_j} & 0 & 0 & 0 \\ * & (-1)^{n_j} I_k & 0_{k, a_{j+1}-k} & 0 & \cdots & 0 & 0 \end{pmatrix}_{i\times n}.$$

Here the first column block has size $n - n_{j+1}$. Using Laplace expansion on the last k rows, we obtain

$$\Delta^{[i]}_{[i-1]\cup\{i+1\}}(z) = \sum_{J\in\binom{[i-1]\cup\{i+1\}}{k}} (-1)^{|[i-n_j+1,i]|+|J|} \epsilon(J) \Delta^{[n_j]}_{[i-1]\cup\{i+1\}\setminus J}(z)\cdot z_J ,$$

where z_J is the determinant of the submatrix with columns from J and last k rows. Since the last k row is $\begin{pmatrix} * & (-1)^{n_{k-1}} I_k & 0_{k,a_{j+1}-k} & 0 & \cdots & 0\end{pmatrix}$. Its last nonzero column is in position $\hat{i} := n - n_{j+1} + k$, so we may assume that $J \subset [l'] \setminus \{i\}$ where $l' = \min\{i+1, \hat{i}\}$, as otherwise $z_J = 0$ for J occurring in the above sum. Similarly and more easily, we set $l = \min\{i, \hat{i}\}$ and have

$$\Delta^{[i]}_{[i]}(z) = \sum_{\tilde{J}\in\binom{[l]}{k}} (-1)^{|[i-n_j+1,i]|+|\tilde{J}|} \Delta^{[n_j]}_{[i]\setminus\tilde{J}}(z)\cdot z_{\tilde{J}}.$$

Since z is of the specified form as above, we have $z_J = \varepsilon p_{J\cup[n-n_{j+1}+k+1,n]}$ and $z_{\tilde{J}} = \varepsilon p_{\tilde{J}\cup[n-n_{j+1}+k+1,n]}$, in which $\varepsilon = \pm 1$ depends only on $\{n_1, \cdots, n_j\}$. Hence, we have $u_{i,i+1} = \frac{\Delta^{[i]}_{[i-1]\cup\{i+1\}}(z)}{\Delta^{[i]}_{[i]}(z)} = S_i^{(j)}$ and the proof is complete. □

Example 3.19. *When $I^P = I$, we have that $P\backslash G$ is a complete flag variety. In this case, there is no $S_i^{(j)}$-term, and the superpotential $\mathcal{F}_-$ is simply given by*

$$\mathcal{F}_- = \sum_{i=1}^{n-1} p_{\frac{[i-1]\cup\{i+1\}}{p_{[i]}}} + \sum_{i=1}^{n-1} q_i \frac{p_{[n-i,n]\setminus\{n-i+1\}}}{p_{[n-i+1,n]}}.$$

Example 3.20. *When $I^P = \{k\}$, we have that $P\backslash G$ is the complex Grasssmannian $Gr(k,n)$. In this case, there are no $S_i^{(j)}$-terms, and the superpotential $\mathcal{F}_-$ is simply given by*

$$\mathcal{F}_- = \sum_{i=1}^{k-1} \frac{p_{[i-1]\cup\{i+1\}\cup[n-k+i+1,n]}}{p_{[i]\cup[n-k+i+1,n]}} + \sum_{i=k+1}^{n-1} \frac{p_{[i-k+1,i+1]\setminus\{i\}}}{p_{[i-k+1,i]}} + \frac{p_{[k-1]\cup\{k+1\}}}{p_{[k]}} + q_k \frac{p_{[n-k,n]\setminus\{n-k+1\}}}{p_{[n-k+1,n]}}.$$

4. Quantum cohomology of partial flag varieties

The main result of this section will be to show that the image of the superpotential in the Jacobi ring agrees with the the first Chern class under the known isomorphism with quantum cohomology (Section 4.1). This result is Theorem 4.14.

On the A-side, we consider the small quantum cohomology ring of $X = G^\vee/P^\vee$, denoted by $QH^*(X)$. It is an associative and commutative algebra over $\mathbb{C}[q_{n_1}, \cdots, q_{n_r}]$ with a basis given by the Schubert classes σ_v, where q_{n_j} are formal variables.

$$QH^*(X) = \mathbb{C}[q_{n_1}, \cdots, q_{n_r}] \otimes H^*(X, \mathbb{Z})$$

The structure constants are defined through the 3-point, genus-zero Gromov-Witten invariants of X. There is also an enumerative meaning of these constants (see e.g. [FP97]). Recall that the number r occurs in the isomorphism $G^\vee/P^\vee \xrightarrow{\cong} \mathbb{F}\ell(n_\bullet) = \mathbb{F}\ell(n; n_r, \cdots, n_1)$.

A permutation $w \in S_n$ is an element in W^P if and only if $w(n_j + 1) < \cdots < w(n_{j+1})$ for all $0 \leq j \leq r$. In particular, if $w = s_{n_j - i+1} \cdots s_{n_j}$, $1 \leq j \leq r$, $1 \leq i \leq n_j$, then σ_w is called a special Schubert class. The following is a special case of the quantum Pieri rule in [C-Fon99].

Proposition 4.1 (Ciocan-Fontanine)**.** *Let w be a Grassmannian permutation with $w(1) < \cdots < w(m)$ and $w(m+1) < \cdots < w(n)$ for some $m = n_j$, $1 \leq j \leq r$. Then in $QH^*(X)$, we have*

$$\sigma_w \cdot \sigma_{s_{n_j}} = \sum_{\substack{a \leq n_j < b, \\ \ell(w t_{ab}) = \ell(w)+1}} \sigma_{w t_{ab}} + \sum_{\ell(w\tau) = \ell(w) - \ell(\tau)} q_{n_j} \sigma_{w\tau},$$

where t_{ab} is the transposition interchanging a and b, $\tau := s_{n_j} \cdot s_{n_j+1} \times \cdots \times s_{n_{j+1}-1} \cdot s_{n_j-1} \cdot s_{n_j-2} \times \cdots \times s_{n_{j-1}+1}$.

Remark 4.2. *Since w is a Grassmannian permutation, there is at most one quantum term in the expansion of the product $\sigma_w \cdot \sigma_{s_{n_j}}$. Note that the condition $\ell(w\tau) = \ell(w) - \ell(\tau)$ is equivalent to $w(n_j) > w(n_{j+1}), w(n_j + 1) < w(n_{j-1} + 1)$.*

4.1. **Peterson isomorphism.** In this section we state three theorems of D. Peterson, of which the proofs may be found in [Rie03, Section 4].

Definition 4.3. *For $1 \leq i \leq m < n$, we define a rational function G_i^m on G/B_- as follows:*

$$G_i^m(gB_-) := \frac{\Delta_{[m+1,n]}^{\{m-i+1\} \cup [m+2,n]}(g)}{\Delta_{[m+1,n]}^{[m+1,n]}(g)}.$$

Definition 4.4 (Peterson variety)**.** *Let $\mathcal{Y}$ denote the (type A) Peterson variety, which is the projective subvariety of G/B_- cut out by the relation*

$$g^{-1} f g \in \mathfrak{b}_- \oplus \left(\bigoplus_{i \in I} \mathfrak{g}_{\alpha_i} \right) = \left\{ \begin{pmatrix} * & * & 0 & \dots & 0 \\ * & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & * \\ * & \dots & \dots & * & * \end{pmatrix} \right\}, \tag{4.1}$$

where g represents gB_- and f is the principal nilpotent

$$f = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

We set

$$\mathcal{Y}_P := \mathcal{Y} \cap B_+ \dot{w}_P B_-/B_-,$$

and refer to this intersection as the Peterson variety associated to the parabolic subgroup P.

Theorem 4.5 (D. Peterson)**.** *Let $\mathcal{Y}_P$ be the Peterson variety associated to the parabolic subgroup P. There is a unique isomorphism*

$$\mathcal{O}(\mathcal{Y}_P) \xrightarrow{\sim} QH^*(G^\vee/P^\vee),$$
$$G_i^{n_j} \mapsto (-1)^i \sigma_{s_{n_j-i+1}\cdots s_{n_j}}$$

where $1 \le j \le r$, $1 \le i \le n_j$ and $G_i^{n_j}$ is as constructed in Definition 4.3.

Remark 4.6. *The isomorphism we are using differs from the one used in* [Rie03] *by signs.*

Remark 4.7. *The rational function G_i^m is a regular function on the Schubert cell $B_+\dot{w}_P B_-/B_-$ if $m \in I^P$.*

More generally, for a Grassmannian permutation w, with $w(1) < \cdots < w(m)$ and $w(m+1) < \cdots < w(n)$ for some $1 < m < n$, we can define a rational function G_w on G/B_- as follows:

$$G_w(gB_-) := \frac{\Delta^{\{w(m+1),\cdots,w(n)\}}_{[m+1,n]}(g)}{\Delta^{[m+1,n]}_{[m+1,n]}(g)}.$$

We use $G^{\{w(m+1),\cdots,w(n)\}}(gB_-) := G_w(gB_-)$ for short. This will not lead to any misunderstanding since our n is fixed throughout the paper. Then we have the following result, by [Rie03, Prop 11.3].

Proposition 4.8. *If w is a Grassmannian permutation with descent m and $m \in I^P$, then the isomorphism in Theorem 4.5 sends G_w to $(-1)^{\ell(w)}\sigma_w$.*

Theorem 4.9 (D. Peterson)**.** *Let $\mathcal{X}_P := \mathcal{Y}_P \cap (B_-\dot{w}_0 B_-/B_-)$. Then the map in Theorem 4.5 induces an isomorphism between $\mathcal{O}(\mathcal{X}_P)$ and $QH^*(G^\vee/P^\vee)[q_{n_1}^{-1}, \cdots, q_{n_r}^{-1}]$.*

We may also refer to $\mathcal{X}_P$ as Peterson variety. We recall that $\mathcal{X}_P$ can be described using Toeplitz matrices using an idea going back to B. Kostant.

Theorem 4.10 (D. Peterson)**.** *Consider the following variety of lower-triangular Toeplitz matrices given by*

$$X_P := \left\{ b_- = \begin{pmatrix} x_1 & & & \\ x_2 & x_1 & & \\ \vdots & \ddots & \ddots & \\ x_n & \cdots & x_2 & x_1 \end{pmatrix} \mid b_- \in B_+\dot{w}_P\dot{w}_0 B_+ \right\}.$$

The map $X_P \to \mathcal{X}_P$ sending b_- to $b_-\dot{w}_0 B_-$ is an isomorphism.

4.2. **Critical points of the superpotential.** We have the following isomorphism ψ_R which is a version of an isomorphism from [Rie08, Section 4.1],

$$\psi_R : B_- \cap U_+\mathcal{T}^{W_P}\dot{w}_P^{-1}\dot{w}_0 U_+ \xrightarrow{\cong} \mathcal{R}^{R^-}_{w_P,w_0} \times \prod_{i\in I^P} \mathbb{C}^*_q;$$
$$b_- = u_1 t \dot{w}_P^{-1}\dot{w}_0 u_2 \mapsto \psi_R(b_-) = (b_-\dot{w}_0 B_-, (\alpha_{n_i}(t))_{i=1}^r).$$

Define $\mathcal{F}_R$ by

$$\mathcal{F}_R := \mathcal{F}_{\mathrm{Lie}} \circ \psi_R^{-1} : \mathcal{R}^{R^-}_{w_P,w_0} \times \prod_{i\in I^P} \mathbb{C}^*_q \longrightarrow \mathbb{C}.$$

We now consider the ring

$$Jac(\mathcal{F}_R) := \mathcal{O}\left(\mathcal{R}^{R^-}_{w_P,w_0} \times \prod_{i\in I^P} \mathbb{C}^*_q\right) / (\partial_{\mathcal{R}^{R^-}_{w_P,w_0}} \mathcal{F}_R), \tag{4.2}$$

where we are taking partial derivatives of $\mathcal{F}_R$ in the $\mathcal{R}^{R^-}_{w_P,w_0}$ directions, and which we refer to as the (fiberwise) Jacobi ring of $\mathcal{F}_R$. This ring describes the critical points of $\mathcal{F}_R$ along the fibres of the projection $\mathrm{pr}_2 : \mathcal{R}^{R^-}_{w_P,w_0} \times \prod_{i\in I^P} \mathbb{C}^*_q \to \prod_{i\in I^P} \mathbb{C}^*_q$.

The next theorem shows that the Jacobi ring of $\mathcal{F}_R$ is isomorphic to the coordinate ring of the Peterson variety $\mathcal{X}_P$. Namely, Consider the subvariety of $\mathcal{R}^{R^-}_{w_P,w_0} \times \prod_{i\in I^P} \mathbb{C}^*_q$ cut out by the ideal $(\partial_{\mathcal{R}^{R^-}_{w_P,w_0}} \mathcal{F}_R)$ of partial derivatives of $\mathcal{F}_R$ along $\mathcal{R}^{R^-}_{w_P,w_0}$. We denote the corresponding subvariety in $Z_P = B_- \cap U_+ \mathcal{T}^{W_P} \dot{w}_P^{-1} \dot{w}_0 U_+$ by Z_P^{crit}. The theorem stated below is a direct translation of [Rie08, Theorem 4.1] with our notation.

Theorem 4.11. *We have that $Z_P^{crit} = X_P$. Moreover, the subvariety*

$$\psi_R(Z_P^{crit}) \subset \mathcal{R}^{R^-}_{w_P,w_0} \times \prod_{i\in I^P} \mathbb{C}^*_q,$$

which is defined by the ideal $(\partial_{\mathcal{R}^{R^-}_{w_P,w_0}} \mathcal{F}_R)$ is isomorphic to $\mathcal{X}_P$ via the restriction of the first projection $\mathrm{pr}_1 : \mathcal{R}^{R^-}_{w_P,w_0} \times \prod_{i\in I^P} \mathbb{C}^*_q \to \mathcal{R}^{R^-}_{w_P,w_0}$.

Proof. We have the following result proved in [Rie08] that we state for the parabolic subgroup $Q := \dot{w}_0 P \dot{w}_0^{-1}$. Let us set $\tilde{\mathcal{T}}^{W_Q} := \dot{w}_0^{-1} \mathcal{T}^{W_P} \dot{w}_0$. We define

$$\begin{array}{llll} \tilde{\mathcal{F}}_Q : & \tilde{Z}_Q = B_- \cap U_+ \tilde{\mathcal{T}}^{W_Q} \dot{w}_Q \dot{w}_0^{-1} U_+ & \to & \mathbb{C}, \\ & \tilde{b}_- = u_1 \tilde{t} \dot{w}_Q \dot{w}_0^{-1} u_2 & \mapsto & \sum_i e_i^*(u_1) + \sum_i e_i^*(u_2), \end{array}$$

and its restrictions

$$\tilde{\mathcal{F}}_{Q,\tilde{t}} : \tilde{Z}_{Q,\tilde{t}} = B_- \cap U_+ \tilde{t} \dot{w}_Q \dot{w}_0^{-1} U_+ \to \mathbb{C}.$$

Then it is shown in [Rie08] that the critical points of $\tilde{\mathcal{F}}_{Q,\tilde{t}}$ lie in X_Q. Namely,

$$\tilde{Z}^{crit}_{Q,\tilde{t}} := \{\tilde{b}_- \in B_- \cap U_+ \tilde{t} \dot{w}_Q \dot{w}_0^{-1} U_+ \mid \partial \tilde{\mathcal{F}}_{Q,\tilde{t}}(\tilde{b}) = 0\} = X_Q \cap \tilde{Z}_{Q,\tilde{t}},$$

where X_Q is as in Theorem 4.10. Moreover the fiberwise critical point variety $\tilde{Z}_Q^{crit}$ (union over all fibers $\tilde{Z}^{crit}_{Q,\tilde{t}}$) agrees with X_Q.

We can now compare $\tilde{\mathcal{F}}_Q$ with our superpotential $\mathcal{F}_{\mathrm{Lie}}$. We have that

$$\mathcal{F}_{\mathrm{Lie}}(b_- = u_1 t \dot{w}_P^{-1} \dot{w}_0 u_2) = \tilde{\mathcal{F}}_Q(b_-^{-1} = u_2^{-1} \tilde{t} \dot{w}_Q \dot{w}_0^{-1} u_1^{-1})$$

where $\tilde{t} = \dot{w}_0^{-1} \dot{w}_P t^{-1} \dot{w}_P^{-1} \dot{w}_0 = \dot{w}_0^{-1} t^{-1} \dot{w}_0$. Therefore, b_- is a critical point of the analogous restriction $\mathcal{F}_{\mathrm{Lie},t}$ if and only if b_-^{-1} is a critical point of $\tilde{\mathcal{F}}_{Q,\tilde{t}}$ and we have that the critical point variety Z_P^{crit} of $\mathcal{F}_{\mathrm{Lie}}$ is equal to the inverse of X_Q. Finally, the inverse of X_Q is precisely X_P (using that the inverse of a Toeplitz matrix is again Toeplitz). It follows that $\psi_R(Z^{crit}_{\mathrm{Lie}})$ projects to $\mathcal{X}_P$, thanks to Theorem 4.10 □

Corollary 4.12. *The fiberwise Jacobi ring $Jac(\mathcal{F}_R)$ of the superpotential $\mathcal{F}_R$ is isomorphic to the quantum cohomology ring $QH^*(X)[q_{n_1}^{-1}, \cdots, q_{n_r}^{-1}]$.*

Proof. The Jacobi ring is related to $QH^*(X)[q_{n_1}^{-1}, \cdots, q_{n_r}^{-1}]$ by

$$\begin{array}{lll} Jac(\mathcal{F}_R) \xrightarrow{\sim} \mathcal{O}(\mathcal{X}_P) & & \text{(Theorem 4.11)} \\ \qquad \xrightarrow{\sim} QH^*(G^\vee/P^\vee)[q_{n_1}^{-1}, \cdots, q_{n_r}^{-1}] & & \text{(Theorem 4.9).} \end{array}$$

□

We have the following corollary of Theorem 4.11 that we record for use later on. Suppose $\mathbf{q} \in \prod_{i\in I^P} \mathbb{C}_q^*$. Define

$$\mathcal{F}_{\mathbf{q}} : B_- \cap U_+\dot{w}_P^{-1}\dot{w}_0U_+ \to \mathbb{C} \tag{4.3}$$

by $\mathcal{F}_{\mathbf{q}}(\hat{b}) = \mathcal{F}_{\mathrm{Lie}} \circ \gamma^{-1}(\hat{b}, \mathbf{q})$. This is precisely the function from (3.2), but now with quantum parameters fixed.

Corollary 4.13. *If $\hat{b}$ is a critical point of $\mathcal{F}_{\mathbf{q}}$, then $t\hat{b} = \gamma^{-1}(\hat{b}, \mathbf{q})$ is a Toeplitz matrix.*

Proof. In the notation from the proof of Theorem 4.11, we have $\hat{b}$ is a critical point of $\mathcal{F}_{\mathbf{q}}$ if and only if $t\hat{b}$ is a critical point of $\mathcal{F}_{\mathrm{Lie},t}$. But the critical points of $\mathcal{F}_{\mathrm{Lie},t}$ lie in Z_P^{crit}, which equals to X_P by Theorem 4.11. Therefore $t\hat{b}$ is a Toeplitz matrix. □

4.3. **Image of first Chern class.**

The following theorem is the main result in this section, the proof of which is in the end of this subsection.

Theorem 4.14. *Let θ be the isomorphism $Jac(\mathcal{F}_R) \xrightarrow{\sim} QH^*(X)[q_{n_1}^{-1}, \cdots, q_{n_r}^{-1}]$ in Corollary 4.12. Let $[\mathcal{F}_R]$ be the class of superpotential $\mathcal{F}_R$ in the Jacobi ring. Then we have*

$$\theta([\mathcal{F}_R]) = c_1(X).$$

The proof of Theorem 4.14 will occupy the rest of the paper.

Fix $\mathbf{q} \in \prod_{i\in I^P} \mathbb{C}_q^*$ and recall the map $\mathcal{F}_{\mathbf{q}} : B_- \cap U_+\dot{w}_P^{-1}\dot{w}_0U_+ \to \mathbb{C}$ defined in (4.3). Let us consider $\hat{b} \in B_- \cap U_+\dot{w}_P^{-1}\dot{w}_0U_+$ and let z, u, v and b be as in Definition 3.7. We then have

$$b = u\dot{w}_0^{-1}\dot{w}_Pv = uz^{-1}. \tag{4.4}$$

Also recall that $b_- = t\hat{b} = tb^{-1}$, with $t \in \mathcal{T}^{W_P}$ corresponding to $\mathbf{q}$ via (3.1). The following lemmas are key ingredients in the proof of Theorem 4.14.

Lemma 4.15. *Let $n_j \in I^P$. Then $v_{n_j,n_j+1} = -\frac{t_{n_j+1}}{t_{n_j}} G_1^{n_j}(b_-\dot{w}_0B_-)$.*

Proof. Since $u, v \in U_+$ and $\dot{w}_0^{-1}\dot{w}_P = \mathrm{antidiag}\{(-1)^{n_r}I_{a_{r+1}}, \cdots, (-1)^{n_1}I_{a_2}, I_{a_1}\}$, we have

$$\begin{aligned} v_{n_j,n_j+1} &= \frac{\Delta^{[n_j]}_{[n_j-1]\cup\{n_j+1\}}(v)}{\Delta^{[n_j]}_{[n_j]}(v)} \\ &= \frac{\Delta^{[n-n_j+1,n]}_{[n_j-1]\cup\{n_j+1\}}(\dot{w}_0^{-1}\dot{w}_Pv)}{\Delta^{[n-n_j+1,n]}_{[n_j]}(\dot{w}_0^{-1}\dot{w}_Pv)} \\ &= \frac{\Delta^{[n-n_j+1,n]}_{[n_j-1]\cup\{n_j+1\}}(b)}{\Delta^{[n-n_j+1,n]}_{[n_j]}(b)} \end{aligned}$$

Note that for the diagonal entries of t we have that $t_{n_j+1} = t_{n_j+2} = \cdots = t_{n_{j+1}}$, where $0 \leq j \leq r$. Therefore, we have

$$\begin{aligned} v_{n_j,n_j+1} &= \frac{\Delta^{[n-n_j+1,n]}_{[n_j-1]\cup\{n_j+1\}}(b)}{\Delta^{[n-n_j+1,n]}_{[n_j]}(b)} \\ &= \frac{t_{n_j+1}}{t_{n_j}} \frac{\Delta^{[n-n_j+1,n]}_{[n_j-1]\cup\{n_j+1\}}(bt^{-1})}{\Delta^{[n-n_j+1,n]}_{[n_j]}(bt^{-1})} \\ &= -\frac{t_{n_j+1}}{t_{n_j}} \frac{\Delta^{[n_j]\cup[n_j+2,n]}_{[n-n_j]}(tb^{-1})}{\Delta^{[n_j+1,n]}_{[n-n_j]}(tb^{-1})} \\ &= -\frac{t_{n_j+1}}{t_{n_j}} G_1^{n_j}(b_-\dot{w}_0 B_-). \end{aligned}$$

where in the second to last equality one needs to apply Corollary 3.16. □

The situation for $u_{i,i+1}$ is slightly more complicated.

Lemma 4.16. *Let* $1 \leq i < n - n_r$*. Suppose* $\hat{b}$ *is a critical point of* $\mathcal{F}_{\mathbf{q}}$*, then* $u_{i,i+1} = u_{n-n_r,n-n_r+1}$*.*

Proof. Since $u \in U_+$, we have $u_{i,i+1} = \Delta^{[n]\setminus\{i+1\}}_{[n]\setminus\{i\}}(u)$. By Collorary 3.16, we have that $\Delta^{[n]\setminus\{i+1\}}_{[n]\setminus\{i\}}(u) = -\Delta^{\{i\}}_{\{i+1\}}(u^{-1})$. Applying generalized Cramer's rule, see Lemma 3.15, to the equation $b^{-1} = zu^{-1}$, we have $\Delta^{\{i\}}_{\{i+1\}}(u^{-1}) = \det z_{b^{-1}}(\{i\},\{i+1\})$. Now since $\hat{b}$ is a critical point of $\mathcal{F}_{\mathbf{q}}$, we have that $t\hat{b} = tb^{-1}$ is a Toeplitz matrix by Corollary 4.13. Note that $t_{n_r+1} = t_{n_r+2} = \cdots = t_n$. Therefore for $1 \leq i < n - n_r$ we have

$$\begin{aligned} \det z_{b^{-1}}(\{i\},\{i+1\}) &= \det z_{t^{-1}(tb^{-1})}(\{i\},\{i+1\}) \\ &= t_n^{-1} \det z_{(tb^{-1})}(\{i\},\{i+1\}) \\ &= t_n^{-1} \det z_{(tb^{-1})}(\{n-n_r\},\{n-n_r+1\}) \\ &= \det z_{b^{-1}}(\{n-n_r\},\{n-n_r+1\}). \end{aligned}$$

Therefore, we have shown that $u_{i,i+1} = u_{n-n_r,n-n_r+1}$ whenever b is a critical point of $\mathcal{F}_{\mathbf{q}}$ for $1 \leq i < n - n_r$.

□

Lemma 4.17. *Let* $i = n - n_j$ *for some* $1 \leq j \leq r$*. Suppose* $\hat{b}$ *is a critical point of* $\mathcal{F}_{\mathbf{q}}$*, then* $u_{i,i+1} = -G_1^{n_j}(b_-\dot{w}_0 B_-)$*.*

Proof. Since $u, v \in U_+$ and $\dot{w}_0^{-1}\dot{w}_P = \mathrm{antidiag}\{(-1)^{n_r} I_{a_{r+1}}, \cdots, (-1)^{n_1} I_{a_2}, I_{a_1}\}$, we have

$$\begin{aligned} u_{n-n_j,n-n_j+1} &= \frac{\Delta^{\{n-n_j\}\cup[n-n_j+2,n]}_{[n-n_j+1,n]}(u)}{\Delta^{[n-n_j+1,n]}_{[n-n_j+1,n]}(u)} \\ &= \frac{\Delta^{\{n-n_j\}\cup[n-n_j+2,n]}_{[n_j]}(u\dot{w}_0^{-1}\dot{w}_P)}{\Delta^{[n-n_j+1,n]}_{[n_j]}(u\dot{w}_0^{-1}\dot{w}_P)} \\ &= \frac{\Delta^{\{n-n_j\}\cup[n-n_j+2,n]}_{[n_j]}(b)}{\Delta^{[n-n_j+1,n]}_{[n_j]}(b)} \\ &= \frac{\Delta^{\{n-n_j\}\cup[n-n_j+2,n]}_{[n_j]}(bt^{-1})}{\Delta^{[n-n_j+1,n]}_{[n_j]}(bt^{-1})}. \end{aligned}$$

Now since $\hat{b}$ is a critical point of $\mathcal{F}_{\mathbf{q}}$, we have that $t\hat{b} = tb^{-1}$ is a Toeplitz matrix by Corollary 4.13. So that its inverse bt^{-1} is also a Toeplitz matrix. Therefore, we have

$$\frac{\Delta^{\{n-n_j\}\cup[n-n_j+2,n]}_{[n_j]}(bt^{-1})}{\Delta^{[n-n_j+1,n]}_{[n_j]}(bt^{-1})} = \frac{\Delta^{[n-n_j+1,n]}_{[n_j-1]\cup\{n_j+1\}}(bt^{-1})}{\Delta^{[n-n_j+1,n]}_{[n_j]}(bt^{-1})}.$$

Then it follows as in the proof of Lemma 4.15 that $u_{n-n_j,n-n_j+1} = -G_1^{n_j}(b_-\dot{w}_0B_-)$. □

Lemma 4.18. *Let $n - n_1 < i \le n-1$. Suppose $\hat{b}$ is a critical point of $\mathcal{F}_{\mathbf{q}}$, then $u_{i,i+1} = -G_1^{n_1}(b_-\dot{w}_0B_-)$.*

Proof. By Proposition 3.17, we have $u_{i,i+1} = \frac{\Delta^{[i]}_{[i-1]\cup\{i+1\}}(z)}{\Delta^{[i]}_{[i]}(z)}$. Since z is of the form (3.12), if we set $d = i-(n-n_1)$, then $d < \min\{i, n_1\}$ and we have

$$\begin{aligned} \frac{\Delta^{[i]}_{[i-1]\cup\{i+1\}}(z)}{\Delta^{[i]}_{[i]}(z)} &= -\frac{\Delta^{\{d\}\cup\{d+2,i]\}}_{[i-d]}(z)}{\Delta^{[d+1,i]}_{[i-d]}(z)} \\ &= -\frac{\Delta^{\{d\}\cup\{d+2,i]\}}_{[i-d]}(b^{-1})}{\Delta^{[d+1,i]}_{[i-d]}(b^{-1})}. \end{aligned}$$

Recall that $b_- = t\hat{b} = tb^{-1}$. Note that $t_1 = t_2 = \cdots = t_{n_1}$ and $1 \le d < d+1 \le n_1$. So we have

$$\begin{aligned} u_{i,i+1} &= -\frac{\Delta^{\{d\}\cup\{d+2,i]\}}_{[i-d]}(b^{-1})}{\Delta^{[d+1,i]}_{[i-d]}(b^{-1})} \\ &= -\frac{\Delta^{\{d\}\cup[d+2,i]}_{[n-n_1]}(tb^{-1})}{\Delta^{[d+1,i]}_{[n-n_1]}(tb^{-1})} \\ &= -\frac{G^{\{d\}\cup[d+2,i]}(b_-\dot{w}_0B_-)}{G^{[d+1,i]}(b_-\dot{w}_0B_-)} \\ &= -G_1^{n_1}(b_-\dot{w}_0B_-). \end{aligned}$$

The last equality follows from Proposition 4.1 and the isomorphism in Corollary 4.12. □

Lemma 4.19. *Suppose $n - n_{j+1} < i < n - n_j$ for some $1 \le j \le r-1$. Suppose $\hat{b}$ is a critical point of $\mathcal{F}_{\mathbf{q}}$, then*

$$u_{i,i+1} = -(G_1^{n_j}(b_-\dot{w}_0B_-) + G_1^{n_{j+1}}(b_-\dot{w}_0B_-)).$$

We leave the proof of this lemma to the next section. Now we are ready to prove Theorem 4.14 assuming the above lemmas.

Proof of Theorem 4.14. Let $\hat{b} \in B_- \cap U_+\dot{w}_P^{-1}\dot{w}_0U_+$ and let z, u, v and b be as in Definition 3.7. Also recall that $b_- = t\hat{b}$ and $q_{n_j} = \frac{t_{n_j}}{t_{n_j+1}}$. By Equation 3.10, we have $\mathcal{F}_R(b_-\dot{w}_0B_-, \mathbf{q}) = \mathcal{F}_{\mathrm{Lie}}(b_-) = \mathcal{F}_-(Pz, \mathbf{q}) = \sum_{i \in I^P} q_iv_{i,i+1} + \sum_{i=1}^{n-1} u_{i,i+1}$. Suppose that $\hat{b}$ is a critical point of $\mathcal{F}_{\mathbf{q}}$. Then by Corollary 4.12 and the above lemmas, we have

$$\theta([\mathcal{F}_R]) = \sum_{j=1}^{r}(n_{j+1} - n_{j-1})\sigma_{s_{n_j}} = c_1(X) \in QH^*(X).$$

□

5. Proof of Lemma 4.19 and Quantum Schubert calculus

The goal of this section is to prove the most difficult of the lemmas, which is Lemma 4.19.

5.1. **Equivalence of Lemma 4.19 and an identity in quantum Schubert calculus.** We first prove that Lemma 4.19 is equivalent to the identity in quantum Schubert calculus stated in the following theorem.

Definition 5.1. *Recall that we are considering $n - n_{j+1} < i < n - n_j$ for some $1 \le j \le r-1$, and let $d := i - (n - n_{j+1})$. We set*

$$\Xi := \left\{ J \in \binom{[i]}{d} \mid J \cap [n_j + d + 1, n] = \emptyset \right\},$$

and define Weyl group elements $w_J \in W^P$ for certain $J \in \Xi$ as follows. For $J = \{\mathrm{j}_1 < \mathrm{j}_2 < \cdots < \mathrm{j}_d\} \in \Xi$, let $\{x_1 < x_2 < \cdots < x_{i-d}\} := [i] \backslash J$.

(1) *If $n_j \ge d$, then w_J is the following permutation*

$$\{w(1) < \cdots < w(n_j)\} = \{\mathrm{j}_1 < \mathrm{j}_2 < \cdots < \mathrm{j}_d < i+1 < i+2 < \cdots < i + n_j - d\}$$
$$\{w(n_j+1) < \cdots < w(n_{j+1})\} = \{x_1 < i + n_j - d + 1 < i + n_j - d + 2 < \cdots < n-1\}$$
$$\{w(n_{j+1}+1) < \cdots < w(n_{j+2})\} = \{x_2 < \cdots < x_{n_{j+2}-n_{j+1}} < n\}$$
$$\{w(n_{j+2}+1) < \cdots < w(n)\} = \{x_{n_{j+2}-n_{j+1}+1} < \cdots < x_{i-d}\}$$

(2) *If $n_j < d$, and $x_1 < \mathrm{j}_{n_j+1}$ then w_J is defined by*

$$\{w(1) < \cdots < w(n_j)\} = \{\mathrm{j}_1 < \cdots < \mathrm{j}_{n_j}\}$$
$$\{w(n_j+1) < \cdots < w(n_{j+1})\} = \{x_1 < \mathrm{j}_{n_j+1} < \cdots < \mathrm{j}_d < i+1 < \cdots < n-1\}$$
$$\{w(n_{j+1}+1) < \cdots < w(n_{j+2})\} = \{x_2 < \cdots < x_{n_{j+2}-n_{j+1}} < n\}$$
$$\{w(n_{j+2}+1) < \cdots < w(n)\} = \{x_{n_{j+2}-n_{j+1}+1} < \cdots < x_{i-d}\}.$$

Example 5.2. *Suppose $n_1 = 2, n_2 = 4, n = 7$. Let $j = 1$ and $i = 4$ (which indeed satisfies $n - n_{j+1} < i < n - n_j$). Then $d = 1$ and*

$$\Xi = \left\{ J \in \binom{[4]}{1} \mid J \cap [4,7] = \emptyset \right\} = \{\{1\},\{2\},\{3\}\}.$$

Since $n_j = 2 \geq d = 1$ we have a Weyl group element w_J for each $J \in \Xi$. Suppose $J = \{\mathrm{j}_1\}$ and $[4] \setminus J = \{x_1, x_2, x_3\}$. Then the definition of w_J is

$$w_J(1) = \mathrm{j}_1, \quad w_J(2) = 5, \quad w_J(3) = x_1, \quad w_J(4) = 6, \quad w_J(5) = x_2, \quad w_J(6) = x_3, \quad w_J(7) = 7.$$

For our three choices of J this gives the following three Weyl group elements,

$$w_{\{1\}} = 1526347, \quad w_{\{2\}} = 2516347, \quad w_{\{3\}} = 3516247,$$

with descents at 2 and 4, so in W^P.

We can now state our theorem. Note that since $w_J \in W^P$, we have an associated Schubert class σ_{w_J}. If a permutation $w \in W^P$ is Grassmannian, so that it is determined by the values $w(1) < \ldots < w(m)$ up to $m = n_k$ for some k, then we may write $\sigma_{\{w(1),\cdots,w(m)\}}$ for σ_w. Recall that we set $|J| := \sum_{i\in J} i$.

Theorem 5.3. *Consider $X = G^\vee/P^\vee$ and fix i such that $n - n_{j+1} < i < n - n_j$ for some $1 \leq j \leq r-1$. Let $d := i - (n - n_{j+1})$. For each w_J defined above we consider $\sigma_{w_J} \in QH^*(X)$, and we set $\sigma_{w_J} := 0$ for $J \in \Xi$ where w_J is not defined. Then the identity*

$$\sum_{J\in\Xi} (-1)^{|J|} \sigma_{w_J} \sigma_{[1,n_j+d]\setminus J} = 0 \tag{5.1}$$

holds in $QH^*(X)$.

Remark 5.4. *The quantum product $\sigma_{J\cup[i+1,n]} \cdot \sigma_{s_{n_{j+1}}}$ consists of at most one quantum part, say $q_{n_{j+1}}\sigma_{w_J}$. The above definition of w_J is an explicit description of such a class.*

Lemma 5.5. *The formula for $u_{i,i+1}$ in Lemma 4.19 is equivalent to the corresponding identity in Theorem 5.3.*

Proof. We use determinantal identities to rewrite the $u_{i,i+1}$ in Lemma 4.19. By Proposition 3.17, we have $u_{i,i+1} = \frac{\Delta^{[i]}_{[i-1]\cup\{i+1\}}(z)}{\Delta^{[i]}_{[i]}(z)}$. Using Laplace expansion on the first $n - n_{j+1}$ columns, we have

$$\frac{\Delta^{[i]}_{[i-1]\cup\{i+1\}}(z)}{\Delta^{[i]}_{[i]}(z)} = \frac{\sum\limits_{J\in\binom{[i]}{d}} (-1)^{|J|} \Delta^{[i]\setminus J}_{[n-n_{j+1}]}(z) \cdot \Delta^{J}_{[n-n_{j+1}+1,i-1]\cup\{i+1\}}(z)}{\sum\limits_{J\in\binom{[i]}{d}} (-1)^{|J|} \Delta^{[i]\setminus J}_{[n-n_{j+1}]}(z) \cdot \Delta^{J}_{[n-n_{j+1}+1,i]}(z)}$$

where $d := i - (n - n_{j+1})$. Since z is of the form (3.12), we have that the determinant $\Delta^J_{[n-n_{j+1}+1,i-1]\cup\{i+1\}}(z)$ vanishes if $J \cap (\{n_j + d\} \cup [n_j + d + 2, n]) \neq \emptyset$, and $\Delta^J_{[n-n_{j+1}+1,i]}(z)$ vanishes if $J \cap [n_j + d + 1, n] \neq \emptyset$. If we set

$$A := \{J \in \binom{[i]}{d} | J \cap (\{n_j + d\} \cup [n_j + d + 2, n]) = \emptyset\}$$

and $\Xi = \{J \in \binom{[i]}{d} | J \cap [n_j+d+1, n] = \emptyset\}$ as already defined, then we have

$$\frac{\sum_{J\in\binom{[i]}{d}} (-1)^{|J|}\Delta^{[i]\setminus J}_{[n-n_{j+1}]}(z)\cdot \Delta^{J}_{[n-n_{j+1}+1,i-1]\cup\{i+1\}}(z)}{\sum_{J\in\binom{[i]}{d}} (-1)^{|J|}\Delta^{[i]\setminus J}_{[n-n_{j+1}]}(z)\cdot \Delta^{J}_{[n-n_{j+1}+1,i]}(z)}$$
$$= -\frac{\sum_{J\in A} \eta(J)(-1)^{|J|}\Delta^{[i]\setminus J}_{[n-n_{j+1}]}(z)\cdot \Delta^{J\cup\{n_j+d\}\cup[n_j+d+2,n]}_{[n-n_j]}(z)}{\sum_{J\in \Xi} (-1)^{|J|}\Delta^{[i]\setminus J}_{[n-n_{j+1}]}(z)\cdot \Delta^{J\cup[n_j+d+1,n]}_{[n-n_j]}(z)}$$

in which $\eta(J)$ is the function

$$\eta(J) = \begin{cases} 1, & \text{if } n_j+d+1 \notin J, \\ -1, & \text{if } n_j+d+1 \in J. \end{cases}$$

Since $b^{-1} = zu^{-1}$ and $u \in U_+$, we have

$$\frac{\sum_{J\in A} \eta(J)(-1)^{|J|}\Delta^{[i]\setminus J}_{[n-n_{j+1}]}(z)\cdot \Delta^{J\cup\{n_j+d\}\cup[n_j+d+2,n]}_{[n-n_j]}(z)}{\sum_{J\in \Xi} (-1)^{|J|}\Delta^{[i]\setminus J}_{[n-n_{j+1}]}(z)\cdot \Delta^{J\cup[n_j+d+1,n]}_{[n-n_j]}(z)}$$
$$= \frac{\sum_{J\in A} \eta(J)(-1)^{|J|}\Delta^{[i]\setminus J}_{[n-n_{j+1}]}(b^{-1})\cdot \Delta^{J\cup\{n_j+d\}\cup[n_j+d+2,n]}_{[n-n_j]}(b^{-1})}{\sum_{J\in \Xi} (-1)^{|J|}\Delta^{[i]\setminus J}_{[n-n_{j+1}]}(b^{-1})\cdot \Delta^{J\cup[n_j+d+1,n]}_{[n-n_j]}(b^{-1})}$$

We recall that $b_- = t\hat{b} = tb^{-1}$. Note that $t_{n_j+1} = t_{n_j+2} = \cdots = t_{n_{j+1}}$ for all $0 \le j \le r$. Since $n_j < n_j+d < n_j+d+1 \le n_{j+1}$, we have

$$u_{i,i+1} = -\frac{\sum_{J\in A} \eta(J)(-1)^{|J|}\Delta^{[i]\setminus J}_{[n-n_{j+1}]}(b^{-1})\cdot \Delta^{J\cup\{n_j+d\}\cup[n_j+d+2,n]}_{[n-n_j]}(b^{-1})}{\sum_{J\in \Xi} (-1)^{|J|}\Delta^{[i]\setminus J}_{[n-n_{j+1}]}(b^{-1})\cdot \Delta^{J\cup[n_j+d+1,n]}_{[n-n_j]}(b^{-1})}$$
$$= -\frac{\sum_{J\in A} \eta(J)(-1)^{|J|}\Delta^{[i]\setminus J}_{[n-n_{j+1}]}(tb^{-1})\cdot \Delta^{J\cup\{n_j+d\}\cup[n_j+d+2,n]}_{[n-n_j]}(tb^{-1})}{\sum_{J\in \Xi} (-1)^{|J|}\Delta^{[i]\setminus J}_{[n-n_{j+1}]}(tb^{-1})\cdot \Delta^{J\cup[n_j+d+1,n]}_{[n-n_j]}(tb^{-1})}$$
$$= -\frac{\sum_{J\in A} \eta(J)(-1)^{|J|}G^{[i]\setminus J}(b_-\dot{w}_0B_-)\cdot G^{J\cup\{n_j+d\}\cup[n_j+d+2,n]}(b_-\dot{w}_0B_-)}{\sum_{J\in \Xi} (-1)^{|J|}G^{[i]\setminus J}(b_-\dot{w}_0B_-)\cdot G^{J\cup[n_j+d+1,n]}(b_-\dot{w}_0B_-)}$$

Applying the isomorphism in Corollary 4.12, we see that the formula for $u_{i,i+1}$ in Lemma 4.19 is equivalent to the following identity in $QH^*(X)$.

$$\sum_{J\in\Xi}(-1)^{|J|}(\sigma_{J\cup[i+1,n]}\cdot\sigma_{s_{n_{j+1}}})\cdot\sigma_{[1,n_j+d]\setminus J} + \sum_{J\in\Xi}(-1)^{|J|}\sigma_{J\cup[i+1,n]}\cdot(\sigma_{[1,n_j+d]\setminus J}\cdot\sigma_{s_{n_j}})$$
$$= \sum_{J\in A}\eta(J)(-1)^{|J|}\sigma_{J\cup[i+1,n]}\cdot\sigma_{([1,n_j+d-1]\cup\{n_j+d+1\})\setminus J}. \tag{5.2}$$

It remains to show that this is exactly the identity in Theorem 5.3.

Assume that $J=\{\mathrm{j}_1<\mathrm{j}_2<\cdots<\mathrm{j}_d\}$, then by Proposition 4.1,

$$\sigma_{J\cup[i+1,n]}\cdot\sigma_{s_{n_{j+1}}}=\sum_{1\le s\le d}\sigma_{\{\mathrm{j}_1,\cdots,\mathrm{j}_{s-1},\mathrm{j}_s+1,\mathrm{j}_{s+1},\cdots,\mathrm{j}_d,i+1,\cdots,n\}}+q_{n_{j+1}}\sigma_{w_J}.$$

Here, we set $\sigma_{\{\mathrm{j}_1,\cdots,\mathrm{j}_{s-1},\mathrm{j}_s+1,\mathrm{j}_{s+1},\cdots,\mathrm{j}_d,i,\cdots,n\}}:=0$ if either $s=d$ and $\mathrm{j}_d=i$ or $\mathrm{j}_s+1=\mathrm{j}_{s+1}$ holds. We divide the above sum into two parts as follows

$$\begin{aligned}C_1(J)&:=\sum_{1\le s\le d-1}\sigma_{\{\mathrm{j}_1,\cdots,\mathrm{j}_{s-1},\mathrm{j}_s+1,\mathrm{j}_{s+1},\cdots,\mathrm{j}_d,i+1,\cdots,n\}}\\ C_2(J)&:=\sigma_{\{\mathrm{j}_1,\cdots,\mathrm{j}_{d-1},\mathrm{j}_d+1,i+1,\cdots,n\}}\end{aligned}$$

Similarly, we have

$$\begin{aligned}\sigma_{[1,n_j+d]\setminus J}\cdot\sigma_{s_{n_j}}&=\sum_{1\le j\le d}\sigma_{[1,n_j+d]\setminus\{\mathrm{j}_1,\cdots,\mathrm{j}_{s-1},\mathrm{j}_s-1,\mathrm{j}_{s+1},\cdots,\mathrm{j}_d\}}+D_2(J)\\ &=D_1(J)+D_2(J)\end{aligned}$$

Here, $D_1(J)$ is defined as

$$D_1(J):=\sum_{1\le j\le d}\sigma_{[1,n_j+d]\setminus\{\mathrm{j}_1,\cdots,\mathrm{j}_{s-1},\mathrm{j}_s-1,\mathrm{j}_{s+1},\cdots,\mathrm{j}_d\}}$$

where we set $\sigma_{[1,n_j+d]\setminus\{\mathrm{j}_1,\cdots,\mathrm{j}_{s-1},\mathrm{j}_s-1,\mathrm{j}_{s+1},\cdots,\mathrm{j}_d\}}:=0$ if either $s=1$ and $\mathrm{j}_1=1$ or $\mathrm{j}_s-1=\mathrm{j}_{s-1}$ holds. And $D_2(J)$ is defined as follows

$$D_2(J):=\begin{cases}\sigma_{[1,n_j+d-1]\cup\{n_j+d+1\}\setminus J}, & \text{if } n_j+d\notin J,\\ 0, & \text{if } n_j+d\in J.\end{cases}$$

Note that since $n_{j+1}>n_j+d$, we have $w(n_{j+1})>w(n_j)$ and therefore there are no quantum terms in the product $\sigma_{[1,n_j+d]\setminus J}\cdot\sigma_{s_{n_j}}$ by the remark after Proposition 4.1.
If $n_j+d\in J$, namely, $\mathrm{j}_d=n_j+d$, then directly from the definition of A and Ξ we have

$$\sum_{\substack{J\in\Xi\\ n_j+d\in J}}(-1)^{|J|}C_2(J)\cdot\sigma_{[1,n_j+d]\setminus J}=\sum_{\substack{J\in A\\ n_j+d+1\in J}}\eta(J)(-1)^{|J|}\sigma_{J\cup[i+1,n]}\cdot\sigma_{([1,n_j+d-1]\cup\{n_j+d+1\})\setminus J}.$$

If $n_j+d\notin J$, then we have

$$\sum_{\substack{J\in\Xi\\ n_j+d\notin J}}(-1)^{|J|}\sigma_{J\cup[i+1,n]}\cdot D_2(J)=\sum_{\substack{J\in A\\ n_j+d+1\notin J}}\eta(J)(-1)^{|J|}\sigma_{J\cup[i+1,n]}\cdot\sigma_{([1,n_j+d-1]\cup\{n_j+d+1\})\setminus J}.$$

Moreover, for $J=\{\mathrm{j}_1<\mathrm{j}_2<\cdots<\mathrm{j}_d\}$, we denote $J_s^+:=\{\mathrm{j}_1,\cdots,\mathrm{j}_{s-1},\mathrm{j}_s+1,\mathrm{j}_{s+1},\cdots,\mathrm{j}_d\}$ and $J_s^-:=\{\mathrm{j}_1,\cdots,\mathrm{j}_{s-1},\mathrm{j}_s-1,\mathrm{j}_{s+1},\cdots,\mathrm{j}_d\}$. Then

$$\sum_{J\in\Xi}(-1)^{|J|}\sigma_{J\cup[i+1,n]}\cdot D_1(J)=\sum_{J\in\Xi}\sum_{s=1}^{d}(-1)^{|J|}\sigma_{J\cup[i+1,n]}\cdot\sigma_{[1,n_j+d]\setminus J_s^-}.$$

Since $\sigma_{[1,n_j+d]\setminus J_s^-}\neq 0$ only if $J_s^-\in J$, in which case $(J_s^-)_s^+=J\in\Xi$, we have

$$\begin{aligned}\sum_{J\in\Xi}\sum_{s=1}^{d}(-1)^{|J|}\sigma_{J\cup[i+1,n]}\cdot\sigma_{[1,n_j+d]\setminus J_s^-}=&\sum_{J\in\Xi}\sum_{s=1}^{d-1}(-1)^{|J|+1}\sigma_{J_s^+\cup[i+1,n]}\cdot\sigma_{[1,n_j+d]\setminus J}+\\ &\sum_{\substack{J\in\Xi\\ n_j+d\notin J}}(-1)^{|J|+1}\sigma_{J_d^+\cup[i+1,n]}\cdot\sigma_{[1,n_j+d]\setminus J}.\end{aligned}$$

Therefore, we have

$$\sum_{J\in\Xi}(-1)^{|J|}C_1(J)\cdot\sigma_{[1,n_j+d]\setminus J}+\sum_{\substack{J\in\Xi\\ n_j+d\notin J}}(-1)^{|J|}C_2(J)\cdot\sigma_{[1,n_j+d]\setminus J}+\sum_{J\in\Xi}(-1)^{|J|}\sigma_{J\cup[i+1,n]}\cdot D_1(J)=0.$$

We therefore see that the identity (5.2) is equivalent to the identity

$$\sum_{J\in\Xi}(-1)^{|J|}\sigma_{w_J}\sigma_{[1,n_j+d]\setminus J}=0$$

in Theorem 5.3. Hence, the statement follows. □

The structure of the proof of Lemma 4.19 and Theorem 5.3 is now the following. We will first prove Theorem 5.3 in the special case where $n_j+n_{j+1}\leq n$. It then follows that Lemma 4.19 holds whenever $n_j+n_{j+1}\leq n$, because of Lemma 5.5. Next, we introduce a symmetry on the domain of the superpotential $\mathcal{F}_R$ that allows us to deduce the statement of Lemma 4.19 for $n_j+n_{j+1}\geq n$ from the one for $n_j+n_{j+1}\leq n$. Finally, we obtain Theorem 5.3 for $n_j+n_{j+1}\leq n$, since this proposition and Lemma 4.19 are equivalent in every case. This strategy also shows an interaction between mirror symmetry and quantum Schubert calculus.

5.2. **A special case of Theorem 5.3.** In this section we prove Theorem 5.3 in the case where $n_j+n_{j+1}\leq n$. To do this we first prove a version of the identity (5.1) in the quantum cohomology of the complete flag variety $\mathbb{F}\ell_n$ in the following key lemma.

Lemma 5.6. *Assume that $n_j+n_{j+1}\leq n$. With notations as in Theorem 5.3, the following identity, obtained simply by replacing the Schubert classes in* (5.1) *with corresponding ones for $\mathbb{F}\ell_n$, holds in the quantum cohomology ring $QH^*(\mathbb{F}\ell_n)$.*

$$\sum_{J\in\Xi}(-1)^{|J|}\sigma^B_{w_J}\sigma^B_{[1,n_j+d]\setminus J}=0.$$

To prove the above lemma, we need some preparation. Recall that a permutation w is called $321-$*avoiding* if there does not exist $i<j<k$ such that $w(i)>w(j)>w(k)$. The key observation in the proof of Lemma 5.6 is the following lemma, that the permutations w_J arising above are all $321-$avoiding.

Lemma 5.7. *The permutations w_J constructed in Definition 5.1 are $321-$avoiding.*

Proof. We will argue by contradiction. Consider the case $n_j\geq d$ first. Suppose that there exists $i_0<j_0<k_0$ satisfying $w_J(i_0)>w_J(j_0)>w_J(k_0)$. Since $i_0<j_0$ and $w_J(i_0)>w_J(j_0)$, we must have $n_j+1\leq j_0$. If we assume that $j_0\leq n_{j+1}$, then we must have $j_0=n_j+1$ since $w_J(i_0)>w_J(j_0)$. So we have $w_J(j_0)=x_1$. However, this is in contradiction with $j_0<k_0$ and $w_J(j_0)>w_J(k_0)$. Therefore we must have $j_0>n_{j+1}$. Since $j_0<k_0$ and $w_J(j_0)>w_J(k_0)$, we have $j_0=n_{j+2}$ and $w_J(j_0)=n$. But this is in contradiction with $w_J(i_0)>w_J(j_0)$. In conclusion, for the case $n_j\geq d$, w_J is a $321-$avoiding permutation. The case $n_j<d$ can be proved similarly. □

Remark 5.8. *For example, the identity in Lemma 5.6 coming from $\mathbb{F}\ell(7;4,2)$, where $n_j=2, n_{j+1}=4$ and $i=4, d=1$, is*

$$\sigma^B_{1526347}\cdot\sigma^B_{2314567}-\sigma^B_{2516347}\cdot\sigma^B_{1324567}+\sigma^B_{3516247}\cdot\sigma^B_{1234567}=0,$$

and it involves only 321*-avoiding permutations.*

Definition 5.9. *Let $w \in S_n$ be a permutation, then the code of w is defined as*

$$c(w) = (c_1, c_2, \cdots, c_n)$$

where $c_i := \sharp\{j | i < j, w(j) < w(i)\}$.

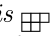
Definition 5.10. *Let w be a $321-$avoiding permutation with code $c(w) = (c_1, \cdots, c_n)$. The flag of the partition is defined as $\phi(w) = \{j_1 < j_2 < \cdots < j_l\} := \{j | c_j > 0\}$. We define a skew partition λ/μ by embedding it into $\mathbb{Z} \times \mathbb{Z}$ as follows:*

$$\lambda_k - \mu_k = c_{j_k}$$
$$\lambda/\mu = \{(k, h) : 1 \leq k \leq l, k - j_k - c_{j_k} + 1 \leq h \leq k - j_k\}$$

Example 5.11. *We continue with the example from Remark 5.8.*

(1) *If $w = 1526347$, then the code of w is $c(w) = (0, 3, 0, 2, 0, 0, 0)$. And we have $\{j_1 < j_2\} = \{2 < 4\}$ with $l = 2$. Then we have*

$$k = 1, 1 - 2 - 3 + 1 \leq h \leq 1 - 2$$
$$k = 2, 2 - 4 - 2 + 1 \leq h \leq 2 - 4$$

Therefore, the skew partition is  *with $\lambda = (3, 2)$ and $\mu = (0, 0)$.*

(2) *If $w = 2516347$, then the code of w is $c(w) = (1, 3, 0, 2, 0, 0, 0)$ with flag $\phi(w) = \{1, 2, 4\}$. Then*

$$k = 1, 1 - 1 - 1 + 1 \leq h \leq 1 - 1$$
$$k = 2, 2 - 2 - 3 + 1 \leq h \leq 2 - 2$$
$$k = 3, 3 - 4 - 2 + 1 \leq h \leq 3 - 4$$

Therefore, the skew partition is $\lambda/\mu =$  *with $\lambda = (3, 3, 2)$ and $\mu = (2, 0, 0)$.*

(3) *If $w = 3516247$, then the code of w is $c(w) = (2, 3, 0, 2, 0, 0, 0)$ with flag $\phi(w) = \{1, 2, 4\}$. Then*

$$k = 1, 1 - 1 - 2 + 1 \leq h \leq 1 - 1$$
$$k = 2, 2 - 2 - 3 + 1 \leq h \leq 2 - 2$$
$$k = 3, 3 - 4 - 2 + 2 + 1 \leq h \leq 3 - 4$$

Therefore, the skew partition is $\lambda/\mu =$  *with $\lambda = (3, 3, 2)$ and $\mu = (1, 0, 0)$.*

In [BJS93], Schubert polynomial $\mathfrak{S}_w$ is explicitly written down in a determinantal formula for a $321-$avoiding permutation as follows.

Theorem 5.12 (Corollary 2.3 of [BJS93])**.** *Let w be a $321-$avoiding permutation with flag $\phi(w) = (\phi_1 < \cdots < \phi_k)$ and skew partition λ/μ. Let $X_i = (x_1, x_2, \cdots, x_i)$. Then we have*

$$\mathfrak{S}_w = \det(h_{\lambda_i - \mu_j - i + j}(X_{\phi_i}))_{1 \leq i, j \leq k}$$

where $h_r(X_i)$ is the complete homogeneous symmetric polynomial of degree r in variables X_i.

In [Kirillov], A.N. Kirillov defines quantum Schubert polynomials and conjectures that the quantum version of the above determinantal formula holds as well, see [Kirillov, Conjecture 1]. We will verify this conjecture in the quantum cohomology ring of the complete flag variety $\mathbb{F}\ell_n$ using the work of [FGP97]. This should have been known to the experts (see e.g. [CK23, formula (6)]). Since we are not aware of this formula appearing in form of a theorem, we state it here as Theorem 5.20 and provide a detailed argument for completeness.

Definition 5.13. *Let G_k be the matrix*

$$\begin{pmatrix} x_1 & q_1 & 0 & \cdots & 0 \\ -1 & x_2 & q_2 & \cdots & 0 \\ 0 & -1 & x_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_k \end{pmatrix}$$

The quantum elementary polynomial E_i^k is defined by the following formula

$$\det(1 + \lambda G_k) = \sum_{i=0}^{k} E_i^k \lambda^i$$

And we set $E_i^k = 0$ if $i < 0$ or $i > k$.

By setting $q_1 = q_2 = \cdots = q_{k-1} = 0$, E_i^k recovers the ordinary elementary symmetric polynomial $e_i^k = e_i^k(x_1, \cdots, x_k)$. Let $e_{i_1 \cdots i_m} := e_{i_1}^1 \cdots e_{i_m}^m$ be standard elementary monomial. The following lemma is a classical result, and can be found in [Macdonald].

Lemma 5.14. *Let I_n be the ideal in $\mathbb{Z}[x_1, \cdots, x_n]$ generated by $e_1^n, \cdots, e_n^n$, then each of the following forms a $\mathbb{Z}-$basis in $\mathbb{Z}[x_1, \cdots, x_n]/I_n$:*

1. *the standard elementary monomials $e_{i_1 \cdots i_{n-1}}$, with $0 \leq i_k \leq k$;*
2. *the Schubert polynomials $\mathfrak{S}_w$ for $w \in S_n$.*

Moreover, each of these families spans the same vector space $L_n \subset \mathbb{Z}[x_1, \cdots, x_n]$ which is complementary to I_n.

Therefore, any Schubert polynomial $\mathfrak{S}_w$ is uniquely a linear combination of standard elementary monomials with integer coefficients. In [FGP97], the quantum Schubert polynomial is defined as the linear combination of the quantum elementary monomials $E_{i_1 \cdots i_m} := E_{i_1}^1 \cdots E_{i_m}^m$ with the same coefficients. Namely, we have

Definition 5.15. *The quantum Schubert polynomial $\mathfrak{S}_w^q$ for a permutation $w \in S_n$ is defined as*

$$\mathfrak{S}_w^q = \sum \alpha_{i_1 \cdots i_{n-1}} E_{i_1 \cdots i_{n-1}}$$

where the coefficients $\alpha_{i_1 \cdots i_{n-1}}$ are the same as the coefficients found in the classical expansion $\mathfrak{S}_w = \sum \alpha_{i_1 \cdots i_{n-1}} e_{i_1 \cdots i_{n-1}}$.

We recall the quantum analogue of Lemma 5.14 proved in [FGP97].

Lemma 5.16. *Let I_n^q be the ideal in $\mathbb{Z}[q_1, \cdots, q_{n-1}][x_1, \cdots, x_n]$ generated by $E_1^n, \cdots, E_n^n$, then each of the following determines a $\mathbb{Z}[q]-$basis in $\mathbb{Z}[q, x]/I_n^q$:*

1. *the quantum standard elementary monomials $E_{i_1 \cdots i_{n-1}}$, with $0 \leq i_k \leq k$;*
2. *the quantum Schubert polynomials $\mathfrak{S}_w^q$ for $w \in S_n$.*

Moreover, each of these families spans the same vector space $L_n^q \subset \mathbb{Z}[q, x]$ which is complementary to I_n^q.

One of the main results in [FGP97] is the following

Theorem 5.17 (Theorem 1.2 of [FGP97])**.** *The map*

$$\pi : \mathbb{Z}[q_1, \cdots, q_{n-1}][x_1, \cdots, x_n] \longrightarrow QH^*(\mathbb{F}\ell_n)$$

sending $x_1 + \cdots + x_i$ to $\sigma_{s_i}^B \in QH^*(\mathbb{F}\ell_n)$ is a surjective ring homomorphism with kernel I_n^q generated by $E_1^n, \cdots, E_n^n$. Under the induced isomorphism $\mathbb{Z}[q, x]/I_n^q \cong QH^*(\mathbb{F}\ell_n)$, the coset of the quantum Schubert polynomial $\mathfrak{S}_w^q$ is sent to the corresponding quantum Schubert class σ_w^B.

Now we are ready to prove the quantum version of the determinantal formula for a 321−avoiding permutation.

Definition 5.18. *We call* $H_l^k := \det(E_{j-i+1}^{k+l-1})_{1\le i,j\le l}$ *the quantum complete homogeneous polynomial in* k *variables of degree* l*. Set* $H_{i_1,\cdots,i_{n-1}} := H_{i_1}^1 \cdots H_{i_{n-1}}^{n-1}$*.*

Remark 5.19. $H_{i_k}^k \in I_n^q$ *if* $i_k > n-k$*.*

Theorem 5.20. *Let* w *be a* 321−*avoiding permutation with flag* $\phi(w) = (\phi_1 < \cdots < \phi_k)$ *and skew partition* λ/μ*. Let* $X_i = (x_1, x_2, \cdots, x_i)$*. Then in* $\mathbb{Z}[q,x]/I_n^q$ *we have*

$$\mathfrak{S}_w^q = \det(H_{\lambda_i-\mu_j-i+j}(X_{\phi_i}))_{1\le i,j\le k}.$$

Proof. We consider the involution ω of $\mathbb{Z}[q_1,\cdots,q_{n-1}][x_1,\cdots,x_n]$ defined by $\omega(x_k) = -x_{n+1-k}$ and $\omega(q_k) = q_{n-k}$, for $1 \le k \le n$. According to [FGP97], I_n^q is an invariant subspace for the involution ω. Therefore ω induces an automorphism on $\mathbb{Z}[q,x]/I_n^q$. Moreover, we have

$$\omega(E_{i_1\cdots i_{n-1}}) = H_{i_{n-1}\cdots i_1};\quad \omega(H_{i_1\cdots i_{n-1}}) = E_{i_{n-1}\cdots i_1};\quad \omega(\mathfrak{S}_w^q) = \mathfrak{S}_{w_0ww_0}^q.$$

Therefore it suffices to show

$$\mathfrak{S}_{w_0ww_0}^q = \det(E_{\lambda_i-\mu_j-i+j}(X_{n-\phi_i}))_{1\le i,j\le k}$$

Note that the right hand side of the equality is a linear combination of quantum standard elementary monomials by the definition of determinants. Then by Lemma 5.16 it suffices to show that the coefficient of any standard elementary monomial $E_{i_1,\cdots,i_{n-1}}$ with $0 \le i_k \le k$ on the right hand side is the same as in the definition of the quantum Schubert polynomial. But by Definition 5.15, it suffices to show this in the classical case. However, by applying involution to Theorem 5.12, we have the following equality in $\mathbb{Z}[x_1,\cdots,x_n]/I_n$

$$\mathfrak{S}_{w_0ww_0} = \det(e_{\lambda_i-\mu_j-i+j}(X_{n-\phi_i}))_{1\le i,j\le k}$$

Since the right hand side is a linear combination of $e_{i_1,\cdots,i_{n-1}}$ and the standard elementary monomials $e_{i_1,\cdots,i_{n-1}}$ with $0 \le i_k \le k$ span a vector space complementary to I_n, by discarding the other monomials in the expansion of the determinant, we get the formula $\mathfrak{S}_{w_0ww_0} = \sum_{0\le i_k\le k} \alpha_{i_1\cdots i_{n-1}} e_{i_1\cdots i_{n-1}}$ as wanted. □

Remark 5.21. *In the proof we used the involution* ω*, therefore we are only able to prove the identity* $\mathfrak{S}_w^q = \det(H_{\lambda_i-\mu_j-i+j}(X_{\phi_i}))_{1\le i,j\le k}$ *in the quotient ring* $\mathbb{Z}[q,x]/I_n^q$*. However, the original conjectural identity in* [Kirillov] *is stated in the ring* $\mathbb{Z}[q,x]$*.*

We now use this theorem to prove Lemma 5.6.

Proof of Lemma 5.6. Using the isomorphism $\mathbb{Z}[q,x]/I_n^q \cong QH^*(\mathbb{F}\ell_n)$, we may identify $\mathfrak{S}_w^q$ with σ_w^B, and treat $H_r(X_i)$ as an element in $QH^*(\mathbb{F}\ell_n)$. Also we use $\times$ for the multiplication. Since w_J is a 321−avoiding permutation by Lemma 5.7, we are able to apply Theorem 5.20. The proof is divided into two cases: $n_j \ge d$ and $n_j < d$.

(1) Consider the case $n_j \ge d$ first. Then for $J = \{\mathrm{j}_1 < \cdots < \mathrm{j}_d\} \in \Xi = \{J \in \binom{[i]}{d} | J \cap [n_j + d+1, n] = \emptyset\}$, let $\{x_1 < x_2 < \cdots < x_{i-d}\} := [i]\backslash J$, w_J is the following permutation

$$\{w(1) < \cdots < w(n_j)\} = \{\mathrm{j}_1 < \mathrm{j}_2 < \cdots < \mathrm{j}_d < i+1 < i+2 < \cdots < i+n_j-d\}$$
$$\{w(n_j+1) < \cdots < w(n_{j+1})\} = \{x_1 < i+n_j-d+1 < i+n_j-d+2 < \cdots < n-1\}$$
$$\{w(n_{j+1}+1) < \cdots < w(n_{j+2})\} = \{x_2 < \cdots < x_{n_{j+2}-n_{j+1}} < n\}$$
$$\{w(n_{j+2}+1) < \cdots < w(n)\} = \{x_{n_{j+2}-n_{j+1}+1} < \cdots < x_{i-d}\}.$$

The code of w_J is $c(w_J) = (\mathrm{j}_1 - 1, \mathrm{j}_2 - 2, \cdots, \mathrm{j}_d - d, i-d, i-d, \cdots, i-d, 0, i-d-1, \cdots, i-d-1, 0, \cdots, 0, n-n_{j+2}, 0, \cdots, 0)$ with flag $\phi(w_J) = (1, 2, \cdots, n_j, n_j+2, n_j+3, \cdots, , n_{j+1}, n_{j+2})$. Then it determines a skew partition λ/μ, where

$$\lambda = (i-d, \cdots, i-d, i-d-1, \cdots, i-d-1, n-n_{j+2}) \text{ with } n_j \text{ many } i-d \text{ and } n_{j+1}-n_j-1 \text{ many } i-d-1;$$
$$\mu = (i-d-(\mathrm{j}_1-1), i-d-(\mathrm{j}_2-2), \cdots, i-d-(\mathrm{j}_d-d), 0, \cdots, 0).$$

Then by Theorem 5.20, we have $\sigma^B_{w_J} = \det(H_{\lambda_r-\mu_s-r+s}(X_{\phi_r}))_{1\le r,s\le n_{j+1}}$. Here we do assume $n_{j+2} < n$, the case $n_{j+2} = n$ can be dealt with similarly. We are going to use Laplace expansion on the first d columns of this determinant. Let $R = (r_1 < \cdots < r_d) \in \binom{[n_{j+1}]}{d}$ be row index for the expansion, and denote M_R for the cofactor (with sign) obtained by removing the first d columns and rows indexed by R. Then Laplace expansion says that

$$\det(H_{\lambda_r-\mu_s-r+s}(X_{\phi_r}))_{1\le r,s\le n_{j+1}} = \sum_R M_R \times \det(H_{\lambda_r-\mu_s-r+s}(X_{\phi_r}))_{1\le s\le d, r\in R}.$$

We observe that M_R is independent of J since it involves only the last $n_{j+1}-d$ columns of $\det(H_{\lambda_r-\mu_s-r+s}(X_{\phi_r}))_{1\le r,s\le n_{j+1}}$ and only the first d items of μ depend on J. Therefore, we have

$$\begin{aligned}&\sum_{J\in\Xi}(-1)^{|J|}\sigma^B_{w_J}\sigma^B_{[1,n_j+d]\setminus J}\\ =&\sum_{J\in\Xi}\sum_R(-1)^{|J|}M_R\times\det(H_{\lambda_r-\mu_s-r+s}(X_{\phi_r}))_{1\le s\le d,r\in R}\times\sigma^B_{[1,n_j+d]\setminus J}\\ =&\sum_R M_R\sum_{J\in\Xi}(-1)^{|J|}\det(H_{\lambda_r-\mu_s-r+s}(X_{\phi_r}))_{1\le s\le d,r\in R}\times\sigma^B_{[1,n_j+d]\setminus J}.\end{aligned}$$

The Schubert class $\sigma^B_{[1,n_j+d]\setminus J}$ is indexed by a Grassmannian permutation, which in particular is a 321−avoiding permutation. Let $\alpha = (\alpha_1, \cdots, \alpha_{n_j})$ be the corresponding partition such that $J \cup \{\alpha_1 + n_j, \cdots, \alpha_{n_j} + 1\} = [1, n_j + d]$. Then we have

$$\sigma^B_{[1,n_j+d]\setminus J} = \det(H_{\alpha_a-a+b}(X_{n_j+1-b}))_{1\le a,b\le n_j}.$$

We will construct an $(n_j + d) \times (n_j + d)$ matrix A_R. We define the first d row vectors of A_R to be

$$(H_{\lambda_r-r-i+d+1}(X_{\phi_r}), H_{\lambda_r-r-i+d+2}(X_{\phi_r}), \cdots, H_{\lambda_r-r-i+d+n_j+d}(X_{\phi_r}))$$

where r runs through $R = (r_1 < \cdots < r_d)$. And we define the last n_j row vectors of A_R to be

$$(H_{1-n_j-1+b}(X_{n_j+1-b}), H_{2-n_j-1+b}(X_{n_j+1-b}), \cdots, H_{n_j+d-n_j-1+b}(X_{n_j+1-b}))$$

where b run through $[1, n_j]$.

Next we show that $\det A_R = 0$. We will prove this by showing that either A_R contains two identical row vectors or A_R contains a zero row vector. We observe that $\lambda_r - r - i + d + \phi_r = 0$, therefore, in order to show that A_R contains two identical row vectors it suffices to prove that $\phi_r = n_j + 1 - b$ for some $r \in R$ and $b \in [1, n_j]$, namely, $R \cap [1, n_j] \neq \emptyset$. Now suppose we have the opposite, namely $R \cap [1, n_j] = \emptyset$. Then we have $r_1 \ge n_j + 1$ and $r_d \ge n_j + d$. So we have $\lambda_{r_d} - r_d - i + d = -\phi_{r_d} < -(n_j + d)$. Therefore the d^{th} row of A_R is a zero vector since $H_m(X) := 0$ for $m < 0$. In conclusion, we have $\det A_R = 0$.

Note that λ_r is independent of $J \in \Xi$ and $\mu_s = i - d - (\mathrm{j}_s - s)$, so $\lambda_r - \mu_s - r + s = \lambda_r - r - i + d + \mathrm{j}_s$. Also note that $\alpha_a - a + b = \alpha_a + (n_j + 1 - a) - n_j - 1 + b$, while $\alpha_a + (n_i + 1 - a)$ lies in the complement of $J \subseteq [1, n_j + d]$. By our assumption that $n_j + n_{j+1} \leq n$, we have $i \geq n_j + d = n_j + i - (n - n_{j+1})$. Therefore we have $\Xi = \binom{[n_j+d]}{d}$. Then by taking the Laplace expansion on the first d rows of A_R, we see that

$$\det A_R = \sum_{J \in \Xi} (-1)^{|J|} \det(H_{\lambda_r - \mu_s - r + s}(X_{\phi_r}))_{1 \leq s \leq d, r \in R} \times \sigma^B_{[1, n_j + d] \setminus J}.$$

Therefore, under the assumption $n_j + n_{j+1} \leq n$, we have

$$\begin{aligned} & \sum_{J \in \Xi} (-1)^{|J|} \sigma^B_{w_J} \sigma^B_{[1, n_j+d] \setminus J} \\ =& \sum_R M_R \sum_{J \in \Xi} (-1)^{|J|} \det(H_{\lambda_r - \mu_s - r + s}(X_{\phi_r}))_{1 \leq s \leq d, r \in R} \times \sigma^B_{[1, n_j + d] \setminus J} \\ =& \sum_R M_R \det A_R \\ =& 0. \end{aligned}$$

(2) For the case $n_j < d$, the proof is similar. Let $J = \{\mathrm{j}_1 < \cdots < \mathrm{j}_d\} \in \Xi = \{J \in \binom{[i]}{d} | J \cap [n_j + d + 1, n] = \emptyset\}$. Let $\{x_1 < \cdots < x_{i-d}\} = [i] \setminus J$. We consider those J with $x_1 < \mathrm{j}_{n_j+1}$ only. Then w_J is defined as

$$\begin{aligned} &\{w(1) < \cdots < w(n_j)\} = \{\mathrm{j}_1 < \cdots < \mathrm{j}_{n_j}\} \\ &\{w(n_j + 1) < \cdots < w(n_{j+1})\} = \{x_1 < \mathrm{j}_{n_j+1} < \cdots < \mathrm{j}_d < i + 1 < \cdots < n - 1\} \\ &\{w(n_{j+1} + 1) < \cdots < w(n_{j+2})\} = \{x_2 < \cdots < x_{n_{j+2} - n_{j+1}} < n\} \\ &\{w(n_{j+2} + 1) < \cdots < w(n)\} = \{x_{n_{j+2} - n_{j+1} + 1} < \cdots < x_{i-d}\}. \end{aligned}$$

The code of w_J is $c(w_J) = (\mathrm{j}_1 - 1, \mathrm{j}_2 - 2, \cdots, \mathrm{j}_{n_j} - n_j, 0, \mathrm{j}_{n_j+1} - n_j - 2, \mathrm{j}_{n_j+2} - n_j - 3, \cdots, \mathrm{j}_d - d - 1, i - d - 1, i - d - 1, \cdots, i - d - 1, 0, \cdots, 0, n - n_{j+2}, 0, \cdots, 0)$ with flag $\phi(w_J) = (1, 2, \cdots, n_j, n_j + 2, n_j + 3, \cdots, n_{j+1}, n_{j+2})$. Then it determines a skew partition λ/μ, where

$$\begin{aligned} \lambda &= (i - d, \cdots, i - d, i - d - 1, \cdots, i - d - 1, n - n_{j+2}) \text{ with } n_j \text{ many } i - d \\ &\quad \text{and } n_{j+1} - n_j - 1 \text{ many } i - d - 1; \\ \mu &= (i - d - (\mathrm{j}_1 - 1), i - d - (\mathrm{j}_2 - 2), \cdots, i - d - (\mathrm{j}_d - d), 0, \cdots, 0). \end{aligned}$$

We notice that the flag $\phi(w_J)$ and the skew partition λ/μ are the same as the case $n_j \geq d$. Therefore, the rest of the proof is similar to the case $n_j \geq d$.

□

Example 5.22. *We demonstrate the idea of the above proof in the following identity.*

$$\sigma^B_{1526347} \cdot \sigma^B_{2314567} - \sigma^B_{2516347} \cdot \sigma^B_{1324567} + \sigma^B_{3516247} \cdot \sigma^B_{1234567} = 0.$$

Applying the determinantal formula, we see that

$$\sigma^B_{1526347} = \det\begin{pmatrix} H_3(X_2) & H_4(X_2) \\ H_1(X_4) & H_2(X_4)\end{pmatrix}, \qquad \sigma^B_{2314567} = \det\begin{pmatrix} H_1(X_2) & H_2(X_1) \\ H_0(X_2) & H_1(X_1)\end{pmatrix},$$

$$\sigma^B_{2516347} = \det\begin{pmatrix} H_1(X_1) & H_4(X_1) & H_5(X_1) \\ H_0(X_2) & H_3(X_2) & H_4(X_2) \\ H_{-2}(X_4) & H_1(X_4) & H_2(X_4)\end{pmatrix}, \quad \sigma^B_{1324567} = \det\begin{pmatrix} H_1(X_2) & H_2(X_1) \\ H_{-1}(X_2) & H_0(X_1)\end{pmatrix},$$

$$\sigma^B_{3516247} = \det\begin{pmatrix} H_2(X_1) & H_4(X_1) & H_5(X_1) \\ H_1(X_2) & H_3(X_2) & H_4(X_2) \\ H_{-1}(X_4) & H_1(X_4) & H_2(X_4)\end{pmatrix}, \quad \sigma^B_{1234567} = \det\begin{pmatrix} H_0(X_2) & H_1(X_1) \\ H_{-1}(X_2) & H_0(X_1)\end{pmatrix}.$$

We write

$$\sigma^B_{1526347} = \det\begin{pmatrix} H_3(X_2) & H_4(X_2) \\ H_1(X_4) & H_2(X_4)\end{pmatrix} = \det\begin{pmatrix} 1 & H_4(X_1) & H_5(X_1) \\ 0 & H_3(X_2) & H_4(X_2) \\ 0 & H_1(X_4) & H_2(X_4)\end{pmatrix}.$$

Notice that the last two columns of these 3×3 matrix are the same, so it suffices to prove that

$$1\times\det\begin{pmatrix} H_1(X_2) & H_2(X_1) \\ H_0(X_2) & H_1(X_1)\end{pmatrix} - H_1(X_1)\det\begin{pmatrix} H_1(X_2) & H_2(X_1) \\ H_{-1}(X_2) & H_0(X_1)\end{pmatrix} + H_2(X_1)\det\begin{pmatrix} H_0(X_2) & H_1(X_1) \\ H_{-1}(X_2) & H_0(X_1)\end{pmatrix} = 0,$$

$$-H_0(X_2)\det\begin{pmatrix} H_1(X_2) & H_2(X_1) \\ H_{-1}(X_2) & H_0(X_1)\end{pmatrix} + H_1(X_2)\det\begin{pmatrix} H_0(X_2) & H_1(X_1) \\ H_{-1}(X_2) & H_0(X_1)\end{pmatrix} = 0.$$

These follow from the Laplace expansion of the following identities respectively.

$$\det\begin{pmatrix} H_2(X_1) & H_1(X_2) & H_2(X_1) \\ H_1(X_1) & H_0(X_2) & H_1(X_1) \\ 1 = H_0(X_1) & H_{-1}(X_2) & H_0(X_1)\end{pmatrix} = 0,$$

$$\det\begin{pmatrix} H_1(X_2) & H_1(X_2) & H_2(X_1) \\ H_0(X_2) & H_0(X_2) & H_1(X_1) \\ 0 = H_{-1}(X_2) & H_{-1}(X_2) & H_0(X_1)\end{pmatrix} = 0.$$

It remains in this section to deduce the identity (5.1) also in the partial flag variety setting. We use the following result due to Dale Peterson.

Proposition 5.23 (Proposition 11.1 in [Rie03])**.** *Let $w \in W$ and let σ^B_w be the corresponding quantum Schubert class regarded as a function on the Peterson variety $\mathcal{Y}_{B_-}$ for the complete flag variety. Let $\widetilde{\sigma}^B_w$ be the rational function on the closure $\mathcal{Y} = \overline{\mathcal{Y}}_{B_-}$ that agrees with σ^B_w on $\mathcal{Y}_B$. If $w \in W^P$, then $\widetilde{\sigma}_w$ restricts to a regular function on $\overline{\mathcal{Y}}_P \subset \mathcal{Y}$, and this restriction represents the quantum Schubert class $\sigma^P_w \in QH^*(G^\vee/P^\vee)$ associated to w.*

This proposition implies that any identity in quantum Schubert calculus for the complete flag variety $G^\vee/B^\vee = \mathbb{F}\ell_n$ involving only Schubert classes of the form σ^B_w for $w \in W^P$ and without quantum parameters, holds also in $QH^*(G^\vee/P^\vee)$ with σ^B_w replaced by σ^P_w. As a consequence we have the following corollary; namely we obtain Theorem 5.3 in the case $n_j + n_{j+1} \le n$.

Corollary 5.24. *Let $n - n_{j+1} < i < n - n_j$ for some $1 \le j \le r-1$ and $d := i - (n - n_{j+1})$. Let Ξ and w_J be as defined in Definition 5.1. Assume that $n_j + n_{j+1} \le n$. Set $\sigma_{w_J} := 0$ if w_J is not defined. Then the following identity holds in $QH^*(X)$,*

$$\sum_{J\in\Xi} (-1)^{|J|}\sigma_{w_J}\sigma_{[1,n_j+d]\setminus J} = 0.$$

5.3. **Proof of Lemma 4.19.**

Lemma 5.25. *Suppose $n - n_{j+1} < i < n - n_j$ for some $1 \leq j \leq r-1$. Assume additionally that $n_j + n_{j+1} \leq n$. If $\hat{b}$ is a critical point of $\mathcal{F}_{\mathbf{q}}$, then*

$$u_{i,i+1} = -(G_1^{n_j}(b_-\dot{w}_0B_-) + G_1^{n_{j+1}}(b_-\dot{w}_0B_-)).$$

Proof. The statement is a direct consequence of Corollary 5.24 combined with Lemma 5.5. □

Definition 5.26. *We define a group involution on $G = GL_n(\mathbb{C})$ using a combination of inverse, transpose and conjugation by $\dot{w}_0$,*

$$g \mapsto \tau(g) := \dot{w}_0(g^{-1})^T\dot{w}_0^{-1}.$$

Let $Q \supseteq B_-$ be the parabolic subgroup with $I^Q = n - I^P = \{n-n_r, \cdots, n-n_1\}$. It is straightforward to check that our involution has the following properties.

(1) $\tau(P) = Q$ and $\tau(U_+) = U_+$.
(2) $\tau(\dot{w}_P) = \dot{w}_Q$.
(3) for $x \in U_+$ we have the relationship $\tau(x)_{i,i+1} = x_{n-i,n-i+1}$, for the entries just above the diagonal.

Lemma 5.27. *Suppose $n - n_{j+1} < i < n - n_j$ for some $1 \leq j \leq r-1$. Assume additionally that $n_j + n_{j+1} \geq n$. If $\hat{b}$ is a critical point of $\mathcal{F}_{\mathbf{q}}$, then*

$$u_{i,i+1} = -(G_1^{n_j}(b_-\dot{w}_0B_-) + G_1^{n_{j+1}}(b_-\dot{w}_0B_-)). \tag{5.3}$$

Proof. Since $\hat{b} \in B_- \cap U_+\dot{w}_P^{-1}\dot{w}_0U_+$, we have that $\tau(\hat{b}) \in B_- \cap U_+\dot{w}_Q^{-1}\dot{w}_0U_+$. We can now apply Lemma 5.25 to $\tau(\hat{b})$, where we must replace P by Q. Namely for $\tau(\hat{b})$, Lemma 5.25 says that, if $n_j = n-(n-n_j) < n-i < n-(n-n_{j+1}) = n_{j+1}$, and $(n-n_j)+(n-n_{j+1}) \leq n$ (which is equivalent to our assumptions on i), then

$$\tau(u)_{n-i,n-i+1} = -(G_1^{n-n_j}(\tau(b_-)\dot{w}_0B_-) + G_1^{n-n_{j+1}}(\tau(b_-)\dot{w}_0B_-)).$$

Recall that

$$G_1^m(gB_-) := \frac{\Delta_{[m+1,n]}^{\{m\}\cup[m+2,n]}(g)}{\Delta_{[m+1,n]}^{[m+1,n]}(g)}.$$

Now we deduce that

$$G_1^{n-m}(\tau(b_-)\dot{w}_0B_-) = G_1^m(b_-\dot{w}_0B_-),$$

using Jacobi's theorem. Moreover by property (3) above, we have $\tau(u)_{n-i,n-i+1} = u_{i,i+1}$. Therefore the identity (5.3) holds. □

Proof of Lemma 4.19 and Theorem 5.3. Lemma 4.19 follows from the combination of Lemmas 5.25 and 5.27. We showed in Lemma 5.5, that Theorem 5.3 is true if and only if Lemma 4.19 holds. Since Lemma 4.19 has now been proved, we are done. □

6. Appendix

In this Appendix we give a translation of the Plücker coordinate formula for the superpotential $\mathcal{F}_-$ using Young diagrams.

For $1 \le k < n$, we consider the set of partitions inside $k \times (n-k)$ rectangle,

$$\mathcal{P}_{k,n} := \{(\lambda_1, \cdots, \lambda_k) \in \mathbb{Z}^k \mid n-k \ge \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k \ge 0\}.$$

There is a bijection

$$\binom{[n]}{k} \to \mathcal{P}_{k,n}; \quad J = (j_1, \cdots, j_k) \mapsto \lambda(J) = (j_k - k, \cdots, j_2 - 2, j_1 - 1).$$

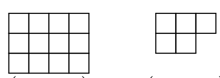
Geometrically, we consider the $k \times (n-k)$ rectangle of $k(n-k)$ unit boxes. A positive path of such rectangle is a path starting from the lower left hand corner and moving either upward or to the right along edges, towards the upper right hand corner. In particular, a Plücker coordinate $p_{j_1 \cdots j_k}$ is naturally viewed as the positive path that moves upwards precisely at the $j_1, j_2, \cdots, j_k$-th steps. Moreover, the boxes above the positive path p_J form the partition $\lambda(J)$. We therefore use the following notation convention

$$p_J = p_\lambda = p^{(k)}_{\mathrm{YD}(\lambda)},$$

where the superscript (k) is used to indicate that the Young diagram $\mathrm{YD}(\lambda)$ of the partition λ is inside $k \times (n-k)$ rectangle. In particular,

$$p_{[k]} = p_{(0,\cdots,0)} = p^{(k)}_{\emptyset}.$$

Example 6.1. *The Young diagrams of the partitions* $(4,4,4)$ *and* $(3,2,0)$ *in* $\mathcal{P}_{3,7}$ *are given as follows.*



$(4,4,4)$ $\qquad$ $(3,2,0)$

The Plücker coordinate p_{146} *for* $Gr(3,7)$ *corresponds to the partition* $(3,2,0)$.

By $(m^l, 0^{k-l})$ we mean the partition $(m, \cdots, m, 0, \cdots, 0) \in \mathcal{P}_{k,n}$ with l copies of m. The Young diagram $\mathrm{YD}(m^l, 0^{k-l})$ is an $l \times m$ rectangle $\square_{l \times m}$, and $\mathrm{YD}(1, 0^{k-1}) = \square$. We call $(m^l, 0^{k-l})$ a *maximal partition* in $\mathcal{P}_{k,n}$ if $l = k$ or $m = n-k$ holds.

Definition 6.2. *Let* $\lambda \in \mathcal{P}_{k,n}$ *and* $\nu \in \mathcal{P}_{k-a,n-a}$. *We define*

$$p^{(k)}_{\square_{k\times m}, \mathrm{YD}(\lambda)} := \begin{cases} p^{(k)}_{\mathrm{YD}(m^k+\lambda)}, & \text{if } m^k + \lambda \in \mathcal{P}_{k,n}, \\ 0, & \text{otherwise;} \end{cases}$$

$$p^{(k)}_{\square_{a\times(n-k)}, \mathrm{YD}(\nu)} := \begin{cases} p^{(k)}_{\mathrm{YD}((n-k)^a, \nu)}, & \text{if } ((n-k)^a, \nu) \in \mathcal{P}_{k,n}, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 6.3. *Let* $k < l < n$ *and* $1 \le m < l - k$. *We define*

$$L(p^{(k)}_{\square_{k\times m}} \cdot p^{(l)}_{\square_{(l-m)\times(n-l)}}) := \sum_{\mu \le m^k} (-1)^{|\mu| + km} p^{(k)}_{\mathrm{YD}(\mu)} \cdot p^{(l)}_{\square_{(l-m)\times(n-l)}, \mathrm{YD}((m^k/\mu)^c)},$$

where $(m^k/\mu)^c \in \mathcal{P}_{m,m+k}$ *denotes the conjugate of* $(m - \mu_k, \cdots, m - \mu_1)$. *We define*

$$L(p^{(k)}_{\square_{k\times m}, \square} \cdot p^{(l)}_{\square_{(l-m)\times(n-l)}}) := \sum_{\mu \le \lambda'; \mu_1 \ne m} (-1)^{|\mu| + km} p^{(k)}_{\mathrm{YD}(\mu)} \cdot p^{(l)}_{\square_{(l-m)\times(n-l)}, \mathrm{YD}((\lambda'/\mu)^c)},$$

where $\lambda' = (m+1, m^{k-1})$*;* $(\lambda'/\mu)^c \in \mathcal{P}_{m,m+k+1}$ *denotes the conjugate of* $(m-\mu_k, \cdots, m-\mu_2, 0)$ *if* $\mu_1 = m+1$*, or of* $(m-\mu_k, \cdots, m-\mu_1, 1)$ *if* $\mu_1 < m$*.*

Theorem 6.4. *In terms of the Plücker coordinates indexed by Young diagrams,*

$$\begin{aligned}\mathcal{F}_- = &\sum_{i=1}^{n_1-1} \frac{p^{(n_1)}_{\square_{i\times(n-n_1)},\square}}{p^{(n_1)}_{\square_{i\times(n-n_1)}}} + \sum_{j=1}^{r-1}\sum_{m=1}^{n_{j+1}-n_j-1} \frac{L(p^{(n_j)}_{\square_{n_j\times m},\square}\cdot p^{(n_{j+1})}_{\square_{(n_{j+1}-m)\times(n-n_{j+1})}})}{L(p^{(n_j)}_{\square_{n_j\times m}}\cdot p^{(n_{j+1})}_{\square_{(n_{j+1}-m)\times(n-n_{j+1})}})}\\ &+\sum_{i=1}^{n-n_r-1}\frac{p^{(n_r)}_{\square_{n_r\times i},\square}}{p^{(n_r)}_{\square_{n_r\times i}}} + \sum_{j=1}^{r}\frac{p^{(n_j)}_{\square}}{p^{(n_j)}_{\emptyset}} + \sum_{j=1}^{r} q_{n_j}\frac{p^{(n_j)}_{\square_{n_j\times(n-n_j)}\setminus q_{n_j}}}{p^{(n_j)}_{\square_{n_j\times(n-n_j)}}}\end{aligned}$$

where $\square_{n_j\times(n-n_j)}\setminus q_{n_j}$ *denotes the Young diagram obtained by removing* $n_j - n_{j-1}$ *boxes from the last column of* $\square_{n_j\times(n-n_j)}$ *and removing* $n_{j+1}-n_j$ *boxes from the last row, with the removal of the box at the bottom-right corner double counted.*

Proof. It suffices to discuss the $S_i^{(j)}$-terms in Theorem 3.18. (Other terms therein are direct translations to Young diagrams.)

For the denominator $L(p^{(n_j)}_{\square_{n_j\times m}}\cdot p^{(n_{j+1})}_{\square_{(n_{j+1}-m)\times(n-n_{j+1})}})$ as above, where $m = i - n_j$, we define a map $\alpha : \{J | J \in \binom{[i]}{m}\} \to \{\mu | \mu \le m^{n_j}\}$ as follows: it sends $J = \{a_1, ..., a_m\}$ to the Young diagram $\alpha(J)$ with $a_1, ..., a_m$ steps horizontal. It follows directly that α is a bijection. It remains to check the following facts:

(1) $J \in \binom{[\min\{i,\hat{i}\}]}{m}$ if and only if the join $\square_{(n_{j+1}-m)\times(n-n_{j+1})}, \mathrm{YD}((m^k/\alpha(J))^c)$ is inside the $n_{j+1}\times(n-n_{j+1})$ rectangle.
(2) For $J \in \binom{[\min\{i,\hat{i}\}]}{m}$, we have $p_{J\cup[\hat{i}+1,n]} = p^{(n_{j+1})}_{\square_{(n_{j+1}-m)\times(n-n_{j+1})},\mathrm{YD}((m^k/\alpha(J))^c)}$ and $p_{[i]\smallsetminus J} = p^{(n_j)}_{\mathrm{YD}(\alpha(J))}$. In particular for $J = [m]$, the corresponding product is the leading term $p^{(n_j)}_{\square_{n_j\times m}}\cdot p^{(n_{j+1})}_{\square_{(n_{j+1}-m)\times(n-n_{j+1})}}$.

By definition, $J \in \binom{[\min\{i,\hat{i}\}]}{m}$ if and only if the numbering of the first m vertical steps of the Young diagram $J \cup [\hat{i}+1, n]$ are $a_1, ..., a_m$ and $J\cup[\hat{i}+1,n]$ is inside the $n_{j+1}\times(n-n_{j+1})$ rectangle. Notice that $\mathrm{YD}((m^{n_j}/\alpha(J))^c)$ is the Young diagram $(a_m - m, ..., a_1 - 1)$. Thus when the join $\square_{(n_{j+1}-m)\times(n-n_{j+1})}, \mathrm{YD}((m^{n_j}/\alpha(J))^c)$ is inside the $n_{j+1}\times(n-n_{j+1})$ rectangle, the numbering of its first m vertical steps are exactly $a_1, ..., a_m$, and hence coincides with the Young diagram of $\alpha(J\cup[\hat{i}+1,n])$. Therefore in this case, the Plücker coordinates are also identified.

The arguments for the numerators are similar. Let $\lambda' = (m+1, m^{n_j-1})$. Here we define $\alpha' : \{J | J \in \binom{[i+1]\smallsetminus i}{m}\} \to \{\mu | \mu \le \lambda', \mu_1 \ne m\}$ as follows: α' sends $\{a_1, ..., a_m\}$ to the (unique) Young diagram $\alpha'(J)$ inside λ' with $[i+1]\smallsetminus\{i, a_1, ..., a_m\}$ steps vertical and $\mu_1 \ne m$. Such map is a bijection. Again we can similarly check the following facts:

(1) $J \in \binom{[\min\{i+1,\hat{i}\}]\smallsetminus i}{m}$ if and only if the join $\square_{(n_{j+1}-m)\times(n-n_{j+1})}, \mathrm{YD}((\lambda'/\alpha'(J))^c)$ is inside the $n_{j+1}\times(n-n_{j+1})$ rectangle.

(2) For $J \in \binom{[\min\{i+1,\hat{i}\}]\smallsetminus i}{m}$, we have $p_{[i-1]\cup\{i+1\}\smallsetminus J} = p^{(n_j)}_{\mathrm{YD}(\alpha'(J))}$ and $p_{J\cup[\hat{i}+1,n]} = p^{(n_{j+1})}_{\square_{(n_{j+1}-m)\times(n-n_{j+1})},\mathrm{YD}((\lambda'/\alpha'(J))^c)}$.

When $\alpha^{'}(J)_1 = m+1$. Let $J = \{a_1, ..., a_m\}$. $(\lambda^{'}/\alpha^{'}(J))^c$ is a partition given by the conjugate of $(m-\mu_k, ..., m-\mu_2, 0)$, and we have the fact that $a_m \neq i+1$ and $\mathrm{YD}((\lambda^{'}/\alpha^{'}(J))^c)$ is the Young diagram $(a_m - m, ..., a_1 - 1)$. Thus when the join $\square_{(n_{j+1}-m)\times(n-n_{j+1})}, \mathrm{YD}((\lambda^{'}/\alpha^{'}(J))^c)$ is inside the $n_{j+1} \times (n - n_{j+1})$ rectangle, the numbering of its first m vertical steps are exactly $a_1, ..., a_m$ and hence it coincides with the Young diagram of $J \cup [\hat{i}+1, n]$. Therefore the Plücker coordinates are also identified. The argument about other parts is similar. □

Example 6.5. *For $F\ell_{2,4;7}$, we have*

$$\begin{aligned}\mathcal{F}_- &= \frac{p_{27}}{p_{17}} + \frac{p_{24}p_{1567} - p_{14}p_{2567} + p_{12}p_{4567}}{p_{23}p_{1567} - p_{13}p_{2567} + p_{12}p_{3567}} + \frac{p_{2346}}{p_{2345}} + \frac{p_{3457}}{p_{3456}} + \frac{p_{13}}{p_{12}} + \frac{p_{1235}}{p_{1234}} + q_2\frac{p_{46}}{p_{67}} + q_4\frac{p_{1467}}{p_{4567}} \\ &= \frac{p^{(2)}_{(5,1)}}{p^{(2)}_{(5)}} + \frac{p^{(2)}_{(2,1)}p^{(4)}_{(3,3,3)} - p^{(2)}_{(2)}p^{(4)}_{(3,3,3,1)} + p^{(2)}_{\emptyset}p^{(4)}_{(3,3,3,3)}}{p^{(2)}_{(1,1)}p^{(4)}_{(3,3,3)} - p^{(2)}_{(1)}p^{(4)}_{(3,3,3,1)} + p^{(2)}_{\emptyset}p^{(4)}_{(3,3,3,2)}} + \frac{p^{(4)}_{(2,1,1,1)}}{p^{(4)}_{(1,1,1,1)}} + \frac{p^{(4)}_{(3,2,2,2)}}{p^{(4)}_{(2,2,2,2)}} + \frac{p^{(2)}_{(1)}}{p^{(2)}_{\emptyset}} + \frac{p^{(4)}_{(1)}}{p^{(4)}_{\emptyset}} + q_2\frac{p^{(2)}_{(4,3)}}{p^{(2)}_{(5,5)}} + q_4\frac{p^{(4)}_{(3,3,2)}}{p^{(4)}_{(3,3,3,3)}}.\end{aligned}$$

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School of Mathematics, Sun Yat-sen University, Guangzhou 510275, P.R. China
E-mail address: `lichangzh@mail.sysu.edu.cn`

Department of Mathematics, King's College London, Strand, London WC2R 2LS, UK
E-mail address: `konstanze.rietsch@kcl.ac.uk`

School of Mathematics, Sun Yat-sen University, Guangzhou 510275, P.R. China
E-mail address: `yangmzh8@mail2.sysu.edu.cn`

Department of Mathematics, Caltech, 1200 East California Boulevard, Pasadena, CA 91125
E-mail address: `czhang5@caltech.edu`