GAMMA CONJECTURE I FOR DEL PEZZO SURFACES

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ABSTRACT. Gamma conjecture I and the underlying Conjecture $\mathcal{O}$ for Fano manifolds were proposed by Galkin, Golyshev and Iritani recently. We show that both conjectures hold for all two-dimensional Fano manifolds. We prove Conjecture $\mathcal{O}$ by deriving a generalized Perron-Frobenius theorem on eigenvalues of real matrices and a vanishing result of certain Gromov-Witten invariants for del Pezzo surfaces. We prove Gamma conjecture I by applying mirror techniques proposed by Galkin-Iritani together with the study of Gamma conjecture I for weighted projective spaces. We also provide applications of our generalized Perron-Frobenius theorem on Conjecture $\mathcal{O}$ for two Fano manifolds of higher dimensions.

CONTENTS

1. INTRODUCTION

Let X be a Fano manifold, namely a compact manifold whose anti-canonical line bundle is ample. Denote by $QH^*(X)$ the small quantum cohomology of X and $J_X(t)$ Givental's small J-function in Gromov-Witten theory of X. In [GGI], Galkin, Golyshev and Iritani proposed the so-called Gamma conjectures and the underlying conjecture $\mathcal{O}$ for any Fano

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manifold X. Conjecture $\mathcal{O}$ concerns with eigenvalues of a linear operator on the small quantum cohomology ring $QH^*(X)$. Gamma conjecture consists of Gamma conjecture I and II. Gamma conjecture I cares about the asymptotic expansion of Givental's small J-function $J_X(t)$ by assuming conjecture $\mathcal{O}$ first. Gamma conjecture II was also proposed in [GGI] as a refinement of a part of the original Dubrovin conjecture [Du1], independent of the new formulation in [Du2, CDG]. In this paper, we will focus on Conjecture $\mathcal{O}$ and Gamma conjecture I.

To be more precise, the quantum cohomology ring $QH^*(X)$ is a deformation of the classical cohomology ring $H^*(X) = H^*(X, \mathbb{Q})$ by incorporating genus zero, three-point Gromov-Witten invariants of X. As vector spaces, we have $QH^*(X) = H^*(X) \otimes \mathbb{Q}[q_1, \cdots, q_m]$, where m is the second Betti number of X and q_i's are quantum variables parameterizing a basis of effective curve classes in $H_2(X, \mathbb{Z})$. The quantum multiplication by the first Chern class $c_1(X)$ of X induces a linear operator $\hat{c}_1 = c_1(X)\star_{\mathbf{q}=\mathbf{1}}$ on the even part $H^\bullet(X) := H^{\mathrm{ev}}(X) = QH^{\mathrm{ev}}(X)|_{\mathbf{q}=(1,\cdots,1)}$, which is a vector space of finite dimension. The so-called *Property $\mathcal{O}$*, introduced for general Fano manifolds, consists of two parts. The first half says that the spectral radius $\rho(\hat{c}_1) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \hat{c}_1\}$ itself is a eigenvalue of $\hat{c}_1$ of multiplicity one. The second half says that $|\lambda| \neq \rho(\hat{c}_1)$ whenever $\lambda \neq \rho(\hat{c}_1)$ is an eigenvalue of $\hat{c}_1$ in the case of Fano manifolds of Fano index one (see section 2.1 for general cases). Galkin, Golyshev and Iritani [GGI] made the following conjecture with the name $\mathcal{O}$ indicating a deep relation with homological mirror symmetry.

Conjecture $\mathcal{O}$. *Every Fano manifold satisfies Property $\mathcal{O}$.*

There have been complete classifications of Fano manifolds of small dimension. Any one-dimensional Fano manifold is isomorphic to the complex projective line $\mathbb{P}^1$. Every two-dimensional Fano manifold, i.e. a del Pezzo surface, is either isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or the blowup X_r of $\mathbb{P}^2$ at r points in general position $(0 \leq r \leq 8)$. As one main result of the present paper, we prove conjecture $\mathcal{O}$ for $X_r (1 \leq r \leq 8)$ (which are all of Fano index one), in addition to the known cases $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$. Namely, we show the following.

Theorem 1.1. *Every del Pezzo surface satisfies Property $\mathcal{O}$.*

The formulation of Gamma conjecture I requires a bit more knowledge on the Gromov-Witten theory. On one hand, the restriction $J_X(t)$ of Givental's big J-function to the anti-canonical line is a (multi-valued) $H^*(X)$-valued function over $\mathbb{C}^*$ defined by incorporating genus zero, one-point Gromov-Witten invariants with gravitational descendents. Under the assumption that X satisfies conjecture $\mathcal{O}$, $J_X(t)$ admits an asymptotic expansion at the infinity, whose leading term $A_X \in H^\bullet(X)$ is unique up to a nonzero scalar and is called the *principal asymptotic class* of X. On the other hand, the Gamma class $\hat{\Gamma}_X = \prod_{i=1}^{\dim X} \Gamma(1 + x_i)$ [HKTY, Libg, Lu, Iri1] of X is a real characteristic class, defined by the Chern root $x_1, \cdots, x_{\dim X}$ of the tangent bundle TX of X and Euler's Γ-function $\Gamma(z) = \int_0^\infty e^t t^{z-1} dt$. It has the following expansion (see e.g. [GGI]):

$$\hat{\Gamma}_X = \exp\big(- C_{\mathrm{eu}} c_1(X) + \sum_{k=2}^{\infty} (-1)^k (k-1)! \zeta(k) \mathrm{ch}_k(TX) \big) \in H^\bullet(X, \mathbb{R})$$

where $C_{\mathrm{eu}} = 0.5772156...$ is the Euler-Mascheroni constant, $\zeta(k) = \sum_{n=1}^\infty \frac{1}{n^k}$ is the value of Riemann zeta function at k, and ch_k denotes the k-th Chern character. Gamma conjecture I relates Gromov-Witten invariants (which are deformation invariants but not topological invariants) to the topology of X, in the way that the Gamma class of X coincides with the principal asymptotic class of X up to normalization by a scalar; namely

Gamma conjecture I. *Let$X$ be a Fano manifold satisfying Property $\mathcal{O}$. Then there exists a constant $C \in \mathbb{C}$ such that $\hat{\Gamma}_X = C \cdot A_X$.*

We refer to [GGI, GaIr] for the various equivalent formulations of the above conjecture. There are close relationships between Gamma conjectures and mirror symmetry. For instance in the study of mirror symmetry for a quintic Calabi-Yau threefold Q in $\mathbb{P}^4$, the inverse of Gamma class $\hat{\Gamma}_Q$ appears in the leading term of a specified solution to the corresponding Picard-Fuchs equation. Hosono's conjecture [Hoso] for Q is also closely related to the truth of Gamma conjecture I for the ambient space $\mathbb{P}^4$. We refer to [GGI, section 1.5], [GaIr, section 1.3] and [AGIS, section 1.2] for a survey of the early history. As another main result of the present paper, we show the following.

Theorem 1.2. *Gamma conjecture I holds for any del Pezzo surface.*

There have been lots of studies on conjecture $\mathcal{O}$ recently. It has been proved for flag varieties G/P of arbitrary Lie type by Cheong and the third named author [ChLi], and was proved earlier in the special cases of cominuscule Grassmannians of classical Lie types [Rie, GaGo, Che]. Conjecture $\mathcal{O}$ has also been proved recently for Fano 3-folds of Picard rank one [GoZa], Fano complete intersections in projective spaces [GaIr, SaSh, Ke], horospherical varieties of Picard rank one [LMS, BFSS] and 3-dimensional Bott-Samelson varieties [Wit]. Gamma conjecture I was less studied, and so far has been proved for complex Grassmannians [GGI], Fano 3-fold of Picard rank one [GoZa], Fano complete intersections in projective spaces [GaIr, SaSh, Ke], and toric Fano manifolds that satisfy B-model analogue of Property $\mathcal{O}$ [GaIr].

The proof of conjecture $\mathcal{O}$ for flag varieties in [ChLi] relied on the well-known Perron-Frobenius theory [Perr, Frob] for non-negative real matrices. It requires at least the positivity of the matrix of the linear operator $\hat{c}_1$ with respect to some good basis of $H^\bullet(X)$. However, this does not hold even for the blowup X_1 of $\mathbb{P}^2$ at one point. In the case of del Pezzo surfaces $X_r(1 \leq r \leq 8)$, we are able to prove conjecture $\mathcal{O}$ by deriving a generalization of Perron-Frobenius theorem in Theorem 3.2 that allows part of the entries of a real matrix to be negative, and providing some vanishing properties of Gromov-Witten invariants in Theorem 3.8. We remark that the conjecture could also be directly proved by analysing the characteristic polynomial of $\hat{c}_1$ as was studied in [BaMa] due to the very well study of the relevant Gromov-Witten invariants of X_r [GöPa, Vakil]. However, we expect our method to have further applications for other Fano manifolds. For one reason, it will be sufficient to apply our generalized Perron-Frobenius theorem by the study of a part of the Gromov-Witten invariants. For another reason, the hypotheses of our generalized Perron-Frobenius theorem could be satisfied for more general Fano manifolds, to demonstrate which we provide two examples of higher dimension in the appendix. Del Pezzo surfaces are either toric Fano or complete intersections in nice ambient spaces of Picard rank one. This fact enables us to prove Gamma conjecture I by using the mirror techniques proposed by Galkin and Iritani [GaIr], where the quantum Lefschetz principle [Lee, CoGi] is a key ingredient. Here we would like to point out that the proof for X_7 and X_8 requires the study of the orbifold version of Gamma conjecture I for certain weighted projective spaces as shown in Theorem 5.9. A more systemic study of these conjectures (with modification if necessary) for general orbifolds will be of independent and great interest.

The present paper is organized as follows. In section 2, we describe the precise statements of conjecture $\mathcal{O}$ and Gamma conjecture I, and review basic facts of del Pezzo surfaces. In section 3, we derive a generalized Perron-Frobenius theorem, and provide vanishing properties of certain Gromov-Witten invariants of X_r. In section 4, we prove conjecture $\mathcal{O}$ by analyzing the corresponding matrix of $\hat{c}_1$ with respect to a specified basis of $H^*(X)$. In section 5, we prove Gamma conjecture I for del Pezzo surfaces as well as for certain weighted projective spaces. Finally in the appendix, we provide two further applications of our generalized Perron-Frobenius theorem on Conjecture $\mathcal{O}$.

2. Preliminaries

2.1. **Conjecture $\mathcal{O}$ and Gamma conjecture I for Fano manifolds.** In this subsection, we review the precise statements of conjecture $\mathcal{O}$ and Gamma conjecture I for Fano manifolds, mainly following [GGI, GaIr].

2.1.1. *Quantum cohomology.* We refer to [CoKa] for more details.

Let X be a Fano manifold, namely a compact complex manifold X whose anticanocial line bundle is ample. Let $\overline{\mathcal{M}}_{0,k}(X,\mathbf{d})$ denote the moduli stack of k-pointed genus-zero stable maps $(f: C\to X; p_1,\cdots,p_k)$ of class $\mathbf{d}\in H_2(X,\mathbb{Z})$, which has a coarse moduli space $\overline{M}_{0,k}(X,\mathbf{d})$. Let $[\overline{M}_{0,k}(X,\mathbf{d})]^{\rm virt}$ be the virtual fundamental class of $\overline{\mathcal{M}}_{0,k}(X,\mathbf{d})$, which is of complex degree $\dim X-3+\int_{\mathbf{d}}c_1(X)+k$ in the Chow group $A_*(\overline{\mathcal{M}}_{0,k}(X,\mathbf{d}))$. Given classes $\gamma_1,\cdots,\gamma_k\in H^*(X)=H^*(X,\mathbb{Q})$ and nonnegative integers a_i for $1\leq i\leq k$, we have the following associated gravitational correlator

$$\langle\tau_{a_1}\gamma_1,\cdots,\tau_{a_k}\gamma_k\rangle_{\mathbf{d}}:=\int_{[\overline{M}_{0,k}(X,\mathbf{d})]^{\rm virt}}\prod_{i=1}^{k}\big(c_1(\mathcal{L}_i)^{a_i}\cup ev_i^*(\gamma_i)\big).$$

Here $\mathcal{L}_i$ denotes the line bundle on $\overline{\mathcal{M}}_{0,k}(X,\mathbf{d})$ whose fiber over the stable map $(f: C\to X; p_1,\cdots,p_k)$ is the cotangent space $T^*_{p_i}C$, and ev_i denotes the i-th evaluation map. When $a_i=0$ for all i, the above gravitational correlator becomes an ordinary k-pointed Gromov-Witten invariant $\langle\gamma_1,\cdots,\gamma_k\rangle_{\mathbf{d}}$ of class $\mathbf{d}$. The gravitational correlators satisfy a number of axioms and the topological recursion relations (**TRR**). For the precise statements, we refer to [CoKa, section 10.1.2] for **Degree Axiom**, **Divisor Axiom** and **Fundamental Class Axiom**, and [CoKa, Lemma 10.2.2] for **TRR**, which will be used in the next two sections.

The (small) quantum cohomogy ring $QH^*(X)=(H^*(X)\otimes_{\mathbb{Q}}\mathbb{Q}[q_1,\cdots,q_m],\star)$ is a deformation of the classical cohomology $H^*(X)$. Here $m=b_2(X)$ is the second Betti number of X, and the quantum product of $\alpha,\beta\in H^*(X)$ is given by[1]

$$\alpha\star\beta:=\sum_{\mathbf{d}\in H_2(X,\mathbb{Z})}\sum_{i=1}^{N}\langle\alpha,\beta,\phi_i\rangle_{\mathbf{d}}\phi^iq^{\mathbf{d}}.$$

Here $\{\phi_i\}_{i=1}^N$ is a homogeneous basis of $H^*(X)$, $\{\phi^i\}$ is its dual basis in $H^*(X)$ with respect to the Poincaré pairing, $q^{\mathbf{d}}=\prod_{j=1}^m q_j^{d_j}$ for $\mathbf{d}=(d_1,\cdots,d_m)$ with a basis of effective curve classes of $H_2(X,\mathbb{Z})$ being fixed a prior. We notice that the Gromov-Witten invariant $\langle\alpha,\beta,\phi_i\rangle_{\mathbf{d}}$ vanishes unless $d_i\geq 0$ for all i and $\deg(\alpha\cup\beta\cup\phi_i)=2(\dim X+\int_{\mathbf{d}}c_1(X))$. In particular, the quantum product is a finite sum and is a polynomial in $\mathbf{q}$. Moreover, the quantum product is independent of choices of the basis $\{\phi_i\}$.

[1]In terms of the notation in [GGI], $q_i=e^{h_i}$; the quantum product $\alpha\star_{\tau=\mathbf{0}}\beta$ defined therein coincides with the product $\alpha\star\beta|_{\mathbf{q}=(1,\cdots,1)}$ here.

2.1.2. *Conjecture $\mathcal{O}$.* Consider the even part of the cohomology $H^\bullet(X) := H^{\mathrm{even}}(X)$ and the finite-dimensional $\mathbb{Q}$-algebra $QH^\bullet(X) = (H^\bullet(X), \bullet)$ with the product defined by $\alpha \bullet \beta := (\alpha \star \beta)|_{\mathbf{q}=\mathbf{1}}$, namely by the evaluation of the quantum product at $\mathbf{1} := (1, \cdots, 1)$. Let $\hat{c}_1$ denote the linear operator induced by the first Chern class:

$$\hat{c}_1 : H^\bullet(X) \longrightarrow H^\bullet(X);\ \beta \mapsto c_1(X) \star \beta|_{\mathbf{q}=\mathbf{1}},$$

which is independent of the choices of bases of effective curve classes.

Definition 2.1 (Property $\mathcal{O}$)**.** *For a Fano manifold X, we denote by $\rho = \rho(\hat{c}_1)$ the spectral radius of the linear operator $\hat{c}_1$, namely*

$$\rho := \max\{|\lambda| \,:\, \lambda \in \mathit{Spec}(\hat{c}_1)\} \quad \textit{where} \quad \mathit{Spec}(\hat{c}_1) := \{\lambda \,:\, \lambda \in \mathbb{C} \textit{ is an eigenvalue of } \hat{c}_1\}.$$

*We say that X satisfies **Property** $\mathcal{O}$ if the following two conditions are satisfied.*

(1) *$\rho \in \mathit{Spec}(\hat{c}_1)$ and it is of multiplicity one.*
(2) *For any $\lambda \in \mathit{Spec}(\hat{c}_1)$ with $|\lambda| = \rho$, we have $\lambda^s = \rho^s$, where s is the Fano index of X, namely $s = \max\{k \in \mathbb{Z} \,:\, \frac{c_1(X)}{k} \in H^2(X, \mathbb{Z})\}$.*

Conjecture $\mathcal{O}$ ([GGI])**.** *Every Fano manifold satisfies Property $\mathcal{O}$.*

As explained in [GGI, Remark 3.1.5], the name "$\mathcal{O}$" indicates the structure sheaf $\mathcal{O}_X$, which is expected to correspond to ρ from the viewpoint of homological mirror symmetry. Although the above statement concerns about the even part of the cohomology only, it is in fact equivalent to a statement for the whole of $H^*(X)$ [SaSh, GaIr], and is also equivalent to that for the smaller part $\bigoplus_p H^{p,p}(X)$ of cohomology of Hodge type [GaIr].

Conjecture $\mathcal{O}$ for flag variety G/P was proved by Cheong and the third named author [ChLi] by using Perron-Frobenius theorem based on a remark due to Kaoru Ono.

2.1.3. *Gamma conjecture I.* On the trivial $H^\bullet(X)$-bundle over $\mathbb{P}^1$, there is a so-called quantum connection, given by

$$\nabla_{z\partial_z} = z\partial_z - \frac{1}{z}(c_1(X)\bullet) + \mu.$$

Here z is an inhomogeneous co-ordinate on $\mathbb{P}^1$ and μ is the Hodge grading operator defined by $\mu : H^\bullet(X) \to H^\bullet(X); \phi \in H^{2p}(X) \mapsto \mu(\phi) = (p - \frac{\dim X}{2})\phi$. The quantum connection is a meromorphic connection, which is logarithmic at $z = \infty$ and irregular at $z = 0$. The space of flat sections can be identified with the cohomology group $H^\bullet(X)$ via the fundamental solution $S(z)z^{-\mu}z^{c_1(X)}$ with a unique holomorphic function $S : \mathbb{P}^1 \setminus \{0\} \to \mathrm{End}(H^\bullet(X))$ (see [GGI, Proposition 2.3.1] for detailed explanations). The Givental's J-function, defined by $J_X(t) = z^{\dim X}\big(S(z)z^{-\mu}z^{c_1(X)}\big)^{-1}1$ where $t = \frac{1}{z}$, has the following expansion around $t = 0$.

$$J_X(t) = e^{c_1(X)\log t}\Big(1 + \sum_{i=1}^N \sum_{\mathbf{d} \in H_2(X,\mathbb{Z})\setminus\{\mathbf{0}\}} \langle \frac{\phi_i}{1 - c_1(\mathcal{L}_1)} \rangle_{0,\mathbf{d}} \phi^i t^{\int_{\mathbf{d}} c_1(X)}\Big).$$

Remark 2.2. *Strictly speaking, Givental's big J-function is a formal function of $\tau \in H^\bullet(X)$ taking values in $H^\bullet(X)$, given by*

$$J_X(\tau, \hbar; Q) = \hbar + \tau + \sum_{i=1}^N \phi^i \sum_{\mathbf{d} \in H_2(X,\mathbb{Z})} Q^{\mathbf{d}} \sum_{m=0}^\infty \frac{1}{m!} \langle \frac{\phi_i}{\hbar - c_1(\mathcal{L}_1)}, \underbrace{\tau, \cdots, \tau}_{m} \rangle_{\mathbf{d}},$$

It can be reduced to the form $J_X(\tau,\hbar;Q)=\hbar e^{\frac{\tau}{\hbar}}\Big(\mathbb{1}+\sum_{i=1}^{N}\phi^i\sum_{\mathbf{d}\in H_2(X,\mathbb{Z})}Q^{\mathbf{d}}e^{\int_{\mathbf{d}}\tau}\langle\frac{\phi_i}{\hbar(\hbar-c_1(\mathcal{L}_1))}\rangle_{\mathbf{d}}\Big)$ *for* $\tau\in H^0(X)\oplus H^2(X)$*, after applying the Fundamental Class Axiom and the Divisor Axiom. The Givental's* J*-function in this paper is obtained from the original one by setting*

$$J_X(t)=J_X(\tau=c_1(X)\log t,\hbar=1;Q=\mathbf{1}).$$

Proposition 2.3 (Proposition 3.8 of [GaIr])**.** *For any Fano manifold* X *satisfying Property* $\mathcal{O}$*, Givental's* J*-function* $J_X(t)$ *has an asymptotic expansion of the form*

$$J_X(t)=Ct^{\frac{-\dim X}{2}}e^{\rho t}(A_X+\alpha_1t^{-1}+\alpha_2t^{-2}+\cdots)\tag{1}$$

as $t\to+\infty$ *on the positive real line, where* C *is a non-zero constant and* $\alpha_i\in H^\bullet(X)$*.*

The Gamma class [Libg, Lu, Iri1] is a real characteristic class defined for an almost complex manifold. It is defined by Chern roots $x_1,\cdots,x_n$ of the tangent bundle TX of X and Euler's Γ-function $\Gamma(z)=\int_0^\infty e^tt^{z-1}dt$, and has the following expansion:

$$\hat{\Gamma}_X:=\prod_{i=1}^{n}\Gamma(1+x_i)=\exp\big(-C_{\rm eu}c_1(X)+\sum_{k=2}^{\infty}(-1)^k(k-1)!\zeta(k)\mathrm{ch}_k(TX)\big)\in H^\bullet(X,\mathbb{R})$$

where $C_{\rm eu}$ is the Euler-Mascheroni constant, $\zeta(k)=\sum_{n=1}^\infty\frac{1}{n^k}$ is the value of Riemann zeta function at k, and ch_k denotes the k-th Chern character. There are various equivalent ways to describe Gamma conjecture I. Here we introduce the one given in [GGI, Corollary 3.6.9 (3)]. For $\alpha,\beta\in H^*(X)$, we say $\alpha\propto\beta$ if $\alpha=b\cdot\beta$ for some $b\in\mathbb{C}$.

Gamma conjecture I. *Let* X *be a Fano manifold satisfying Property* $\mathcal{O}$*. Then*

$$\hat{\Gamma}_X\propto\lim_{t\to+\infty}t^{\frac{\dim X}{2}}e^{-\rho t}J_X(t).$$

2.2. **Del Pezzo surfaces.** It is well-known that any one-dimensional Fano manifold is isomorphic to the complex projective line $\mathbb{P}^1$. A Fano manifold of dimension 2 is called a *del Pezzo surface*. It is either isomorphic to $\mathbb{P}^1\times\mathbb{P}^1$, or the blowup X_r of $\mathbb{P}^2$ at r points in general position ($0\le r\le 8$). We will exclude $\mathbb{P}^1\times\mathbb{P}^1$ and $\mathbb{P}^2$ in this subsection.

2.2.1. *Basic topology.* Curves in a surface are divisors. For convenience, we will use the same notation for anyone of a divisor of X_r, its divisor class in $H^2(X_r,\mathbb{Z})$ and its curve class in $H_2(X_r,\mathbb{Z})$, whenever there is no confusion. For instance in $D\cdot D'=\int_X D\cup D'$, we can easily read off the left (resp. right) hand side as the intersection product (resp. Poincaré pairing) of the divisors (resp. divisor classes) D,D'.

Let H denote the pullback to X_r of the hyperplane class of $\mathbb{P}^2$, and let $E_1,\cdots,E_r$ be the exceptional divisors. Together with the Poincaré dual $\mathbb{1}:=[X_r]\in H^0(X_r,\mathbb{Z})$ and $[pt]\in H^4(X_r,\mathbb{Z})$ of the corresponding homology classes, they form a $\mathbb{Z}$-basis:

$$H^*(X_r,\mathbb{Z})=H^{\rm even}(X_r,\mathbb{Z})=\mathbb{Z}\mathbb{1}\oplus\mathbb{Z}[pt]\oplus\mathbb{Z}H\oplus\mathbb{Z}E_1\oplus\cdots\oplus\mathbb{Z}E_r$$

The first Chern class of X_r is given by $c_1:=c_1(X_r)=3H-\sum_{i=1}^rE_i$. We have

$$H\cdot E_i=E_i\cdot E_j=0,\quad H\cdot H=-E_i\cdot E_i=1,\quad \text{for all } i,j\in\{1,\cdots,r\} \text{ with } i\neq j.$$

It follows that X_r is of degree $c_1\cdot c_1=9-r$, and its Fano index equals 1 as $c_1\cdot E_1=1$.

2.2.2. *Geometric interpretations.* There are various geometric descriptions for del Pezzo surfaces X_r, where $1 \leq r \leq 8$. For instance, all of them can be realized as complete intersections in nice spaces as follows.

X_1: degree $(1,1)$ hypersurface in $\mathbb{P}^1 \times \mathbb{P}^2$;
X_2: complete intersection of divisors of degree $(1,0,1)$ and $(0,1,1)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$;
X_3: complete intersection of two divisors of degree $(1,1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$;
X_4: complete intersection of four hyperplanes in Grassmannian $Gr(2,5) \subset \mathbb{P}^9$ (embedded by Plücker);
X_5: complete intersection of two quadrics in $\mathbb{P}^4$;
X_6: cubic surface in $\mathbb{P}^3$;
X_7: hypersurface of degree 4 in the weighted projective space $\mathbb{P}(1,1,1,2)$;
X_8: hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1,1,2,3)$.

We refer to [IsPr, Chapter 3.2] and [Cord] for the above descriptions of X_r with $4 \leq r \leq 8$. Toric del Pezzo surfaces are precisely those X_r with $1 \leq r \leq 3$, and the descriptions are also known to the experts. Here we provide a proof for X_2, which is relatively less studied. The method works for X_1 and X_3 as well.

Proof of X_2 being a complete intersection. The complete intersection Z of two generic divisors of degree $(1,0,1)$ and $(0,1,1)$ in $Y := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ is a smooth projective variety. By the adjunction formula, we obtain the anti-canonical divisor $-K_Z = \mathcal{O}_Y(H_1 + H_2 + H_3)|_Z$, where H_i is the natural pull-back of the hyperplane class of the i-th factor to Y. Clearly, $-K_Z$ is ample, and hence Z is a del Pezzo surface.

The degree $(-K_Z)^2$ of Z can be computed by the intersection product of divisors in Y as follows. Notice that $H_1^2 = H_2^2 = H_3^3 = 0$, $H_1H_2H_3^2 = 1$ and $H_iH_j = H_jH_i$, we have

$$
\begin{aligned}
(-K_Z)^2 &= (H_1 + H_2 + H_3)^2(H_1 + H_3)(H_2 + H_3) \\
&= (H_1^2 + H_2^2 + H_3^2 + 2H_1H_2 + 2H_1H_3 + 2H_2H_3)(H_1H_2 + H_1H_3 + H_2H_3 + H_3^2) \\
&= 7H_1H_2H_3^2 = 7
\end{aligned}
$$

It follows that Z is a del Pezzo surface of degree 7. □

2.2.3. *Quantum cohomology.* There has been well study of the quantum cohomology ring $QH^*(X_r)$ of X_r in [CrMi, GöPa]. Therein we can immediately read off or easily deduce the multiplication table of the quantum product for small r. We notice that $\mathbb{1}$ is the identity element in $QH^*(X_r)$ for any X_r.

Example 2.4. *A basis of $H^*(X_1)$ is given by $\{\mathbb{1}, H, E_1, [pt]\}$. In $QH^*(X_1)$, we have*

$$
\begin{aligned}
&H \star H = [pt] + q^{H-E_1}, \quad H \star E_1 = q^{H-E_1}, \quad H \star [pt] = (H - E_1)q^{H-E_1} + q^H, \\
&E_1 \star E_1 = -[pt] + E_1 q^{E_1} + q^{H-E_1}, \qquad E_1 \star [pt] = (H - E_1)q^{H-E_1}.
\end{aligned}
$$

For conjecture $\mathcal{O}$, we concern about the operator $\hat{c}_1$ on $QH^\bullet(X_1)$ induced by the quantum multiplication by $c_1 = 3H - E_1$ with evaluation of all quantum variables at 1. We have

$$
\hat{c}_1[\mathbb{1}, H, E_1, [pt]] = [\mathbb{1}, H, E_1, [pt]] \begin{bmatrix} 0 & 2 & 2 & 3 \\ 3 & 0 & 0 & 2 \\ -1 & 0 & -1 & -2 \\ 0 & 3 & 1 & 0 \end{bmatrix}.
$$

We denote by $\tilde{M}_1$ the matrix above, while denote by M_1 the matrix with respect to another $\mathbb{Z}$-basis $[\mathbb{1}, H - E_1, E_1, [pt]]$. It follws that

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \cdot \tilde{M} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 & 3 \\ 3 & 0 & 0 & 2 \\ 2 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}.$$

Example 2.5. *A basis of $H^*(X_2)$ is given by $\{\mathbb{1}, H, E_1, E_2, [pt]\}$. In $QH^*(X_2)$, we have*

$$\begin{aligned} H \star H &= [pt] + (H - E_1 - E_2)q^{H-E_1-E_2} + q^{H-E_1} + q^{H-E_2}, \\ H \star E_1 &= (H - E_1 - E_2)q^{H-E_1-E_2} + q^{H-E_1}, \\ H \star [pt] &= (H - E_1)q^{H-E_1} + (H - E_2)q^{H-E_2} + q^H, \\ E_1 \star E_1 &= -[pt] + E_1 q^{E_1} + (H - E_1 - E_2)q^{H-E_1-E_2} + q^{H-E_1}, \\ E_1 \star E_2 &= (H - E_1 - E_2)q^{H-E_1-E_2}, \\ E_1 \star [pt] &= (H - E_1)q^{H-E_1}. \end{aligned}$$

Let $\tilde{M}_2$ and M_2 denote the matrices with respect to the corresponding $\mathbb{Z}$-bases:

$$\hat{c}_1[\mathbb{1}, H, E_1, E_2, [pt]] = [\mathbb{1}, H, E_1, E_2, [pt]]\tilde{M}_2,$$
$$\hat{c}_1[\mathbb{1}, 2H - E_1 - E_2, E_1, E_2, [pt]] = [\mathbb{1}, 2H - E_1 - E_2, E_1, E_2, [pt]]M_2.$$

Then $\tilde{M}_2$ is directly read off from the above table (by setting all quantum variables as 1), and M_2 is obtained after a simple base change from $\tilde{M}_2$. They are precisely given by

$$\tilde{M}_2 = \begin{bmatrix} 0 & 4 & 2 & 2 & 3 \\ 3 & 1 & 1 & 1 & 4 \\ -1 & -1 & -2 & -1 & -2 \\ -1 & -1 & -1 & -2 & -2 \\ 0 & 3 & 1 & 1 & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 0 & 4 & 2 & 2 & 3 \\ \frac{3}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 2 \\ \frac{1}{2} & 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 4 & 1 & 1 & 0 \end{bmatrix}.$$

Example 2.6. *The quantum multiplication table of $QH^*(X_3)$ with respect to the basis $\{\mathbb{1}, H, E_1, E_2, E_3, [pt]\}$ can be read off from the following together with a permutation symmetry among the exceptional divisor classes E_i.*

$$\begin{aligned} H \star H &= [pt] + \sum_{1 \le i \le 3} (H - \sum_{j \ne i} E_j) q^{H - \sum_{j \ne i} E_j} + \sum_{1 \le i \le 3} q^{H - E_i}, \\ H \star E_1 &= (H - E_1 - E_2)q^{H-E_1-E_2} + (H - E_1 - E_3)q^{H-E_1-E_3} + q^{H-E_1}, \\ H \star [pt] &= \sum_{1 \le i \le 3} (H - E_i)q^{H-E_i} + q^H + 2q^{2H-E_1-E_2-E_3}, \\ E_1 \star E_1 &= -[pt] + E_1 q^{E_1} + (H - E_1 - E_2)q^{H-E_1-E_2} + (H - E_1 - E_3)q^{H-E_1-E_3} + q^{H-E_1}, \\ E_1 \star E_2 &= (H - E_1 - E_2)q^{H-E_1-E_2}, \\ E_1 \star [pt] &= (H - E_1)q^{H-E_1} + q^{2H-E_1-E_2-E_3}. \end{aligned}$$

We again denote by $\tilde{M}_3$ the matrix of $\hat{c}_1$ with respect to the above basis, while denote by M_3 the matrix with respect to the following $\mathbb{Q}$-basis, in contrast to that for X_1, X_2.

$$\hat{c}_1[\mathbb{1}, c_1, E_1, E_2, E_3, [pt]] = [\mathbb{1}, c_1, E_1, E_2, E_3, [pt]]M_3.$$

Since $c_1 = 3H - E_1 - E_2 - E_3$, the matrices $\tilde{M}_3$ and M_3 are respectively given by

$$\tilde{M}_3 = \begin{bmatrix} 0 & 6 & 2 & 2 & 2 & 6 \\ 3 & 3 & 2 & 2 & 2 & 6 \\ -1 & -2 & -3 & -1 & -1 & -2 \\ -1 & -2 & -1 & -3 & -1 & -2 \\ -1 & -2 & -1 & -1 & -3 & -2 \\ 0 & 3 & 1 & 1 & 1 & 0 \end{bmatrix},\ M_3 = \begin{bmatrix} 0 & 12 & 2 & 2 & 2 & 6 \\ 1 & 1 & 2/3 & 2/3 & 2/3 & 2 \\ 0 & 0 & -7/3 & -1/3 & -1/3 & 0 \\ 0 & 0 & -1/3 & -7/3 & -1/3 & 0 \\ 0 & 0 & -1/3 & -1/3 & -7/3 & 0 \\ 0 & 6 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

3. Main technical results

3.1. **A generalization of Perron-Frobenius theorem.** By a matrix in this section, we always mean a square real finite matrix. The spectral radius of a matrix M is given by

$$\rho(M) := \max\{|\lambda| \mid \lambda \text{ is an eigenvalue of } M\}.$$

The matrix M is called *reducible*, if there exists a permutation matrix P such that P^tMP is of the form $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ where A, D are square submatrices. The matrix M is called *irreducible* if it is not reducible. The well-known Perron-Frobenius theory [Perr, Frob] concerns about properties on eigenvalues and eigenvectors of nonnegative irreducible matrices including the following proposition. It plays an essential role in the proof of conjecture $\mathcal{O}$ for flag varieties in [ChLi].

Proposition 3.1 (Theorem 1.5 of [Sene])**.** *Let M be an irreducible nonnegative matrix. Then the spectral radius $\rho(M)$ itself is an eigenvalue of M with multiplicity one.*

There have been various extensions of Perron-Frobenius theory (see [ElSz] and references therein). Here we give one more extension, which is new to our knowledge.

Theorem 3.2 (Generalized Perron-Frobenius Theorem)**.** *Let $T = (t_{ij})$ be an $n \times n$ real matrix that satisfies the following:*

1. $\sum_{i=1}^n t_{ij} > 0$ *for* $j = 1, \cdots, n$;
2. T^k *is an irreducible nonnegative matrix for some positive integer k.*

Then the spectral radius $\rho(T)$ is an eigenvalue of T with multiplicity one.

Remark 3.3. *The condition (1) can be replaced by $\sum_{j=1}^n t_{ij} > 0$ for $i = 1, \cdots, n$. Furthermore if the integer k in condition (2) is odd, then condition (1) is redundant, and the statement is an easy consequence of the Perron-Frobenius Theorem.*

Condition (1) was discovered by the fourth named author, who also showed the existence of a positive eigenvector corresponding to $\rho(T)$ in his Bachelor Thesis [Yang].

For column vectors $\mathbf{x} = (x_i)_{n\times 1}, \mathbf{y} = (y_i)_{n\times 1}$ in $\mathbb{R}^n$, we say $\mathbf{x} \le \mathbf{y}$ (resp. $\mathbf{x} < \mathbf{y}$) if $x_i \le y_i$ (resp. $x_i < y_i$) for all $i \in \{1, \cdots, n\}$.

Lemma 3.4. *Let T be an $n \times n$ irreducible nonnegative matrix.*

1. *There exists $m \in \mathbb{Z}_{>0}$ such that $(I_n + T)^m$ is a matrix with all entries positive.*
2. *Let $s \in \mathbb{R}_{>0}$, and $\mathbf{y} \ge \mathbf{0}$ be a nonzero vector satisfying $T\mathbf{y} \le s\mathbf{y}$. Then* (a) $\mathbf{y} > \mathbf{0}$; (b) $s \ge \rho(T)$. *Moveover, $s = \rho(T)$ if and only if $T\mathbf{y} = s\mathbf{y}$.*

Here the first statement is a well-known property for irreducible nonnegative matrices (see e.g. [Sene, Theorem 1.4]); the second statement is referred to as “The Sub-invariance Theorem” (see e.g. [Sene, Theorem 1.6]), which asserts that any irreducible nonnegative matrix has a unique nonnegative eigenvector up to a positive scalar.

Proof of Theorem 3.2. We first show that T has a positive eigenvalue as follows. The argument is similar to that for Theorem 1.1 (a) of [Sene].

Consider the upper-semicontinuous function defined by

$$r:\mathbb{R}^n_{\geq 0}\setminus\{\mathbf{0}\}\to\mathbb{R};\quad \mathbf{x}=(x_1,\cdots,x_n)^t\mapsto r(\mathbf{x}):=\min_{\substack{1\leq j\leq n\\ x_j\neq 0}}\frac{\sum_i x_it_{ij}}{x_j}.$$

By assumption (1) and the definition of $r(\mathbf{x})$, $r(\mathbf{1})>0$ where $\mathbf{1}:=(1,\cdots,1)^t$. Denote $K:=\max\limits_{1\leq i\leq n}\sum_j|t_{ij}|$. Since $x_jr(\mathbf{x})\leq\max\{0,\sum_i x_it_{ij}\}\leq\sum_i x_i|t_{ij}|$ for all j, we have $r(\mathbf{x})\sum_j x_j\leq\sum_j\sum_i x_i|t_{ij}|\leq\sum_i x_iK$ and hence $r(\mathbf{x})\leq K$. Therefore for

$$\rho:=\sup_{\mathbf{x}\in\mathbb{R}^n_{\geq 0}\setminus\{\mathbf{0}\}}r(\mathbf{x})=\sup_{\substack{\mathbf{x}\geq\mathbf{0}\\ \mathbf{x}^t\mathbf{x}=1}}r(\mathbf{x}),$$

we have $0<r(\mathbf{1})\leq\rho\leq K<+\infty$. Since $r(\mathbf{x})$ is upper-semicontinuous on the compact subset $\{\mathbf{x}\mid\mathbf{x}\geq 0,\mathbf{x}^t\mathbf{x}=1\}$, the supremum ρ is actually attained for some $\hat{\mathbf{x}}$ in this compact region. Consequently, we have $\sum_i\hat{x}_it_{ij}\geq\hat{x}_jr(\hat{\mathbf{x}})=\rho\hat{x}_j$ for all j, namely $\mathbf{z}^t:=\hat{\mathbf{x}}^tT-\rho\hat{\mathbf{x}}^t\geq\mathbf{0}^t$.

Assume $\mathbf{z}\neq\mathbf{0}$. By Lemma 3.4, all entries of $(I_n+T^k)^m$ are positive for some m. Thus

$$\mathbf{0}^t<\mathbf{z}^t(I_n+T^k)^m=\hat{\mathbf{x}}^t(I_n+T^k)^mT-\rho\hat{\mathbf{x}}^t(I_n+T^k)^m\quad\text{and}\quad\tilde{\mathbf{x}}:=\hat{\mathbf{x}}^t(I_n+T^k)^m>\mathbf{0},$$

ahence $\frac{\sum_i\tilde{x}_it_{ij}}{\tilde{x}_j}>\rho$ for all $1\leq j\leq n$. It follows that $r(\hat{\mathbf{x}})>\rho$, which is a contradiction to the definition of ρ. Hence, we have $\mathbf{z}=\mathbf{0}$, implying that ρ is an eigenvalue of T.

Now we have $\hat{\mathbf{x}}^tT=\rho\hat{\mathbf{x}}^t$. It follows that $(T^k)^t\hat{\mathbf{x}}=\rho^k\hat{\mathbf{x}}$, where we notice $\rho^k>0$ and $\hat{\mathbf{x}}\geq\mathbf{0}$. Therefore by assumption (2) and Lemma 3.4 (2), we have $\rho^k=\rho((T^k)^t)=\rho(T^k)$. For any eigenvalue λ of T, λ^k is an eigenvalue of T^k. It follows that $|\lambda^k|\leq\rho(T^k)=\rho^k$, and hence $|\lambda|\leq\rho$. That is, $\rho(T)=\rho$. Moreover, ρ^k is an eigenvalue of T^k of multiplicity one by Proposition 3.1. Thus ρ is an eigenvalue of T of multiplicity one. $\square$

The next property is a well-known consequence of the Perron-Frobenius theory on nonnegative matrices. The result was due to Perron when all the entries m_{ij} are positive.

Proposition 3.5. *Let $M=\big(m_{ij}\big)$ be an irreducible nonnegative matrix. Suppose M has exactly h eigenvalues of modulus $\rho(M)$ with multiplicities counted. If there exists i such that $m_{ij}>0$ for any j, then $h=1$, namely M has a unique eigenvalue with modulus $\rho(M)$ given by the simple eigenvalue $\rho(M)$ itself.*

Proof. The number h is traditionally called the *index of imprimitivity* (or *period*) of the irreducible matrix M. If $h>1$, then there exists a permutation matrix P such that PMP^t is of the following form (see e.g. Theorems 3.1 of Chapter 3 of [Minc]):

$$\begin{bmatrix}0&M_{12}&0&\cdots&0&0\\0&0&M_{2,3}&\cdots&0&0\\\vdots&&&\ddots&0&\vdots\\0&0&&\cdots&0&M_{h-1,h}\\M_{h1}&0&&\cdots&&0\end{bmatrix}.$$

However, the hypothesis implies that PMP^t always contains a row of positive numbers, which makes a contradiction. $\square$

Example 3.6. *The matrix M_1 (resp. M_2, M_3) in Example 2.4 (resp. 2.5, 2.6) obviously satisfies the condition (1) in Proposition 3.2. By direct calculations, we have*

$$M_1^3 = \begin{bmatrix} 26 & 1 & 7 & 28 \\ 28 & 26 & 1 & 8 \\ 7 & 21 & 5 & 1 \\ 1 & 7 & 21 & 26 \end{bmatrix}, \quad M_2^2 = \begin{bmatrix} 8 & 16 & 1 & 1 & 8 \\ \frac{1}{2} & 15 & 4 & 4 & \frac{9}{2} \\ \frac{1}{2} & 0 & 4 & 3 & \frac{7}{2} \\ \frac{1}{2} & 0 & 3 & 4 & \frac{7}{2} \\ 7 & 2 & 0 & 0 & 8 \end{bmatrix},$$

and

$$M_3^2 = \begin{bmatrix} 12 & 48 & 8 & 8 & 8 & 24 \\ 1 & 25 & 8/3 & 8/3 & 8/3 & 8 \\ 0 & 0 & 17/3 & 5/3 & 5/3 & 0 \\ 0 & 0 & 5/3 & 17/3 & 5/3 & 0 \\ 0 & 0 & 5/3 & 5/3 & 17/3 & 0 \\ 6 & 6 & 1 & 1 & 1 & 12 \end{bmatrix}.$$

Both M_1^3 and M_2^2 are irreducible nonnegative matrices, whose first row consist of positive numbers. Thus for $i \in \{1,2\}$, $\rho(M_i)$ is an eigenvalue of M_i of multiplicity one by Theorem 3.2, and it is the unique eigenvalue of M_i with modulus $\rho(M_i)$ by Proposition 3.5. Although the matrix M_3^2 is reducible, we can still conclude the same property for $\rho(M_3)$, by applying a consequence of Theorem 3.2 (see Proposition 4.1).

Remark 3.7. *The matrix $\tilde{M}_1$ in Example 2.4 or another matrix $\tilde{M}_1'$ with respect to the $\mathbb{Q}$-basis $\{\mathbb{1}, c_1, E_1, [pt]\}$ of $H^*(X_1)$ both satisfy the condition (1) in Theorem 3.2. However, there does not exist positive integer k such that $\tilde{M}_1^k$ or $(\tilde{M}_1')^k$ is a nonnegative matrix. The situation for X_2 is the same. The condition (1) in Theorem 3.2 even fails for $\tilde{M}_3$.*

3.2. **Vanishing properties of Gromov-Witten invariants of del Pezzo surfaces.** In this subsection, we assume $3 \le r \le 8$ and simply denote $c_1 := c_1(X_r)$. We denote

$$\mathrm{Amb}_r := \mathbb{Q}\mathbb{1} \oplus \mathbb{Q}c_1 \oplus \mathbb{Q}[pt] \quad \text{and} \quad \mathrm{Prim}_r := \{\gamma \in H^2(X_r) : c_1 \cup \gamma = 0\}.$$

Then we obtain $H^*(X_r) = \mathrm{Amb}_r \oplus \mathrm{Prim}_r$ as an orthogonal decomposition of vector spaces with respect to the Poincaré pairing. Moreover, Amb_r is the subalgebra of $H^*(X_r)$ generated by c_1.

The main result of this subsection is the following, which will be used to simplify part of the entries of the matrix corresponding to the operator $\hat{c}_1$ in section 4.1.

Theorem 3.8. *For any $m, a, a_i \in \mathbb{Z}_{\geq 0}, \gamma \in \mathrm{Prim}_r$ and $\gamma_i \in \mathrm{Amb}_r$, $i = 1, \cdots, m$, we have*

$$\sum_{A \in H_2(X_r,\mathbb{Z})} \langle \tau_a(\gamma) \prod_{i=1}^{m} \tau_{a_i}(\gamma_i) \rangle_A = 0. \tag{2}$$

Remark 3.9. *There are only finitely many nonzero terms in the above summation by* [KoMo, Corollary 1.19] *together with the observation that $\langle \tau_a(\gamma) \prod_{i=1}^{m} \tau_{a_i}(\gamma_i) \rangle_A$ is nonzero only if A is an effective class with $\int_A c_1 = (a + \frac{1}{2}\deg\gamma) + \sum_{i=1}^{m}(a_i + \frac{1}{2}\deg\gamma_i) - m$.*

The rest of this subsection is devoted to a proof of Theorem 3.8.

Lemma 3.10. *For any $m, a, a_i \in \mathbb{Z}_{\geq 0}, \gamma \in \mathrm{Prim}_r$ and $\gamma_i \in \mathrm{Amb}_r$, $i = 1, \cdots, m$, we have*

$$\langle \tau_a(\gamma) \prod_{i=1}^{m} \tau_{a_i}(\gamma_i) \rangle_0 = 0$$

Proof. Note that the obstruction theory on $\overline{M}_{0,m+1}(X_r, 0) = X_r \times \overline{M}_{0,m+1}$ is trivial. So the vanishing result follows from the definition of Prim_r. $\square$

Gromov-Witten invariants of X_r admit two types of symmetry. The classical Cremona transformation of $\mathbb{P}^2$ induces an involution σ on $H_2(X_r, \mathbb{Z})$, given by

$$\sigma(dH - \sum_{k=1}^{r} d_k E_k) = (2d - d_1 - d_2 - d_3)H - \sum_{1 \leq k \leq 3} (d - d_1 - d_2 - d_3 + d_k) E_k - \sum_{4 \leq k \leq r} d_k E_k.$$

This leads to the first type of symmetry below, by the argument in [GöPa, Section 5.1].

Lemma 3.11. *For any $m, a, a_i \in \mathbb{Z}_{\geq 0}$ and $\gamma_i \in \mathrm{Amb}_r$, $i = 1, \cdots, m$, we have*

$$\langle \prod_{i=1}^{m} \tau_{a_i}(\gamma_i) \rangle_A = \langle \prod_{i=1}^{m} \tau_{a_i}(\gamma_i) \rangle_{\sigma(A)} \quad \textit{for any } A \in H_2(X_r, \mathbb{Z}).$$

For each $j \in \{1, \cdots, r-1\}$, we define an involution σ_j on $H_2(X_r, \mathbb{Z})$ by

$$\sigma_j(dH - \sum_{k=1}^{r} d_k E_k) = dH - \sum_{\substack{1 \leq k \leq r \\ k \neq j, j+1}} d_k E_k - d_{j+1} E_j - d_j E_{j+1}.$$

Inspired by [GöPa, Section 5.1], we obtain the second type of symmetry as follows.

Lemma 3.12. *For $1 \leq j \leq r-1, m, a, a_i \in \mathbb{Z}_{\geq 0}$ and $\gamma_i \in \mathrm{Amb}_r$, $i = 1, \cdots, m$, we have*

$$\langle \prod_{i=1}^{m} \tau_{a_i}(\gamma_i) \rangle_A = \langle \prod_{i=1}^{m} \tau_{a_i}(\gamma_i) \rangle_{\sigma_j(A)} \quad \textit{for any } A \in H_2(X_r, \mathbb{Z}).$$

Proof. Let X_0 and $\bar{X}_0$ be two copies of $\mathbb{P}^2$, and we fix an isomorphism $\varphi_0 : X_0 \cong \bar{X}_0$. Suppose that $p_1, \cdots, p_r \in X_0$ and $\bar{p}_1, \cdots, \bar{p}_r \in \bar{X}_0$ are in general position, such that

$$\varphi_0(p_k) = \begin{cases} \bar{p}_k, & k \neq j, j+1, \\ \bar{p}_{j+1}, & k = j, \\ \bar{p}_j, & k = j+1. \end{cases}$$

Let X_r (resp. $\bar{X}_r$) be the blow-up of X_0 at p_k (resp. $\bar{X}_0$ at $\bar{p}_k$), $k = 1, \cdots, r$, with H (resp. $\bar{H}$) being the pullback of a line in X_0 (resp. $\bar{X}_0$) in general position, and E_k (resp. $\bar{E}_k$) being the exceptional divisor corresponding to p_k (resp. $\bar{p}_k$). For $\gamma = x\mathbb{1}_{X_r} + yc_1(X_r) + z[pt]_{X_r} \in \mathrm{Amb}(X_r)$, we denote $\bar{\gamma} = x\mathbb{1}_{\bar{X}_r} + yc_1(\bar{X}_r) + z[pt]_{\bar{X}_r} \in \mathrm{Amb}(\bar{X}_r)$. On one hand, by deformation invariance of Gromov-Witten invariants, we have

$$\langle \prod_{i=1}^{m} \tau_{a_i}(\gamma_i) \rangle^{X_r}_{dH - \sum_{k=1}^{r} d_k E_k} = \langle \prod_{i=1}^{m} \tau_{a_i}(\bar{\gamma}_i) \rangle^{\bar{X}_r}_{d\bar{H} - \sum_{k=1}^{r} d_k \bar{E}_k}. \tag{3}$$

On the other hand, we have an isomorphism $\varphi_r : X_r \xrightarrow{\cong} \bar{X}_r$ naturally induced from φ_0. The induced isomorphisms $\varphi_r^* : H^*(\bar{X}_r, \mathbb{Z}) \to H^*(X_r, \mathbb{Z})$ and $(\varphi_r)_* : H_*(X_r, \mathbb{Z}) \to$

$H_*(\bar{X}_r,\mathbb{Z})$ satisfy the following properties: $\varphi_r^*(\bar{H})=H$, $\quad (\varphi_r)_*(H)=\bar{H}$

$$\varphi_r^*(\bar{E}_i)=\begin{cases} E_i, & i\neq j,j+1,\\ E_{j+1}, & i=j,\\ E_j, & i=j+1,\end{cases} \quad \text{and} \quad (\varphi_r)_*(E_i)=\begin{cases} \bar{E}_i, & i\neq j,j+1,\\ \bar{E}_{j+1}, & i=j,\\ \bar{E}_j, & i=j+1.\end{cases}$$

Here we remind of our notation convention that divisors are naturally treated as (co)homology classes in the corresponding setting. Consequently, we have $\varphi_r^*(c_1(\bar{X}_r))=c_1(X_r)$, which implies $\varphi_r^*(\mathbb{1}_{\bar{X}_r})=\mathbb{1}_{X_r}$ and $\varphi_r^*([pt]_{\bar{X}_r})=[pt]_{X_r}$. Therefore

$$\begin{aligned}(4)\quad \langle\prod_{i=1}^m\tau_{a_i}(\gamma_i)\rangle^{X_r}_{dH-\sum_{k=1}^r d_kE_k} &= \langle\prod_{i=1}^m\tau_{a_i}((\varphi_r^*)^{-1}(\gamma_i))\rangle^{\bar{X}_r}_{(\varphi_r)_*d(\bar{H}-\sum_{k=1}^r d_k\bar{E}_k)}\\ &= \langle\prod_{i=1}^m\tau_{a_i}(\bar{\gamma}_i)\rangle^{\bar{X}_r}_{d\bar{H}-\sum_{\substack{1\le k\le r\\ k\neq j,j+1}} d_k\bar{E}_k-d_{j+1}\bar{E}_j-d_j\bar{E}_{j+1}}.\end{aligned}$$

Now the required result follows from (3) and (4). $\square$

Lemma 3.13. *For $m,a,a_i\in\mathbb{Z}_{\geq 0},\gamma\in\mathrm{Prim}_r$ and $\gamma_i\in\mathrm{Amb}_r$, $i=1,\cdots,m$, we have*

$$\sum_{A\in H_2(X_r,\mathbb{Z})}(\int_A\gamma)\langle\prod_{i=1}^m\tau_{a_i}(\gamma_i)\rangle_A=0.$$

Proof. It suffices to show the statement for $\gamma=H-E_1-E_2-E_3$ and $\gamma=E_j-E_{j+1}(1\le j\le r-1)$, since these classes form a basis of Prim_r.

For $\gamma=H-E_1-E_2-E_3$, by Lemma 3.11, we have

$$\begin{aligned}&\sum_A(\int_A\gamma)\langle\prod_{i=1}^m\tau_{a_i}(\gamma_i)\rangle_A\\ =\;&\frac12\sum_A(\int_A\gamma)\langle\prod_{i=1}^m\tau_{a_i}(\gamma_i)\rangle_A+\frac12\sum_A(\int_{\sigma(A)}\gamma)\langle\prod_{i=1}^m\tau_{a_i}(\gamma_i)\rangle_{\sigma(A)}\\ =\;&\sum_A\frac12(\int_{A+\sigma(A)}\gamma)\langle\prod_{i=1}^m\tau_{a_i}(\gamma_i)\rangle_A.\end{aligned}$$

Now the required vanishing result follows from $\int_{A+\sigma(A)}(H-E_1-E_2-E_3)=0$.

For $\gamma=E_j-E_{j+1}(1\le j\le r-1)$, by Lemma 3.12, we have

$$\begin{aligned}&\sum_A(\int_A\gamma)\langle\prod_{i=1}^m\tau_{a_i}(\gamma_i)\rangle_A\\ =\;&\frac12\sum_A(\int_A\gamma)\langle\prod_{i=1}^m\tau_{a_i}(\gamma_i)\rangle_A+\frac12\sum_A(\int_{\sigma_j(A)}\gamma)\langle\prod_{i=1}^m\tau_{a_i}(\gamma_i)\rangle_{\sigma_j(A)}\\ =\;&\sum_A\frac12(\int_{A+\sigma_j(A)}\gamma)\langle\prod_{i=1}^m\tau_{a_i}(\gamma_i)\rangle_A.\end{aligned}$$

Now the required vanishing result follows from $\int_{A+\sigma_j(A)}(E_j-E_{j+1})=0$. $\square$

Proposition 3.14. *Let $m, a, a_i \in \mathbb{Z}_{\geq 0}, \gamma \in \mathrm{Prim}_r$ and $\gamma_i \in \mathrm{Amb}_r$, $i = 1, \cdots, m$. If $\gamma_i \in H^{>0}(X_r)$ whenever $a_i \neq 0$, then we have*

$$\sum_{A\in H_2(X_r,\mathbb{Z})} \langle \tau_0(\gamma) \prod_{i=1}^m \tau_{a_i}(\gamma_i)\rangle_A = 0.$$

Proof. It suffices to show the vanishing of $\sum_{A\neq 0} \langle \tau_0(\gamma) \prod_{i=1}^m \tau_{a_i}(\gamma_i)\rangle_A$ due to Lemma 3.10. We first assume $\gamma_i \in H^{>0}(X_r)$ for all i, so that $\gamma \cup \gamma_i = 0$ always holds. Hence, we have

$$\sum_{A\neq 0}\langle \tau_0(\gamma) \prod_{i=1}^m \tau_{a_i}(\gamma_i)\rangle_A = \sum_{A\neq 0}(\int_A \gamma)\langle \prod_{i=1}^m \tau_{a_i}(\gamma_i)\rangle_A = 0.$$

by using the Divisor Axiom and Lemma 3.13. Then the general statement follows immediately from the Fundamental Class Axiom. □

Proposition 3.15. *Let $m, a, a_i \in \mathbb{Z}_{\geq 0}, \gamma \in \mathrm{Prim}_r$ and $\gamma_i \in \mathrm{Amb}_r$, $i = 1, \cdots, m$. If $\gamma_i \in H^{>0}(X_r)$ for all i, then we have*

$$\sum_{A\in H_2(X_r,\mathbb{Z})} \langle \tau_a(\gamma) \prod_{i=1}^m \tau_{a_i}(\gamma_i)\rangle_A = 0.$$

Proof. We use induction on a. The case $a = 0$ is done in Proposition 3.14. Now assume that the case $a = a' \geq 0$ is verified. For $a = a' + 1$, there are the following three cases.

(i) $\underline{m \geq 2}$. Extend $\{\phi_1 = \mathbb{1}, \phi_2 = c_1, \phi_3 = [pt]\}$ to a basis $\{\phi_\alpha\}_\alpha$ of $H^*(X_r)$ such that $\phi_\alpha \in \mathrm{Prim}_r$ for $\alpha > 3$, and let $\{\phi^\alpha\}_\alpha$ be its dual basis. Using TRR, we conclude that the quantity $\sum_A \langle \tau_{a'+1}(\gamma) \prod_{i=1}^m \tau_{a_i}(\gamma_i)\rangle_A$ is equal to

$$\sum_I \sum_\alpha \Big(\sum_A \langle \tau_0(\phi_\alpha)\tau_{a'}(\gamma) \prod_{i\in I} \tau_{a_i}(\gamma_i)\rangle_A\Big)\cdot\Big(\sum_A \langle \tau_0(\phi^\alpha)\tau_{a_1}(\gamma_1)\tau_{a_2}(\gamma_2) \prod_{j\in\{3,\cdots,m\}\setminus I} \tau_{a_j}(\gamma_j)\rangle_A\Big),$$

where the first summation is over subsets $I \subset \{3, \cdots, m\}$. For $\alpha \leq 3$, the first big parentheses on RHS is zero by induction (together with the Fundamental Class Axiom when $\alpha = 1$); for $\alpha > 3$, we note $\phi^\alpha \in \mathrm{Prim}_r$, and hence the second big parentheses on RHS is zero by Proposition 3.14. So LHS is vanishing.

(ii) $\underline{m = 1}$. By (i), Lemma 3.10, the Divisor Axiom and the Degree Axiom, we have

$$0 = \sum_A \langle \tau_{a'+1}(\gamma)\tau_{a_1}(c_1^2)\tau_0(c_1)\rangle_A = (a' + a_1 + \frac{\deg\gamma}{2} + 2)\sum_A \langle \tau_{a'+1}(\gamma)\tau_{a_1}(c_1^2)\rangle_A.$$

Here we have assumed γ to be homogenous without loss of generality. Similarly,

$$\begin{aligned} 0 &= \sum_A \langle \tau_{a'+1}(\gamma)\tau_{a_1}(c_1)\tau_0(c_1)\rangle_A \\ &= (a' + a_1 + \frac{\deg\gamma}{2} + 1)\sum_A \langle \tau_{a'+1}(\gamma)\tau_{a_1}(c_1)\rangle_A + \sum_A \langle \tau_{a'+1}(\gamma)\tau_{a_1-1}(c_1^2)\rangle_A \end{aligned}$$

It follows that $\sum_A \langle \tau_{a'+1}(\gamma)\tau_{a_1}(\gamma_1)\rangle_A = 0$ whenever $\gamma_1 \in \mathrm{Amb}_r \cap H^{>0}(X_r)$.

(iii) $\underline{m = 0}$. By using (ii), Lemma 3.10, the Divisor Axiom and the Degree Axiom, we are done:

$$0 = \sum_A \langle \tau_{a'+1}(\gamma)\tau_0(c_1)\rangle_A = (a' + \frac{\deg\gamma}{2} + 1)\sum_A \langle \tau_{a'+1}(\gamma)\rangle_A.$$

This proves the case $a = a' + 1$. $\square$

Proof of Theorem 3.8. By Proposition 3.15, it remains to show that for any $m_0, m_1 \geq 0$,

$$\sum_A \langle \tau_a(\gamma) \prod_{i=1}^{m_0} \tau_{a_i}(\mathbb{1}) \prod_{j=1}^{m_1} \tau_{b_j}(\gamma_j) \rangle_A = 0$$

holds for any $a, a_i, b_j \geq 0$, $\gamma \in \mathrm{Prim}_r$ and $\gamma_j \in \mathrm{Amb}_r \cap H^{>0}(X_r)$, where $1 \leq i \leq m_0$ and $1 \leq j \leq m_1$.

We use induction on m_0. The case $m_0 = 0$ is proved in Proposition 3.15. Now assume that the cases $m_0 \leq m_0'$ are verified. For $m_0 = m_0' + 1$, by Lemma 3.10 and the Fundamental Class Axiom, we can assume $a_i \geq 1$ for all i. There are two cases as follows.

(i) $\underline{m_1 > 0}$. We use the same (dual) basis $\{\phi_\alpha\}_\alpha$ (resp. $\{\phi^\alpha\}_\alpha$) as in case (i) in the proof of Proposition 3.15. Using TRR, we have

$$\begin{aligned} &\sum_A \langle \tau_a(\gamma) \prod_{i=1}^{m_0'+1} \tau_{a_i}(\mathbb{1}) \prod_{i=1}^{m_1} \tau_{b_i}(\gamma_i) \rangle_A \\ = &\sum_{\substack{I_0 \sqcup J_0 = \{2, \cdots, m_0'+1\} \\ I_1 \sqcup J_1 = \{2, \cdots, m_1\}}} \sum_\alpha \Big(\sum_A \langle \tau_0(\phi_\alpha) \tau_{a_1 - 1}(\mathbb{1}) \prod_{i \in I_0} \tau_{a_i}(\mathbb{1}) \prod_{i \in I_1} \tau_{b_i}(\gamma_i) \rangle_A \Big) \\ &\cdot \Big(\sum_A \langle \tau_0(\phi^\alpha) \tau_a(\gamma) \tau_{b_1}(\gamma_1) \prod_{j \in J_0} \tau_{a_j}(\mathbb{1}) \prod_{j \in J_1} \tau_{b_j}(\gamma_j) \rangle_A \Big). \end{aligned}$$

If $\phi_\alpha \in \mathrm{Prim}_r$, then the first big parentheses on RHS is zero by induction on $\sum_{i=1}^{m_0} a_i$; if $\phi_\alpha \in \mathrm{Amb}_r$, then the second big parentheses on RHS is zero by induction on m_0. So LHS is vanishing.

(ii) $\underline{m_1 = 0}$. We use Lemma 3.10, the Divisor Axiom and the Degree Axiom to get

$$\begin{aligned} &\sum_A \langle \tau_a(\gamma) \prod_{i=1}^{m_0'+1} \tau_{a_i}(\mathbb{1}) \cdot \tau_0(c_1) \rangle_A \\ = \; &(a + \frac{\deg \gamma}{2} + \sum_{i=1}^{m_0'+1} (a_i - 1)) \sum_A \langle \tau_a(\gamma) \prod_{i=1}^{m_0'+1} \tau_{a_i}(\mathbb{1}) \rangle_A \\ &+ \sum_{i=1}^{m_0'+1} \sum_A \langle \tau_a(\gamma) \tau_{a_i - 1}(c_1) \prod_{\substack{1 \leq j \leq m_0'+1 \\ j \neq i}} \tau_{a_j}(\mathbb{1}) \rangle_A. \end{aligned}$$

Note that LHS is zero by case (i), and the second term on RHS is zero by induction on m_0. This gives the required vanishing result.

This proves the case $m_0 = m_0' + 1$. $\square$

4. CONJECTURE $\mathcal{O}$ FOR DEL PEZZO SURFACES

In this section, we will prove Theorem 1.1, namely the conjecture $\mathcal{O}$ for del Pezzo surfaces, by using the generalized Perron-Frobenius theorem in section 3.1. We will just discuss X_r with $1 \leq r \leq 8$, as it has been known for $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$. We remark that Theorem 1.1 could also be directly proved by analysing the characteristic polynomial of $\hat{c}_1$ which was studied in [BaMa] based on explicit descriptions of the relevant Gromov-Witten invariants [GöPa]. However, we expect our method to have further applications for other Fano manifolds. As we will see, our proof will only require information on parts of the Gromov-Witten invariants.

4.1. **Proof of Theorem 1.1.** Eigenvalues of $\hat{c}_1$ on $QH^\bullet(X_r)$ coincide with that of the matrix of $\hat{c}_1$ with respect to any choice of bases of $H^*(X_r) = H^\bullet(X_r)$. We take the $\mathbb{Z}$-basis as in Examples 2.4 and 2.5 if $r \in \{1, 2\}$, and take the $\mathbb{Q}$-basis $\{\mathbb{1}, c_1, E_1, \cdots, E_r, [pt]\}$ of $H^*(X_r)$ for $3 \leq r \leq 8$. The corresponding matrices M_r have been explicitly described in section 2.2.3 for $1 \leq r \leq 3$. Let us achieve our aim by assuming the next proposition first.

Proposition 4.1. *Write $M_r = \big(m_{ij}\big)$ and $M_r^2 = \big(m_{ij}^{(2)}\big)$. For $3 \leq r \leq 8$, we have*

(1) $\sum_{i=1}^{r+3} m_{ij} > 0$ *for any $j \in \{1, \cdots, r+3\}$, and $m_{2j} \geq 0$ for any j;*
(2) $m_{ij}^{(2)} \geq 0$ *for any i, j, and $m_{ij}^{(2)} > 0$ holds if $i \in \{1, 2, r+3\}$;*
(3) $m_{22}^{(2)} > \sum_{k=3}^{r+2} m_{ik}^{(2)}$ *if $3 \leq i \leq r+2$;*

Now we set $P := (a-1)E_{22} + \sum_{k=1}^{r+3} E_{kk} + b\sum_{k=3}^{r+2} E_{k2}$ where $a, b > 0$ and E_{ij} denotes the $(r+3)\times(r+3)$ matrix whose entries are all zero but the (i,j)-entry given by 1. It follows that $P^{-1} = (c-1)E_{22} + \sum_{k=1}^{r+3} E_{kk} + d\sum_{k=3}^{r+2} E_{k2}$ with $d = -\frac{b}{a} = -bc$. Write $PM_rP^{-1} = \hat{M}_r = \big(\hat{m}_{ij}\big)$ and $\hat{M}_r^2 = \big(\hat{m}_{ij}^2\big)$. By direct calculations, we have

$$\hat{m}_{ij} = \begin{cases} m_{ij}, & \text{if } i = 1, 3 \text{ and } j \neq 2, \\ am_{2j}, & \text{if } i = 2 \text{ and } j \neq 2, \\ m_{ij} + bm_{2j}, & \text{if } 3 \leq i \leq r+2 \text{ and } j \neq 2 \end{cases}$$

and

$$\hat{m}_{i2} = \begin{cases} cm_{i2} + d\sum_{k=3}^{r+2} m_{ik}, & \text{if } i = 1, 3, \\ acm_{22} + da\sum_{k=3}^{r+2} m_{2k}, & \text{if } i = 2, \\ cm_{i2} + cbm_{22} + d\sum_{k=3}^{r+2}(m_{ik} + bm_{i,k}), & \text{if } 3 \leq i \leq r+2. \end{cases}$$

Clearly, M_r and $\hat{M}_r$ have same eigenvalues for any $a \neq 0$. In particular, we can choose sufficiently small positive numbers b, $\frac{b}{c}$ with $c < 1$. It follows that $a > 1$, and

$$\sum_{i=1}^{r+3} \hat{m}_{ij} \geq \sum_{i=1}^{r+3} m_{ij} + b\sum_{k=3}^{r+2} m_{2j} > 0 \text{ if } j \neq 2;$$
$$\sum_{i=1}^{r+3} \hat{m}_{i2} \geq c\big(\sum_{i=1}^{r+3} m_{i2} + brm_{22} - (b + \frac{b}{c} + b + b^2)\sum_{i=1}^{r+3}\sum_{k=3}^{r+2} |m_{ik}|\big) > 0.$$

The expression of $\hat{m}_{ij}^{(2)}$ can be read off directly from that of $\hat{m}_{ij}$ by replacing m_{ij} with $m_{ij}^{(2)}$. It follows that $\hat{m}_{ij}^{(2)} > 0$ for any i, j. (Indeed for $3 \leq i \leq r+2$, $\hat{m}_{i2}^{(2)} = cm_{i2}^{(2)} + cb(m_{22}^{(2)} - \sum_{k=3}^{r+2} m_{ik}^{(2)}) - b\sum_{k=3}^{r+2} m_{ik}^{(2)} > 0$, as $cm_{i2}^{(2)} \geq 0, cb > 0, m_{22}^{(2)} - \sum_{k=3}^{r+2} m_{ik}^{(2)} > 0$ and $b > 0$ is sufficiently small. Arguments for the remaining cases are similar and easier.) Applying Theorem 3.2 for $\hat{M}$, we conclude that $\rho(\hat{c}_1) = \rho(M_r) = \rho(\hat{M})$ is an eigenvalue of $\hat{c}_1$ of multiplicity one. Moreover, it is the unique eigenvalue of $\hat{c}_1$ with modulus $\rho(\hat{M})$ by Proposition 3.5. That is, X_r satisfies Property $\mathcal{O}$.

4.2. **Proof of Proposition 4.1.** By Example 2.6, Proposition 4.1 holds for M_3. Therefore we just consider $4 \leq r \leq 8$ in this subsection. The dual basis of $[\phi_1, \cdots, \phi_{r+3}] := [\mathbb{1}, c_1, E_1, \cdots, E_r, [pt]]$ in $H^*(X_r)$is given by

$$[\mathbb{1}^\vee, c_1^\vee, E_1^\vee, \cdots, E_r^\vee, [pt]^\vee] = [[pt], \frac{H}{3}, \frac{H}{3} - E_1, \cdots, \frac{H}{3} - E_r, \mathbb{1}].$$

We will prove Proposition 4.1 by showing that the matrix M_r is in fact of the following form with required properties

$$M_r = \begin{bmatrix} 0 & m_{12} & m_{13} & m_{14} & \cdots & m_{1,r+2} & m_{1,r+3} \\ 1 & m_{22} & m_{23} & m_{24} & \cdots & m_{2,r+2} & m_{2,r+3} \\ 0 & 0 & d_r & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & d_r & \ddots & \vdots & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & d_r & \vdots \\ 0 & 9-r & 1 & 1 & \cdots & 1 & 0 \end{bmatrix}. \tag{5}$$

Let us start with simple calculations. Entries m_{ij} of M_r are genus zero Gromov-Witten invariants. Clearly, we have $m_{i1} = \delta_{i,2}$ since $\mathbb{1}$ is the identity element in $QH^*(X_r)$. For the last row, we have $m_{r+3,j} = \sum_{A \in H_2(X_r,\mathbb{Z})} \langle c_1, \phi_j, \mathbb{1} \rangle_A$ and hence $m_{r+3,j} = \langle c_1, \phi_j, \mathbb{1} \rangle_0$ by the Fundamental Class Axiom, namely it is given by the coefficient of $[pt]$ in the classical cup product $c_1 \cup \phi_i$. Consequently, we have $m_{r+3,1} = m_{r+3,r+3} = 0$ for the degree reason, $m_{r+3,2} = c_1 \cdot c_1 = 9-r$, and $m_{r+3,j} = c_1 \cdot E_{j-2} = 1$ for $3 \leq j \leq r+2$.

A smooth rational curve E of X_r is called exceptional if $E \cdot E = -1$ and $c_1 \cdot E = 1$. A genus zero Gromov-Witten invariant $\langle \cdots \rangle_A$ for X_r is nonzero only if $A \in H_2(X_r, \mathbb{Z})$ is effective, which can be characterized in terms of effective divisors as follows.

Proposition 4.2 (Corollary 3.3 of [BaPo])**.** *The semigroup of classes of effective divisors on X_r is generated by the classes of exceptional curves if $4 \leq r \leq 7$ and by the classes of exceptional curves together with c_1 for $r = 8$.*

Proposition 4.3 ([Manin])**.** *The classes of exceptional curves are precisely as follows.*

1. E_i, $1 \leq i \leq r$;
2. $H - E_i - E_j$, $1 \leq i < j \leq r$;
3. $2H - E_{i_1} - \cdots - E_{i_5}$, $\quad 1 \leq i_1 < i_2 < \cdots < i_5 \leq r$;
4. $3H - E_k - \sum_{j=1}^{7} E_{i_j}$, $\quad 1 \leq i_1 < i_2 < \cdots < i_7 \leq r$ *and* $k \in \{i_1, \cdots, i_7\}$;
5. $4H - \sum_{j=1}^{8} E_j - E_{i_1} - E_{i_2} - E_{i_3}$, $\quad 1 \leq i_1 < i_2 < i_3 \leq 8 = r$;
6. $5H - \sum_{j=1}^{8} 2E_j + E_{i_1} + E_{i_2}$, $\quad 1 \leq i_1 < i_2 \leq 8 = r$;
7. $6H - \sum_{j=1}^{8} 2E_j - E_i$, $\quad 1 \leq i \leq 8 = r$.

For convenience, we will simply denote a summation $\sum_{A \,:\, A \in H_2(X_r,\mathbb{Z}) \text{ is effective}; ***}$ by $\sum_{***}$ in the rest of this subsection. The next lemma ensures that the summation to be taken does contain one term of positive Gromov-Witten invariants. The first statement holds since Gromov-Witten invariants for X_r are enumerative [GöPa]; the second statement can be easily deduced from the results therein (or by using Theorems 1.2 and 1.4 of [Hu]).

Lemma 4.4. (1) *For any $A \in H_2(X_r, \mathbb{Z})$, we have $\langle [pt] \rangle_A \geq 0$, $\langle [pt], [pt] \rangle_A \geq 0$.*

(2) $\langle [pt]\rangle_{H-E_1-E_2} = \langle [pt]\rangle_{H-E_1} = \langle [pt],[pt]\rangle_H = 1$; $\langle [pt]\rangle_{E_i} = 0$ *for* $1\le i\le r$.

Here we notice $\int_{H-E_1-E_2} c_1 = 1$, $\int_{H-E_1} c_1 = 2$ and $\int_H c_1 = 3$.

Lemma 4.5. $c_1 \bullet c_1 = m_{12}\mathbb{1} + m_{22}c_1 + (9-r)[pt]$ *where* $m_{12} = \sum_{\int_A c_1=2} 4\langle [pt]\rangle_A > 0$ *and* $m_{22} = \sum_{\int_A c_1=1}\int_A \frac{H}{3} > 0$.

Proof. Recall the notation $\alpha\bullet\beta = \alpha\star\beta|_{\mathbf{q}=\mathbf{1}}$. Noting $\mathbb{1}^\vee = [pt]\in H^4(X_r)$, we have $m_{21} = \sum_{A\,:\,\int_A c_1=2}\langle c_1, c_1, \mathbb{1}^\vee\rangle_A$ by the Degree Axiom, and hence $m_{21} = \sum\limits_{\int_A c_1=2} 4\langle [pt]\rangle_A > 0$ by the Divisor Axiom and Lemma 4.4.

The computation for m_{22} is the same. The quantity $m_{r+3,2} = 9-r$ has been discussed above. Since $c_1\cup E_j^\vee = \big(\int_{[X_r]} c_1\cup E_j^\vee\big)[pt] = 0$, we have $E_j^\vee\in \mathrm{Prim}_r$, and consequently $m_{2,2+j} = \sum_A \langle c_1, c_1, E_j^\vee\rangle_A = 0$ for $1\le j\le r$ by Proposition 3.8. ☐

Lemma 4.6. *For any* $1\le j\le r$, *we have*

(1) $m_{1,j+2} = \sum\limits_{\int_A c_1=2} 2\big(\int_A E_j\big)\langle [pt]\rangle_A, \qquad m_{1,r+3} = \sum_{\int_A c_1=3} 3\langle [pt],[pt]\rangle_A$;

(2) $m_{2,j+2} = \sum\limits_{\int_A c_1=1}\big(\int_A E_j\big)\big(\int_A \frac{H}{3}\big), \qquad m_{2,r+3} = \sum_{\int_A c_1=2} 2\big(\int_A \frac{H}{3}\big)\langle [pt]\rangle_A$.

Furthermore, they are all positive.

Proof. Calculations for m_{ij} in the statement are the same as that for m_{21} in Lemma 4.5.

Write $A\in H_2(X_r,\mathbb{Z})$ as $A = a_0H - \sum_{i=1}^r a_iE_i$. Every effective curve class A is a nonnegative combination of classes of effective curve classes (together with c_1 if $r=8$). The hypotheses $\int_A c_1 = 2$ and $\langle [pt]\rangle_A \neq 0$ will imply that $a_0 > 0$ and $a_i\ge 0$ for $1\le i\le r$, in which case we have $\int_A E_j \ge 0$. Therefore we conclude $m_{1,j+2} > 0$ for $1\le j\le r$ by Lemma 4.4. Arguments for the remaining cases are similar. ☐

Lemma 4.7. *For any* $1\le i\le r$, *we have* $m_{2+i,2+i} = d_r$, *where* $d_r := -3, -4, -6, -12, -60$ *for* $r = 4,5,6,7,8$ *respectively.*

Proof. We have $m_{2+i,2+i} = \sum_{\int_A c_1=1}\big(\int_A E_i\big)\big(\int_A \frac{H}{3} - E_i\big)$ by definition and the Divisor Axiom. Without loss of generality, we can assume $i = r$. Since $\int_A c_1 = 1$, the summation can be reduced to those over the classes of exception curves. We discuss all the possibilities with respect to the cases in Proposition 4.3.

(1) The only nonzero contribution is given by $A = E_r$, and $\int_A E_r \int_A(\frac{H}{3} - E_r) = -1$.
(2) The nonzero contributions come from those $H - E_{i_1} - E_r$ with $1\le i_1 < r$, each of which contributes a same quantity; therefore the total contribution is given by $(r-1)\int_{H-E_1-E_r} E_r\big(\int_{H-E_1-E_r}\frac{H}{3} - E_r\big) = -\frac{2(r-1)}{3}$.
(3) Here $r\ge 5$. The total nonzero contribution is given by

$$C_{r-1}^4\cdot\Big(\int_{2H-E_1-E_2-E_3-E_4-E_r} E_r\Big)\Big(\int_{2H-E_1-E_2-E_3-E_4-E_r}\frac{H}{3} - E_r\Big) = -\frac{1}{3}C_{r-1}^4.$$

(4) Here $r\ge 7$. The total nonzero contribution is as follows. (We notice that classes of the form $3H - 2E_1 - \sum_{j=2}^6 E_j - E_r$ does not make nonzero contributions.)

$$C_{r-1}^6\Big(\int_{3H-\sum_{j=1}^6 E_j-2E_r} E_r\Big)\Big(\int_{3H-\sum_{j=1}^6 E_j-2E_r}\frac{H}{3} - E_r\Big) = -2C_{r-1}^6.$$

(5) Here $r = 8$. Denote $A_1 := 4H - \sum_{j=1}^{8} E_j - E_1 - E_2 - E_8$ and $A_2 := 4H - \sum_{j=1}^{8} E_j - E_1 - E_2 - E_3$. The total nonzero contribution is given by

$$C_7^2\Big(\int_{A_1} E_8\Big)\Big(\int_{A_1} \frac{H}{3} - E_8\Big) + C_7^3\Big(\int_{A_2} E_8\Big)\Big(\int_{A_2} \frac{H}{3} - E_8\Big) = -\frac{49}{3}.$$

(6) Here $r = 8$. Denote $A_1 := 5H - \sum_{j=1}^{8} 2E_j + E_1 + E_8$ and $A_2 := 5H - \sum_{j=1}^{8} 2E_j + E_1 + E_2$. The total nonzero contribution is given by

$$C_7^1\Big(\int_{A_1} E_8\Big)\Big(\int_{A_1} \frac{H}{3} - E_8\Big) + C_7^2\Big(\int_{A_2} E_8\Big)\Big(\int_{A_2} \frac{H}{3} - E_8\Big) = -\frac{28}{3}.$$

(7) Here $r = 8$. There is only one nonzero contribution given by

$$\Big(\int_{6H - \sum_{j=1}^{8} 2E_j - E_r} E_r\Big)\Big(\int_{6H - \sum_{j=1}^{8} E_j - E_r} \frac{H}{3} - E_r\Big) = -3.$$

We should have also considered $A = c_1$ when $r = 8$, while it does not make contributions since $\int_{c_1} \frac{H}{3} - E_r = 0$.

Taking the summation of all the nonzero contributions deduces the required result. $\square$

Remark 4.8. *The number d_r is related with the quantum period $G_{X_r}(t)$ of X_r ($4 \le r \le 8$) computed in* [CCGK, section G] *in the way: $G_{X_r}(t) = e^{d_r t}(1 + O(t))$.*

Lemma 4.9. *For any $3 \le i \le r + 2$ and $3 \le j \le r + 3$, $m_{i,j} = 0$ if $i \ne j$.*

Proof. By 3.8, we have $m_{i,r+3} = \sum_A \langle c_1, [pt], E_{i-2}^\vee \rangle_A = 0$ as $c_1, [pt] \in \mathrm{Amb}_r$ and $E_{i-2}^\vee \in \mathrm{Prim}_r$. The vanishing $m_{i1} = m_{i2} = 0$ have been shown earlier.

Now we compute $m_{2+i,2+j} = \sum_{\int_A c_1 = 1} \big(\int_A E_i\big)\big(\int_A \frac{H}{3} - E_j\big)$ for $1 \le i, j \le r$ with $i \ne j$. It suffices to deal with the case when $(i, j) = (1, 2)$. We discuss all the cases as in the proof of Lemma 4.7. Clearly, there is no contribution in case (1) or when $A = c_1$ and $r = 8$. Denote by $C(A_1, A_2)$ the contributions to $m_{3,4}$ from classes of the form A_1 or A_2. We mean $C_a^b = 0$ if $b > a$. We have

Cases	A_1	A_2	$C(A_1, A_2)$	
(2)	$a_1 = a_2 = 1$	$a_1 = 1, a_2 = 0$	$(-\frac{2}{3}) + (r-2) \cdot \frac{1}{3}$	$r \ge 4$
(3)	$a_1 = a_2 = 1$	$a_1 = 1, a_2 = 0$	$C_{r-2}^3 \cdot (-\frac{1}{3}) + C_{r-2}^4 \cdot \frac{2}{3}$	$r \ge 5$
(4)	$a_1 = 1, a_2 = 2$	$a_2 = 0$	$C_{r-2}^5 \cdot (-1) + C_{r-1}^7 \cdot r$	$r \ge 7$
(5)	$a_1 = a_2 = 1$	$a_1 = a_2 = 2$	$C_6^3 \cdot (\frac{1}{3}) + C_6^1 \cdot \frac{-4}{3}$	$r = 8$
(5)	$a_1 = 1, a_2 = 2$	$a_1 = 2, a_2 = 1$	$C_6^2 \cdot 1 \cdot (\frac{-2}{3}) + C_6^2 \cdot 2 \cdot \frac{1}{3}$	$r = 8$
(6)	$a_1 = a_2 = 1$	$a_1 = a_2 = 2$	$\frac{2}{3} + C_6^4 \cdot \frac{-2}{3}$	$r = 8$
(6)	$a_1 = 1, a_2 = 2$	$a_1 = 2, a_2 = 1$	$C_6^5 \cdot (\frac{-1}{3}) + C_6^5 \cdot \frac{4}{3}$	$r = 8$
(7)	$6H - \sum_{k=1}^{8} 2E_k - E_2$		-2	$r = 8$

By direct calculation for each r, the summation of all $C(A_1, A_2)$ equals 0. $\square$

Lemma 4.10. *$m_{1,j+2} + d_r > 0$ and $(9 - r)m_{2,j+2} + d_r > 0$ for any $j \in \{1, \cdots, r\}$.*

Proof. For $A = 2H - E_j - E_{i_1} - E_{i_2} - E_{i_3}$ where $1 \le i_1 < i_2 < i_3 \le r$ are distinct with j, we have $\int_A c_1 = 2$, $\int_A E_j = 1$ and $\langle [pt] \rangle_A = 1$ (namely there exists a unique conic passing through 5 points in X_r). Moreover, we note $\int_{H-E_j} c_1 = 2$, $\int_{H-E_j} E_j = 1$ and

$\langle [pt]\rangle_{H-E_j} = 1$ (namely there exists a unique line passing through 2 points in X_r). Hence

$$m_{1,j+2} = \sum_{\int_A c_1 = 2} 2\int_A E_j \langle [pt]\rangle_A \geq 2\cdot 1\cdot 1 + C^3_{r-1}\cdot 2\cdot 1\cdot 1 = 2 + 2C^3_{r-1} > -d_r.$$

Write $A = a_0H - \sum_{i=1}^8 a_iE_i$. We may assume $j \neq 1$. By Lemmas 4.6, 4.7, we have

$$\begin{aligned}(9-r)m_{2,j+2} + h_r &= (9-r)\sum_{\int_A c_1=1}\Big(\int_A E_j\Big)\Big(\int_A \frac{H}{3}\Big) + \sum_{\int_A c_1=1}\Big(\int_A E_j\Big)\Big(\int_A \frac{H}{3} - E_j\Big)\\ &= -1 + (r-1)\Big(\int_{H-E_1-E_j} E_j\Big)\Big(\int_{H-E_1-E_j}\frac{(10-r)}{3}H - E_j\Big)\\ &\quad + \sum_{a_0>1;\int_A c_1=1}\Big(\int_A E_j\Big)\Big(\int_A \frac{(9-r)+1}{3}H - E_j\Big)\\ &= \frac{(7-r)(r-1)-3}{3} + \sum_{a_0>1;\int_A c_1=1}(a_j)\Big(\frac{10-r}{3}a_0 - a_j\Big)\end{aligned}$$

For $4 \leq r \leq 7$, both parts are positive. For $r = 8$, the second part is much larger than $10/3$. Therefore the above quantity is always positive. ☐

Corollary 4.11. *For any $j \in \{1, \cdots, r+3\}$, we have $\sum_{i=1}^{r+3} m_{ij} > 0$ and $m^{(2)}_{r+3,j} > 0$.*

Lemma 4.12. *For $1 \leq i \leq r+2$, we have $m^{(2)}_{ij} \geq 0$ for any j. Furthermore if $i \in \{1, 2\}$, then $m^{(2)}_{ij} > 0$ for any j.*

Proof. So far we have shown that M_r is of form (5). Due to Lemma 4.6, the statement obviously holds except for $m^{(2)}_{ij}$ with $i \in \{1, 2\}$ and $j \in \{3, \cdots, r+2\}$. Recall the basis $[\phi_1, \cdots, \phi_{r+3}] = [\mathbb{1}, c_1, E_1, \cdots, E_r, [pt]]$. By definition, the entry m_{ij} in the matrix M_r is the coefficient of ϕ_i in the product $c_1 \bullet \phi_j$, namely $m_{ij} = \sum_A \langle c_1, \phi^i, \phi_j\rangle_A$ where $[\phi^1, \cdots, \phi^{r+3}] = [[pt], \frac{H}{3}, \frac{H}{3} - E_1, \cdots, \frac{H}{3} - E_r, \mathbb{1}]$ are the dual basis of $\{\phi_i\}$. Therefore,

$$\begin{aligned}\sum_{k=1}^{r+3} m_{1k}m_{kj} &= \sum_{k=1}^{r+3}\sum_{A\in H_2(X_r,\mathbb{Z})}\langle c_1, \phi^1, \phi_k\rangle_A \sum_{A'\in H_2(X_r,\mathbb{Z})}\langle c_1, \phi^k, \phi_j\rangle_{A'}\\ &= \sum_{B\in H_2(X_r,\mathbb{Z})}\ \sum_{A+A'=B;A,A'\in H_2(X_r,\mathbb{Z})}\ \sum_{k=1}^{r+3}\langle c_1, \phi^1, \phi_k\rangle_A\langle c_1, \phi^k, \phi_j\rangle_{A'}\\ &= \sum_{B\in H_2(X_r,\mathbb{Z})}\ \sum_{A+A'=B;A,A'\in H_2(X_r,\mathbb{Z})}\ \sum_{k=1}^{r+3}\langle c_1, \phi^1, \phi^k\rangle_A\langle c_1, \phi_k, \phi_j\rangle_{A'}\\ &= \sum_{k=1}^{r+3}\sum_{A\in H_2(X_r,\mathbb{Z})}\langle c_1, \phi^1, \phi^k\rangle_A \sum_{A'\in H_2(X_r,\mathbb{Z})}\langle c_1, \phi_k, \phi_j\rangle_{A'}\\ &= \sum_{A\in H_2(X_r,\mathbb{Z})}\langle c_1, [pt], [pt]\rangle_A \sum_{A'\in H_2(X_r,\mathbb{Z})}\langle c_1, \mathbb{1}, E_{j-2}\rangle_{A'}\\ &\quad + \sum_{A\in H_2(X_r,\mathbb{Z})}\langle c_1, [pt], \frac{H}{3}\rangle_A \sum_{A'\in H_2(X_r,\mathbb{Z})}\langle c_1, c_1, E_{j-2}\rangle_{A'}\end{aligned}$$

$$+\sum_{k=3}^{r+2}\sum_{A\in H_2(X_r,\mathbb{Z})}\langle c_1,[pt],\frac{H}{3}-E_{k-2}\rangle_A\sum_{A'\in H_2(X_r,\mathbb{Z})}\langle c_1,E_{k-2},E_{j-2}\rangle_{A'}$$
$$+\sum_{A\in H_2(X_r,\mathbb{Z})}\langle c_1,[pt],\mathbb{1}\rangle_A\sum_{A'\in H_2(X_r,\mathbb{Z})}\langle c_1,[pt],E_{j-2}\rangle_{A'}.$$

The above expressions make sense, as there are only finitely many nonzero terms in each summation. The third equality holds by a change of bases. By Theorem 3.8, we have $\sum_A\langle c_1,[pt],\frac{H}{3}-E_{k-2}\rangle_A=0$ for any $3\leq k\leq r+2$ (as $\frac{H}{3}-E_{k-2}\in\mathrm{Prim}_r$ and $c_1,[pt]\in\mathrm{Amb}_r$). Thus the last equality produces a positive number, due to the enumerative meaning of 3-pointed genus zero Gromov-Witten invariants together with the non-negativity of the pairing $\int_A E_i$ between exceptional divisor classes and effective curve classes.

By WDVV equations, we have the following equality.

$$\begin{aligned}\sum_{k=1}^{r+3}m_{2k}m_{kj}&=\sum_{k=1}^{r+3}\sum_{A\in H_2(X_r,\mathbb{Z})}\langle c_1,\phi^2,\phi_k\rangle_A\sum_{A'\in H_2(X_r,\mathbb{Z})}\langle c_1,\phi^k,\phi_j\rangle_{A'}\\&=\sum_{k=1}^{r+3}\sum_{A\in H_2(X_r,\mathbb{Z})}\langle c_1,c_1,\phi_k\rangle_A\sum_{A'\in H_2(X_r,\mathbb{Z})}\langle \phi^2,\phi^k,\phi_j\rangle_{A'}\end{aligned}$$

By the same arguments as above, we conclude $\sum_{k=1}^{r+3}m_{2k}m_{kj}>0$ for $3\leq j\leq r+2$. □

Remark 4.13. *Since* $\sum_{k=1}^{r+3}m_{2k}m_{kj}=m_{2j}(m_{22}+d_r)+m_{21}m_{1j}+m_{2,r+3}$, *we can also conclude the second inequality by easily checking* $m_{22}+d_r\geq 0$ *for* $r>4$ *and simple calculation for* $r=4$.

Lemma 4.14. $m_{22}^{(2)}>d_r^2$.

Proof. We have $m_{22}^{(2)}>m_{12}+m_{22}^2$. For $4\leq r\leq 7$, we have $m_{12}=\sum_{\int_A=2}4\langle[pt]\rangle_A\geq C_r^1 4\langle[pt]\rangle_{H-E_1}+C_r^4 4\langle[pt]\rangle_{2H-E_1-E_2-E_3-E_4}=4r+\frac{1}{6}r(r-1)(r-2)(r-3)>d_r^2$. For $r=8$, we have $m_{22}=\sum_{\int_A c_1=1}\int_A\frac{H}{3}\geq\sum_{1\leq i_1<i_2<i_3\leq 8}\int_{4H-\sum_{j=1}^8 E_j-E_{i_1}-E_{i_2}-E_{i_3}}\frac{H}{3}=\frac{4}{3}C_8^3>60=|d_8|$. □

Proof of Proposition 4.1. The statement is a direct consequence of the combination of the lemmas and Corollary 4.11 in this subsection. □

5. Gamma conjecture I for del Pezzo surfaces

In this section, we prove Gamma conjecture I for del Pezzo surfaces, by using the mirror techniques proposed by Galkin and Iritani [GaIr] together with the study of Gamma conjecture I for certain weighted projective spaces.

5.1. **Toric del Pezzo surfaces.** Let X be an n-dimensional toric Fano manifold X. In the context of mirror symmetry, the Landau-Ginzburg potential f mirror to X is a Laurent polynomial [Gi1, Gi2] of the form

$$f:(\mathbb{C}^*)^n\to\mathbb{C};\ \mathbf{z}\mapsto f(\mathbf{z})=\mathbf{z}^{b_1}+\cdots+\mathbf{z}^{b_n},$$

where $b_1,\cdots,b_n\in\mathbb{Z}^n$ are primitive generators of the 1-dimensional cones of the fan of X. As one remarkable property, the small quantum cohomology $QH^*(X)$ is isomorphic to the Jacobian ring $\mathrm{Jac}(f)$ as algebras. The restriction $f|_{(\mathbb{R}_{>0})^n}$ is a real function on $(\mathbb{R}_{>0})^n$

that admits a global minimum at a unique point $\mathbf{z}_{con} \in (\mathbb{R}_{>0})^n$ [Ga1, GaIr]; such point $\mathbf{z}_{con} \in (\mathbb{R}_{>0})^n$ is called the *conifold point* of f.

Proposition 5.1. *([GaIr, Theorem 6.3]) Suppose that X is a toric Fano manifold satisfying the B-model analogue of Property $\mathcal{O}$, namely for $T_{con} := f(\mathbf{z}_{con})$,*

(1) *every critical value u of f satisfies $|u| \leq T_{con}$;*
(2) *$\mathbf{z}_{con}$ is the unique critical point of f contained in $f^{-1}(T_{con})$.*

Then X satisfies Gamma conjecture I.

We remark that the proof [GaIr] of the above proposition uses the integral representation of the central charge (see [Hoso] and references therein for the notion of central charge).

There have been lots of studies on mirror symmetry for toric del Pezzo surfaces, namely for X_r with $1 \leq r \leq 3$. The on-shelf Landau-Ginzburg potential f_r mirror to X_r can be read off for instance from [Jer, Example 2.3]. Precisely, we have

$$f_1 = z_1+z_2+\frac{1}{z_1}+\frac{1}{z_1z_2};\, f_2 = z_1+z_2+\frac{1}{z_1}+\frac{1}{z_1z_2}+\frac{1}{z_2};\, f_3 = z_1+z_2+\frac{1}{z_1}+\frac{1}{z_1z_2}+\frac{1}{z_2}+z_1z_2.$$

Therefore, we could have been done by easily verifying that these functions satisfy the hypotheses in Proposition 5.1 due to Galkin and Iritani. Nevertheless, we can also restrict to the study of quantum cohomology by using the following consequence.

Corollary 5.2. *Let X be an n-dimensional toric Fano manifold. If Spec$(\hat{c}_1) \cap \mathbb{R}_{>0} = \{\rho\}$ and the multiplicity of ρ is one, then Gamma conjecture I holds for X.*

Proof. It has been proved in [Au, Iri1] that the set Spec$(\hat{c}_1)$ of eigenvalues of $\hat{c}_1$ coincides with the set of critical values of the potential f mirror to X, and that the multiplicities also coincide. (This is also a general expectation in mirror symmetry for Fano manifolds.) Since $\rho \in$ Spec$(\hat{c}_1)$, the condition (1) in Proposition 5.1 holds. Since $f|_{\mathbb{R}^n_{>0}}$ is a real function with positive real values and f is holomorphic, any critical point $\mathbf{x} \in \mathbb{R}^n_{>0}$ of $f|_{\mathbb{R}^n_{>0}}$ is also a critical point of f via the natural inclusion $\mathbb{R}^n_{>0} \subset (\mathbb{C}^*)^n$. Consequently, $f(\mathbf{x}) \in$ Spec$(\hat{c}_1) \cap \mathbb{R}_{>0}$. It follows that $\rho = T_{con}$ and $f^{-1}(T_{con})$ is a single point set since the multiplicity of ρ is one. Hence, the statement follows by Proposition 5.1. □

By calculating the eigenvalues of the matrices $M_r(1 \leq r \leq 3)$ in section 2.2, we can see that the hypotheses in the above corollary hold for toric del Pezzo surfaces, where we include the known cases $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$. Hence, we have the following.

Proposition 5.3. *Toric del Pezzo surfaces satisfy Gamma conjecture I.*

5.2. **Non-toric cases.** Non-toric del Pezzo surfaces can be described as complete intersections in nice ambient spaces of Picard rank one. As proposed in [GaIr, Section 8], we shall use the quantum Lefschetz principle to prove Gamma conjecture I in these cases.

Let us start with the precise quantum Lefschetz principle in Proposition 5.4 as well as its proof following [GaIr, Theorem 8.3]. We give the details here since our assumption is slightly weaker.

Proposition 5.4. *Let X be a Fano manifold of index $r_X \geq 2$ and write $-K_X = r_X h$. Let $\iota : Y \hookrightarrow X$ be a Fano hypersurface in the linear system $|ah|$ with $0 < a < r$. Assume that*

$$\hat{\Gamma}_X \propto \lim_{t\to+\infty} t^{\frac{\dim X}{2}} e^{-C_X t} J_X(t)$$

for some constant C_X. Then

$$\hat{\Gamma}_Y \propto \lim_{t\to+\infty} t^{\frac{\dim Y}{2}} e^{-C_Y t} J_Y(t) \tag{6}$$

for some constant C_Y. In particular, if Y satisfies Property $\mathcal{O}$, then Y satisfies Gamma conjecture I.

Proof. Write $J_X(t) = e^{r_X h \log t} \sum_{n=0}^{\infty} J_{r_X n} t^{r_X n}$, with $J_{r_X n} \in H^\bullet(X)$.

From [Iri2, Proposition 2.4], $J_Y(t)$ takes values in $\iota^* H^\bullet(X)$. It follows from the quantum Lefschetz principle [Lee, CoGi] that

$$J_Y(t) = e^{(r_X - a)h \log t - C_0 t} \sum_{n=0}^{\infty} \frac{\Gamma(1+ah+an)}{\Gamma(1+ah)} (\iota^* J_{r_X n}) t^{(r_X - a)n}, \tag{7}$$

where C_0 is some constant determined by Y. Set

$$\tilde{J}(t) = t^{\frac{\dim X}{2}} e^{-C_X t} J_X(t), \quad T_0 = \max_{q \geq 0}\{-q + C_X q^{\frac{a}{r_X}}\}, \quad \text{and} \quad C_Y = T_0 - C_0.$$

We have

$$t^{\frac{\dim Y}{2}} e^{-C_Y t} J_Y(t) = \frac{\sqrt{t}}{\Gamma(1+ah)} \int_0^\infty q^{-\frac{a \dim X}{2 r_X}} e^{-(q - C_X q^{\frac{a}{r_X}} + T_0)t} \tilde{J}(t q^{\frac{a}{r_X}}) dq.$$

Using the stationary phase approximation, we conclude that RHS of (6) is proportional to $\frac{\iota^* \hat{\Gamma}_X}{\Gamma(1+ah)}$. Now the first required result follows from the equality $\hat{\Gamma}_Y = \frac{\iota^* \hat{\Gamma}_X}{\Gamma(1+ah)}$, as obtained from the adjunction formula.

If Y satisfies Property $\mathcal{O}$, then Y satisfies Gamma conjecture I by Proposition 2.3. □

We remark that the above proof is more like an outline, and refer to [GaIr, Theorem 8.3] for the details of estimations on the asymptotics skipped here. The assumption on X above is slightly weaker than the requirement of X satisfying Gamma conjecture I.

5.2.1. *Case $X_r (4 \leq r \leq 6)$.*

Proposition 5.5. *For each $4 \leq r \leq 6$, X_r satisfies Gamma conjecture I.*

Proof. For $r = 4$, we consider the embedding of X_4 as complete intersections in complex Grassmannian $Gr(2,5)$ [Cord]. More precisely,

$$X_4 = Gr(2,5) \cap H_1 \cap H_2 \cap H_3 \cap H_4,$$

where $Gr(2,5)$ is viewed as a subvariety in $\mathbb{P}^9$ via the Plücker embedding and H_i's are hyperplanes in $\mathbb{P}^9$ in general position, so that the above intersection makes sense. Denote

$$Y_k := Gr(2,5) \bigcap \bigcap_{i=1}^{k} H_i, \quad 0 \leq k \leq 4.$$

The Picard rank of $Y_0 = Gr(2,5)$ equals one, and Y_0 satisfies Gamma conjecture I by [GGI, Theorem 6.1.1]. Thus Y_0 satisfies the hypotheses on the ambient space in Proposition 5.4. By applying Proposition 5.4 and using induction on k, we conclude that for any $1 \leq k \leq 4$,

$$\hat{\Gamma}_{Y_k} \propto \lim_{t \to +\infty} t^{\frac{\dim Y_k}{2}} e^{-C_{Y_k} t} J_{Y_k}(t), \text{ for some constant } C_{Y_k}.$$

Since $Y_4 = X_4$ satisfies Property $\mathcal{O}$, it follows that X_4 satisfies Gamma conjecture I.

The arguments for cases $r = 5$ and $r = 6$ are similar. □

5.2.2. *Case* $X_r (7 \leq r \leq 8)$. The weighted projective space $\mathcal{X} := \mathbb{P}(1, w_1, \cdots, w_N)$ is the quotient stack $[(\mathbb{C}^{N+1} \setminus \{0\})/\mathbb{C}^*]$ with $\mathbb{C}^*$-action with weights $-1, -w_1, \cdots, -w_N$. The del Pezzo surfaces X_7, X_8 are smooth hypersurfaces in $\mathcal{X}$ respectively in the special cases $(w_1, w_2, w_3) = (1, 1, 2)$ and $(1, 2, 3)$. In order to show Gamma conjecture I for X_7, X_8, we will first study that for $\mathcal{X}$, and then apply the corresponding quantum Lefschetz principle. We refer our readers to [CCLT, Iri1] for basic materials of orbifold Gromov-Witten theory of weighted projective spaces.

To ease notations, in the rest of this section, we denote

$$r_{\mathcal{X}} := 1 + w_1 + \cdots + w_N, \quad c = r_{\mathcal{X}}(\prod_{i=1}^{N} w_i^{-w_i})^{\frac{1}{r_{\mathcal{X}}}} \quad \text{and} \quad h = c_1(\mathcal{O}(1)) \in H^\bullet(\mathcal{X}).$$

Notice that the first Chern class of $\mathcal{X}$ is given by $c_1(\mathcal{X}) = r_{\mathcal{X}} h$.

Following [CCLT, section 1], we let $F := \{\frac{k}{w_i} | 1 \leq i \leq N, 0 \leq k < w_i\}$. For $f \in F$, we let $\mathcal{X}_f$ be the locus of points of $\mathcal{X}$ with isotropic group containing $e^{2\pi\sqrt{-1}f}$. The Chen-Ruan orbifold cohomology group of $\mathcal{X}$ (denoted also by $H^\bullet_{\mathrm{CR}}(\mathcal{X})$) is given by

$$H^\bullet_{orb}(\mathcal{X}) := H^\bullet(I\mathcal{X}) = H^\bullet(\mathcal{X}) \oplus \bigoplus_{f \in F\setminus\{0\}} H^\bullet(\mathcal{X}_f), \text{ where } I\mathcal{X} := \bigsqcup_{f \in F} \mathcal{X}_f.$$

Here $I\mathcal{X}$ is called the inertia stack of $\mathcal{X}$, and we notice $\mathcal{X}_0 = \mathcal{X}$. The bullet '$\bullet$' means that we only consider classes with even topological degree.

The quantum connection ∇ on the trivial $H^\bullet_{orb}(\mathcal{X})$-bundle over $\mathbb{P}^1$ is given by

$$\nabla_{z\partial_z} = z\partial_z - \frac{1}{z}(c_1(\mathcal{X})\bullet) + \mu,$$

where $\bullet$ is the orbifold small quantum product of $\mathcal{X}$ with all Novikov variables setting to 1, and μ is the Hodge grading operator respecting the Chen-Ruan degree of classes in $H^\bullet_{orb}(\mathcal{X})$. The quantum connection is a meromorphic connection, which is logarithmic at $z = \infty$ and irregular at $z = 0$. Consider the holomorphic function

$$S : \mathbb{P}^1 \setminus \{0\} \quad \to \quad \mathrm{End}(H^\bullet_{orb}(\mathcal{X}))$$

defined by

$$(S(z)(\alpha), \beta)^{orb}_{\mathcal{X}} = (\alpha, \beta)^{orb}_{\mathcal{X}} + \sum_{m \geq 0} \frac{1}{z^{m+1}} \sum_d (-1)^{m+1} \langle \alpha\psi^m, \beta \rangle^{\mathcal{X}}_d.$$

Then from [Iri1, Section 2.3], the space of flat sections can be identified with the Chen-Ruan orbifold cohomology group $H^\bullet_{orb}(\mathcal{X})$ via the fundamental solution $S(z)z^{-\mu}z^{c_1(\mathcal{X})}$. In particular, we have the following isomorphism to the space of ∇-flat sections over $\mathbb{R}_{>0}$.

$$\begin{aligned} \Phi : H^\bullet_{orb}(\mathcal{X}) \quad &\xrightarrow{\cong} \quad \{s : \mathbb{R}_{>0} \to H^\bullet_{orb}(\mathcal{X}) : \nabla s = 0\}; \\ \alpha \quad &\mapsto \quad (2\pi)^{-\frac{N}{2}} S(z) z^{-\mu} z^{c_1(\mathcal{X})} \alpha. \end{aligned}$$

It follows from [CCLT, Theorem 1.1] that the characteristic polynomial of $(c_1(\mathcal{X})\bullet)$ is $\lambda^r - c^r$. By parallel discussions to that in [GGI, Section 3.2], we have

Proposition 5.6 (Analogue to Proposition 3.3.1 of [GGI])**.** *Let*

$\mathcal{A} := \{s : \mathbb{R}_{>0} \to H^\bullet_{orb}(\mathcal{X}) | \nabla s = 0, ||e^{\frac{c}{z}} s(z)|| = O(z^m)$ *as* $z \to +0$ *for some* $m \in \mathbb{Z}_{\geq 0}\}$,

$E_c :=$ *eigenspace of* $(c_1(\mathcal{X})\bullet)$ *in* $H^\bullet_{orb}(\mathcal{X})$ *with eigenvalue* c.

Then both $\mathcal{A}$ *and* E_c *are one-dimensional complex vector spaces, and they are isomorphic to each other via the map* $\mathcal{A} \to E_c$ *defined by* $s \mapsto \lim_{z \to +0} e^{\frac{c}{z}} s(z)$

Definition 5.7. *A* ***principal asymptotic class*** *of* $\mathcal{X}$ *is a nonzero class* $A_{\mathcal{X}} \in H^\bullet_{orb}(\mathcal{X})$ *such that* $\Phi(A_{\mathcal{X}}) \in \mathcal{A}$.

By Proposition 5.6, the principal asymptotic class of $\mathcal{X}$ is unique up to a nonzero scalar. Actually, it is given by the Gamma class of $\mathcal{X}$ as below. The Gamma class could be defined for an almost complex orbifold, and lives in the orbifold cohomology group [Iri1, (23)]. The Gamma class of $\mathcal{X}$ is of the form

$$\hat{\Gamma}_{\mathcal{X}} = \Gamma(1+h)\prod_{i=1}^{N}\Gamma(1+w_i h) + \text{ terms in } \bigoplus_{f\in F\setminus\{0\}} H^\bullet(\mathcal{X}_f).$$

The following can be viewed as one (equivalent) version of Gamma conjecture I for $\mathcal{X}$.

Proposition 5.8. $\hat{\Gamma}_{\mathcal{X}}$ *is a principal asymptotic class of* $\mathcal{X}$.

Proof. Consider the Laurant polynomial

$$f = x_1 + \cdots + x_N + \frac{1}{x_1^{w_1}\cdots x_N^{w_N}}, \quad (x_1, \cdots, x_N) \in (\mathbb{C}^*)^N.$$

The next mirror identity follows from the argument in [Iri1, Section 4.3.1]:

$$\left(\phi, S(z)z^{-\mu}z^{rh}\hat{\Gamma}_{\mathcal{X}}\right)^{orb}_{\mathcal{X}} = z^{-\frac{N}{2}}\int_{(\mathbb{R}_{>0})^N} e^{-\frac{f(x)}{z}}\varphi(x,z)\frac{dx_1\cdots dx_N}{x_1\cdots x_N}, \tag{8}$$

where $z > 0$, $\phi \in H^\bullet_{orb}(\mathcal{X})$, and $\varphi(x,z) \in \mathbb{C}[x_1^{\pm}, \cdots, x_N^{\pm}, z]$ is such that the class of the integrand on RHS corresponds to ϕ under the mirror isomorphism in [Iri1]. To study the oscillatory integral on RHS, we find all the critical points p_k of f by direct calculations:

$$p_k := \left(\frac{w_1\xi^k}{\sqrt[r_{\mathcal{X}}]{(\prod_{j=1}^{N} w_j^{w_j})}}, \cdots, \frac{w_N\xi^k}{\sqrt[r_{\mathcal{X}}]{(\prod_{j=1}^{N} w_j^{w_j})}}\right), \quad k = 0, 1, \cdots, r_{\mathcal{X}}-1,$$

where ξ is a primitive $r_{\mathcal{X}}$th root of unity. Moreover, we can check that $f|_{(\mathbb{R}_{>0})^N}$ admits a global minimum at the unique point p_0 with $c = f(p_0)$, satisfying the following properties:

(1) every critical value u of f satisfies $|u| \le c$;
(2) p_0 is the unique critical point of f contained in $f^{-1}(c)$.

Therefore by the stationary phase approximation, we have

$$||e^{\frac{c}{z}}S(z)z^{-\mu}z^{r_{\mathcal{X}}h}\hat{\Gamma}_{\mathcal{X}}|| = O(1), \quad z \to +0.$$

This implies that the flat section $S(z)z^{-\mu}z^{r_{\mathcal{X}}h}\hat{\Gamma}_{\mathcal{X}}$ is in $\mathcal{A}$. □

As for Fano manifolds, we can also interpret the principal asymptotic class of $\mathcal{X}$ in terms of Givental's J-function of $\mathcal{X}$ defined by $J_{\mathcal{X}}(t) = z^{\frac{N}{2}}\left(S(z)z^{-\mu}z^{c_1(\mathcal{X})}\right)^{-1}\mathbb{1}$ with $t = \frac{1}{z}$.

Theorem 5.9. $\mathcal{X}$ *satisfies Gamma conjecture I, namely* $\hat{\Gamma}_{\mathcal{X}} \propto \lim_{t\to+\infty} t^{\frac{N}{2}}e^{-ct}J_{\mathcal{X}}(t)$.

Proof. Denote by $\langle\cdot,\cdot\rangle$ the natural pairing between $H^\bullet_{orb}(\mathcal{X})^\vee$ and $H^\bullet_{orb}(\mathcal{X})$, where $H^\bullet_{orb}(\mathcal{X})^\vee$ is the dual space of $H^\bullet_{orb}(\mathcal{X})$. The dual connection $\nabla^\vee$ of ∇ is a meromorphic connection on the trivial $H^\bullet_{orb}(\mathcal{X})^\vee$-bundle over $\mathbb{P}^1$ given by

$$\nabla^\vee_{z\partial_z} = z\partial_z + \frac{1}{z}(c_1(\mathcal{X})\bullet)^\vee - \mu^\vee,$$

where $(c_1(\mathcal{X})\bullet)^\vee$ and $\mu^\vee$ are dual maps of $(c_1(\mathcal{X})\bullet)$ and μ respectively. Here $\nabla^\vee$ is dual to ∇ in the sense that

$$d\langle f(z), s(z)\rangle = \langle \nabla^\vee f(z), s(z)\rangle + \langle f(z), \nabla s(z)\rangle. \tag{9}$$

For the space of $\nabla^\vee$-flat sections over $\mathbb{R}_{>0}$, we have the following isomorphism:

$$\begin{array}{rcl} \Phi^\vee : H^\bullet_{orb}(\mathcal{X})^\vee & \xrightarrow{\cong} & \{f : \mathbb{R}_{>0} \to H^\bullet_{orb}(\mathcal{X})^\vee : \nabla^\vee f = 0\}; \\ \alpha & \mapsto & (2\pi)^{\frac{N}{2}}((S(z)z^{-\mu}z^{c_1(\mathcal{X})})^{-1})^\vee \alpha. \end{array}$$

By Proposition 5.6, we obtain a nonzero element ϕ_0 in E_c defined by

$$\phi_0 := \lim_{z\to+0} e^{\frac{c}{z}}\Phi(\hat{\Gamma}_{\mathcal{X}})(z) \in E_c.$$

Let $E_c^\vee$ be the eigenspace of $(c_1(\mathcal{X})\bullet)^\vee$ in $H^\bullet_{orb}(\mathcal{X})^\vee$ with eigenvalue c. Then $E_c^\vee$ can be viewed naturally as the dual space of E_c, since the pairing $\langle\cdot,\cdot\rangle|_{E_c^\vee\times E_c}$ is nondegenerate by direct verification. Let $\phi_0^\vee \in E_c^\vee$ be the dual of ϕ_0 in the sense that

$$\langle \phi_0^\vee, \phi_0\rangle = 1.$$

For any $\alpha \in H^\bullet_{orb}(\mathcal{X})^\vee$, we obtain the following formula (cf. [GGI, Proposition 3.6.2]):

$$\lim_{z\to+0} e^{-\frac{c}{z}}\Phi^\vee(\hat{\Gamma}_{\mathcal{X}})(z) = \langle \alpha, \hat{\Gamma}_{\mathcal{X}}\rangle\phi_0^\vee, \tag{10}$$

by using the parallel disccusions to that in [GGI, Section 3.5, Section 3.6]. Note that

$$t^{\frac{N}{2}}e^{-ct}J_{\mathcal{X}}(t) = (2\pi)^{-\frac{N}{2}}e^{-\frac{c}{z}}\Phi^{-1}(\mathbb{1})(z) \text{ with } t = \frac{1}{z}.$$

Let $C = \langle\phi_0^\vee, \mathbb{1}\rangle$. For any $\alpha \in H^\bullet_{orb}(\mathcal{X})^\vee$, we have

$$\begin{array}{rcl} \lim\limits_{t\to+\infty}\langle\alpha, t^{\frac{N}{2}}e^{-ct}J_{\mathcal{X}}(t)\rangle & = & \lim\limits_{z\to+0}\langle\alpha, (2\pi)^{-\frac{N}{2}}e^{-\frac{c}{z}}\Phi^{-1}(\mathbb{1})(z)\rangle \\ & = & \lim\limits_{z\to+0}\langle\Phi^\vee(\alpha)(z), (2\pi)^{-\frac{N}{2}}e^{-\frac{c}{z}}\mathbb{1}\rangle \\ & = & \lim\limits_{z\to+0}\langle e^{-\frac{c}{z}}\Phi^\vee(\alpha)(z), (2\pi)^{-\frac{N}{2}}\mathbb{1}\rangle \\ & = & \langle\alpha, C\cdot(2\pi)^{-\frac{N}{2}}\hat{\Gamma}_{\mathcal{X}}\rangle. \end{array}$$

Here the second (resp. third) equality follows from (9) (resp. (10)). This implies that

$$\lim_{t\to+\infty} t^{\frac{N}{2}}e^{-ct}J_{\mathcal{X}}(t) = C\cdot(2\pi)^{-\frac{N}{2}}\hat{\Gamma}_{\mathcal{X}}.$$

So we only need to show that $C \neq 0$.

Note that $H' := \text{Image}(c - (c_1(\mathcal{X})\bullet))$ is a complementary subspace of E_c in $H^\bullet_{orb}(\mathcal{X})$. Therefore there exists $\gamma \in H^\bullet_{orb}(\mathcal{X})$ such that

$$\mathbb{1} = C'\phi_0 + (c\gamma - c_1(\mathcal{X})\bullet\gamma) \text{ for some } C' \in \mathbb{C}.$$

Moreover,

$$\begin{array}{rcl} C & = & \langle\phi_0^\vee, \mathbb{1}\rangle \\ & = & C' + \langle\phi_0^\vee, c\gamma\rangle - \langle\phi_0^\vee, c_1(\mathcal{X})\bullet\gamma\rangle \\ & = & C' + \langle\phi_0^\vee, c\gamma\rangle - \langle(c_1(\mathcal{X})\bullet)^\vee\phi_0^\vee, \gamma\rangle \\ & = & C' + \langle\phi_0^\vee, c\gamma\rangle - \langle c\phi_0^\vee, \gamma\rangle \\ & = & C'. \end{array}$$

Therefore,

$$\phi_0 \quad = \quad C\phi_0\bullet\phi_0 + \phi_0\bullet(c\gamma - c_1(\mathcal{X})\bullet\gamma)$$

$$\begin{aligned} &= C\phi_0 \bullet \phi_0 + c\phi_0 \bullet \gamma - c_1(\mathcal{X}) \bullet \phi_0 \bullet \gamma \\ &= C\phi_0 \bullet \phi_0 + c\phi_0 \bullet \gamma - c\phi_0 \bullet \gamma \\ &= C\phi_0 \bullet \phi_0. \end{aligned}$$

Since $\phi_0 \neq 0$, it follows that that $C \neq 0$. This finishes the proof. □

Now we discuss hypersurfaces in $\mathcal{X}$ by the quantum Lefschetz principle for orbifolds.

Proposition 5.10. *Let Y be a smooth Fano variety given as a hypersurface in $\mathcal{X}$ defined by a section of $\mathcal{O}(d)$ for a positive integer d. Assume that $1 + w_1 + \cdots + w_N - d > 0$, and $w_i \mid d$ for $1 \leq i \leq N$. Then*

$$\hat{\Gamma}_Y \propto \lim_{t\to+\infty} t^{\frac{\dim Y}{2}} e^{-C_Y t} J_Y(t)$$

for some constant C_Y.

Proof. The argument is similar to that for [GaIr, Theorem 8.3] again. Write

$$J_{\mathcal{X}}(t) = e^{r_{\mathcal{X}} h \log t} \sum_{n=0}^{\infty} J_{r_{\mathcal{X}} n} t^{r_{\mathcal{X}} n}, \text{ with } J_{r_{\mathcal{X}} n} \in H^{\bullet}_{orb}(\mathcal{X}).$$

From [Iri2, Proposition 2.4], $J_Y(t)$ takes values in $\iota^* H^{\bullet}(\mathcal{X})$, where $\iota : Y \to \mathcal{X}$ is the natural inclusion. It follows from [CCLT, Corollary 1.9] and the discussion in the proof of [CCGK, Proposition D.9] that

$$J_Y(t) = e^{(r_{\mathcal{X}} - d)h \log t - C_0 t} \sum_{n=0}^{\infty} \frac{\Gamma(1 + dh + dn)}{\Gamma(1 + dh)} (\iota^* pr J_{r_{\mathcal{X}} n}) t^{(r_{\mathcal{X}} - d)n}, \tag{11}$$

where C_0 is some constant determined by Y, and $pr : H^{\bullet}_{orb}(\mathcal{X}) \to H^{\bullet}(\mathcal{X})$ is the natural projection. Set

$$\tilde{J}(t) = t^{\frac{N}{2}} e^{-ct} J_{\mathcal{X}}(t), \quad T_0 = \max_{q\geq 0}\{-q + cq^{\frac{d}{r_{\mathcal{X}}}}\} \quad \text{and} \quad C_Y = T_0 - C_0.$$

We have

$$t^{\frac{\dim Y}{2}} e^{-C_Y t} J_Y(t) = \frac{\sqrt{t}}{\Gamma(1 + dh)} \int_0^{\infty} q^{-\frac{dN}{2r_{\mathcal{X}}}} e^{-(q - cq^{\frac{d}{r_{\mathcal{X}}}} + T_0)t} \tilde{J}(tq^{\frac{d}{r_{\mathcal{X}}}}) dq.$$

Again using the stationary phase approximation as in the proof of [GaIr, Theorem 8.3], we conclude that RHS of (11) is proportional to $\frac{\iota^* pr \hat{\Gamma}_{\mathcal{X}}}{\Gamma(1+dh)}$. Now the required result follows from the next equality:

$$\hat{\Gamma}_Y = \frac{\iota^* pr \hat{\Gamma}_{\mathcal{X}}}{\Gamma(1 + dh)} = \frac{\Gamma(1 + h) \prod_{i=1}^{N} \Gamma(1 + w_i h)}{\Gamma(1 + dh)}.$$

which is obtained by the adjunction formula. □

Corollary 5.11. *Let Y be as in the assumptions of Proposition 5.10. If Y satisfies Property $\mathcal{O}$, then Y satisfies Gamma conjecture I.*

Proposition 5.12. *X_7 and X_8 satisfy Gamma conjecture I.*

Proof. The del Pezzo surface X_7 can be realized as a smooth hypersuface in $\mathbb{P}(1, 1, 1, 2)$ defined by a section of $\mathcal{O}(4)$. So Gamma conjecture I holds for X_7 by Theorem 5.9, Proposition 5.10 and Corollary 5.11.

The proof for X_8 is similar. □

In a summary, we achieve Theorem 1.2 by combining Propositions 5.3, 5.5 and 5.12.

6. Appendix

In this section, we provide two further applications of our generalized Perron-Frobenius theorem (Theorem 3.2) on Conjecture $\mathcal{O}$ for Fano manifolds of Fano index one in higher dimensions. We hope to investigate more systematical applications in the future study.

6.1. **Bott-Samelson variety.** Let Z be the Bott-Samelson resolution of the complete flag variety $F\ell_3 = \{V_1 \leqslant V_2 \leqslant \mathbb{C}^3 \mid \dim V_1 = 1, \dim V_2 = 2\}$. The Bott-Samelson variety Z is a 3-dimensional Fano manifold of Fano index one. With the notation in [Wit], it is an iterated projected line bundle and each $\mathbb{P}^1$-bundle π_i is equipped with a natural section:

$$Z = Z(\alpha_1, \alpha_2, \alpha_1) \xrightarrow{\pi_3} Z(\alpha_1, \alpha_2) \xrightarrow{\pi_2} Z(\alpha_1) \xrightarrow{\pi_1} \{pt\}.$$

Here $Z(\alpha_1, \alpha_2)$ is isomorphic to the del Pezzo surface X_1.

In [Wit], Withrow studied the quantum cohomology of Z, and calculated the matrix of $\hat{c}_1$ with respect to a natural basis $\{\sigma_{000}, \sigma_{100}, \sigma_{010}, \sigma_{001}, \sigma_{110}, \sigma_{101}, \sigma_{011}, \sigma_{111}\}$ of $H^*(Z)$, which is given by

$$M = \begin{bmatrix} 0 & 2 & 2 & -2 & 0 & 0 & 3 & 4 \\ 3 & -1 & 2 & 1 & 2 & 4 & 0 & 3 \\ 1 & 1 & -1 & -1 & 0 & 0 & 2 & 0 \\ 2 & 0 & 1 & 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 4 & 0 & -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 3 & 2 & 0 \end{bmatrix}.$$

As a consequence, Withrow verified Conjecture $\mathcal{O}$ for Z by checking the approximated numerical eigenvalues. This can be alternatively proved by applying Theorem 3.2 directly. Indeed, M satisfies the hypothesis (1) of Theorem 3.2. By direct calculations (for instance by Mathematica), we find that M^{10} is a positive matrix given by

$$\begin{bmatrix} 10864936 & 15316300 & 22405398 & 1606005 & 14828972 & 24157043 & 20494668 & 26635352 \\ 14727669 & 16777595 & 24284091 & 1827215 & 19331760 & 31483681 & 25762659 & 30245233 \\ 3922308 & 6982588 & 9298432 & 106510 & 5900032 & 8917360 & 8362961 & 10769511 \\ 7878209 & 10242471 & 14164779 & 622465 & 10771768 & 17027288 & 14589084 & 17304004 \\ 7527551 & 7594020 & 12170058 & 1581596 & 9343508 & 16091860 & 12382348 & 15245229 \\ 2714920 & 5850921 & 8257471 & 289763 & 4174204 & 6522612 & 6227600 & 9215640 \\ 8333564 & 9513188 & 14858821 & 1653409 & 10622807 & 18050731 & 14298764 & 18177956 \\ 6255520 & 9096760 & 12114088 & 231311 & 8913729 & 13662541 & 12176736 & 14525144 \end{bmatrix}.$$

Then we are done by noting that Z is of Fano index one.

6.2. **Blow-up of $\mathbb{P}^4$ at one point.** Let X be the blow-up of $\mathbb{P}^4$ at one point P. Let $H \in H^2(X)$ be the pullback of the hyperplane class in $\mathbb{P}^4$, and E be the exceptional divisor of the blow-up. Then we have $c_1(X) = 5H - 3E$, which implies that the Fano index of X is 1. Denote $E(i) := (-1)^{i-1}E^i \in H^{2i}(X)$. For $1 \leq i \leq 3$, the Poincaré dual of $E(i)$ can be represented by a codimension-$(i-1)$ linear subspace in $E \cong \mathbb{P}^3$, and $H^4 = E(4)$ is the Poincaré dual of the point class. One basis of $H^*(X)$ is given by

$$\mathfrak{B} = \{H^i\}_{0 \le i \le 4} \cup \{E(i)\}_{1 \le i \le 3},$$

whose dual basis in $H^*(X)$ with respect to the Poincaré pairing is given by

$$(H^i)^\vee := H^{4-i}, \quad 0 \le i \le 4, \text{ and } E(i)^\vee := -E(4-i), \quad 1 \le i \le 3.$$

Let $e \in H_2(X, \mathbb{Z})$ be the class of a line in the exceptional divisor, and let $l \in H_2(X, \mathbb{Z})$ be the class of a line in $X \setminus E = \mathbb{P}^4 \setminus \{P\}$. Then the semigroup of effective curve classes in X is freely generated by e and $f := l - e$ over $\mathbb{Z}_{\ge 0}$. We have the following simple observations.

Lemma 6.1. *If $A \in H_2(X, \mathbb{Z})$ is an effective curve class supporting a non-zero two-point invariant, then A must belong to one of the following three cases:*

$$\textit{(i) } A = af \textit{ with } a \in \mathbb{Z}_{>0}; \quad \textit{(ii) } A = be \textit{ with } b \in \mathbb{Z}_{>0}; \quad \textit{(iii) } A = f + e = l.$$

Proof. For $A = af + be$ supporting a non-zero two-point invariant, we have either $a > 0$ or $b > 0$. Moreover, by Degree Axiom, we have $3+2a+3b \le 8$, implying $2a+3b \le 2+3$. Thus if $a, b \ne 0$, then we have $a = b = 1$. □

By the above lemma, we can calculate all primary two-point invariants of X by case-by-case analysis. We list the results in the following, leaving proofs to interested readers.

Lemma 6.2.

$$\begin{aligned} \langle H^i, H^j \rangle_{af} &= \delta_{i+j,5}\delta_{a,1}, \quad 0 \le i, j \le 4, \\ \langle H^i, E(j) \rangle_{af} &= \delta_{i+j,5}\delta_{a,1}, \quad 0 \le i \le 4, 1 \le j \le 3, \\ \langle E(i), E(j) \rangle_{af} &= \delta_{i+j,5}\delta_{a,1}, \quad 1 \le i, j \le 3. \end{aligned}$$

Lemma 6.3.

$$\begin{aligned} \langle H^i, H^j \rangle_{be} &= 0, \quad 0 \le i, j \le 4, \\ \langle H^i, E(j) \rangle_{be} &= 0, \quad 0 \le i \le 4, 1 \le j \le 3, \\ \langle E(i), E(j) \rangle_{be} &= \delta_{i,3}\delta_{j,3}\delta_{b,1}, \quad 1 \le i, j \le 3. \end{aligned}$$

Lemma 6.4.

$$\begin{aligned} \langle H^i, H^j \rangle_{l} &= \delta_{i,4}\delta_{j,4}, \quad 0 \le i, j \le 4, \\ \langle H^i, E(j) \rangle_{l} &= 0, \quad 0 \le i \le 4, 1 \le j \le 3, \\ \langle E(i), E(j) \rangle_{l} &= 0, \quad 1 \le i, j \le 3. \end{aligned}$$

Proposition 6.5. *The blow-up of $\mathbb{P}^4$ at one point satisfies Conjecture $\mathcal{O}$.*

Proof. From the above two-point invariants of X, we can write down the matrix M of $(c_1(X)\bullet)$ with respect to the basis $\mathfrak{B}' = \{\mathbb{1}, H-E(1), \cdots, H^3-E(3), E(1), \cdots, E(4)\}$:

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 2 & 0 & 0 & 5 \\ 5 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 3 & 0 & 0 & -3 & 0 \\ 0 & 2 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 3 & 0 \end{bmatrix}.$$

Clearly, M satisfies the hypothesis (1) of Theorem 3.2. By direct calculations (for instance by Mathematica), we find that M^{14} is a positive matrix given by

$$\begin{bmatrix} 11776531513 & 5909045424 & 2887777125 & 10817096903 & 440618668 & 5384452386 & 5355341065 & 1154333460 \\ 1154333460 & 11776531513 & 5909045424 & 2887777125 & 10817096903 & 440618668 & 5384452386 & 16172437968 \\ 16172437968 & 1154333460 & 11776531513 & 5909045424 & 2887777125 & 10817096903 & 440618668 & 8272229511 \\ 8272229511 & 16172437968 & 1154333460 & 11776531513 & 5909045424 & 2887777125 & 10817096903 & 6349664092 \\ 440618668 & 5384452386 & 5355341065 & 713714792 & 6392079127 & 553704359 & 2174062333 & 10817096903 \\ 10817096903 & 440618668 & 5384452386 & 5355341065 & 713714792 & 6392079127 & 553704359 & 2887777125 \\ 2887777125 & 10817096903 & 440618668 & 5384452386 & 5355341065 & 713714792 & 6392079127 & 5909045424 \\ 5909045424 & 2887777125 & 10817096903 & 440618668 & 5384452386 & 5355341065 & 713714792 & 11776531513 \end{bmatrix}.$$

By Theorem 3.2, the spectral radius ρ of $(c_1(X)\bullet)$ is a simple eigenvalue of $(c_1(X)\bullet)$. By Proposition 3.5, ρ is the unique eigenvalue of $(c_1(X)\bullet)$ whose modulus is equal to ρ. This verifies Conjecture $\mathcal{O}$ for X since the Fano index of X is 1. $\square$

Remark 6.6. *It is wellknown that X can be realized as the projectivization of $\mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}$, and the chosen basis $\mathfrak{B}'$ in the above proof is natural from this point of view. For example, $H - E(1)$ is the pullback of the hyperplane class in $\mathbb{P}^3$, and $E(1)$ is the relative hyperplane class.*

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