ON THE QUANTUM COHOMOLOGY OF BLOW-UPS OF FOUR-DIMENSIONAL QUADRICS

JIANXUN HU, HUA-ZHONG KE, CHANGZHENG LI, AND LEI SONG

ABSTRACT. We propose a conjecture relevant to Galkin's lower bound conjecture, and verify it for the blow-ups of a four-dimensional quadric at a point or along a projective plane. We also show that Conjecture $\mathcal{O}$ holds in these two cases.

1. INTRODUCTION

The (small) quantum cohomology $QH^*(X) = (H^*(X)\otimes_{\mathbb{Q}}\mathbb{Q}[\mathbf{q}], \star)$ of a Fano manifold X is a deformation of the classical cohomology ring $H^*(X) = H^*(X, \mathbb{Q})$, by incorporating three-pointed genus zero Gromov-Witten invariants. The quantum multiplication by the first Chern class of X induces a linear operator $\hat{c}_1$ on the finite-dimensional vector space $QH^\bullet(X) := QH^{\rm ev}(X)|_{\mathbf{q}=1}$. Attention has been drawn to the spectral properties of $\hat{c}_1$ in the past five years, mainly due to the following two conjectures. One is the remarkable Conjecture $\mathcal{O}$ proposed by Galkin, Golyshev and Iritani [GGI], which concerns with eigenvalues of $\hat{c}_1$ of modulus equal to the spectral radius $\rho(\hat{c}_1)$ and underlies their Gamma conjecture I. The other one is an interesting conjecture proposed by Galkin [Ga1]:

Galkin's lower bound conjecture . $\rho(\hat{c}_1) \geq \dim X + 1$, *with equality if and only if X is isomorphic a projective space $\mathbb{P}^n$.*

Here and throughout this paper, we always consider the complex dimension. Conjecture $\mathcal{O}$ has been verified in various cases (see [HKLY] and references therein). Galkin's lower bound conjecture was verified for few cases, including complex Grassmannians [ESSSW], Lagrangian and orthogonal Grassmannians [ChHa], and Fano complete intersections in projective spaces [Ke].

Due to the lack of functorial properties of the quantum cohomology, individual studies of relevant information on $QH^*(X)$ have to be taken in general. Nevertheless, when $\pi : X \to Y$ is a blow-up of Fano manifolds, the changing behavior of Gromov-Witten invariants under blowing up may be analyzed via the absolute-relative correspondence (see e.g. [MaPa, HLR]). In some circumstances, the Gromov-Witten invariants for X can be read off from that for Y by explicit blow-up formulae [Gath, Hu1, Hu2, Lai]. Here we propose the following conjecture.

Conjecture 1.1. *Let Y be a Fano manifold, and Z be an irreducible smooth Fano subvariety of Y with* codim$Z \geq 2$*. If the blow-up $X = Bl_Z Y$ of Y along Z is Fano, then $\rho(\hat{c}_1(X)) > \rho(\hat{c}_1(Y))$.*

We start with codim$Z \geq 2$, since the blow-up of Y along a smooth irreducible divisor is isomorphic to Y itself. We even wish to remove the assumption of the smoothness of Z.

Key words and phrases. Quantum cohomology. Blow-up. Quadric hypersurface.
2010 Mathematics Subject Classification. Primary 14N35. Secondary 14J45, 14J33.

When $Y = \mathbb{P}^n$, the Fano submanifold Z can be taken as $\mathbb{P}^r$ for any $0 \leq r \leq n-2$. The blow-up $Bl_{\mathbb{P}^r}\mathbb{P}^n$ is a toric Fano manifold, and the above conjecture is equivalent to Galkin's lower bound conjecture in this case. By using mirror symmetry [GaIr] and analysing the mirror superpotential, Conjecture $\mathcal{O}$ was verified for $Bl_{\mathbb{P}^r}\mathbb{P}^n$ in [Yang]. Numerical computations of the critical values of the same superpotential, which coincide with the eigenvalues of $\hat{c}_1$, provide a class of evidences for the above conjecture. When Y is a del Pezzo surface given by the blow-up of $\mathbb{P}^2$ at r points in general position with $0 \leq r \leq 7$, we can consider a point Z in Y. The blow-up $Bl_Z Y$ is again a del Pezzo surface, and Conjecture 1.1 can be directly verified by numerically solving the characteristic polynomial in [BaMa]. This provides another main class of evidences, which support us to formulate Conjecture 1.1. Our conjecture gives a refined lower bound $\rho(\hat{c}_1(Y))$ of $\rho(\hat{c}_1(Bl_Z Y))$ for blow-ups $Bl_Z Y$ than Galkin's lower bound $\dim Bl_Z Y + 1 = \dim Y + 1$.

Let Q^4 denote a four-dimensional smooth quadric in $\mathbb{P}^5$. To further support our conjecture, we obtain the following main result of the present paper.

Theorem 1.2. *Conjecture $\mathcal{O}$, Conjecture 1.1 and Galkin's lower bound conjecture all hold for the blow-ups $Bl_{\mathbb{P}^0}Q^4$ and $Bl_{\mathbb{P}^2}Q^4$.*

The quadric Q^4 can be interpreted as the complex Grassmannian $Gr(2,4)$ via the Plücker embedding. The blow-ups in the above theorem, together with the blow-ups $Bl_{\mathbb{P}^r}\mathbb{P}^n$, can be uniformly treated as Fano manifolds that arise from the blow-up of a complex Grassmannian $Gr(k,n)$ along a complex sub-Grassmannian. We will study the general case systemically elsewhere.

For $r \in \{0,2\}$, that Galkin's lower bound conjecture holds for $X_r = Bl_{\mathbb{P}^r}Q^4$ is an immediate consequence of Conjecture 1.1 (see Corollary 2.2). We will prove the other two conjectures in Theorem 3.13 and Theorem 4.6 respectively. The proofs rely on explicit computations of the quantum multiplication by the first Chern class $c_1(X_r)$. Thanks to the divisor axiom in Gromov-Witten theory, this reduces to the study of genus zero, two-point Gromov-Witten invariants with insertions in $H^*(X_r)$. To simplify the calculations, we explore the geometric structures of the Fano manifolds by noting that X_0 can also be realized the blow-up of $\mathbb{P}^4$ along a two-dimensional quadric in a 3-plane of $\mathbb{P}^4$, and that X_2 is a $\mathbb{P}^2$-bundle over $\mathbb{P}^2$. Part of the Gromov-Witten invariants are therefore easily obtained from the enumerative information of Q^4 by making use of the blow-up formula in [Hu1]. The proof of Conjecture $\mathcal{O}$ also relies on a good choice of bases of $H^*(X_r)$, so that Perron's theorem [Perr] on positive matrices becomes applicable. We remark that all the three conjectures also hold for $Bl_{\mathbb{P}^1}(Q^4)$, which is a natural case to study. While the geometric structure of $Bl_{\mathbb{P}^1}(Q^4)$ is more complicated than that of $Bl_{\mathbb{P}^0}(Q^4)$ and $Bl_{\mathbb{P}^2}(Q^4)$, determination of two-point Gromov-Witten invariants of $Bl_{\mathbb{P}^1}(Q^4)$ requires ad hoc arguments with WDVV equations involved. We therefore exclude it in this paper.

The present paper is organized as follows. In section 2, we review basic facts of quantum cohomology and introduce Conjecture $\mathcal{O}$. In section 3, we study the quantum cohomology of the blow-up of Q^4 at a point, and prove both Conjecture 1.1 and Conjecture $\mathcal{O}$ in this case. In section 4, we study the quantum cohomology of the blow-up of Q^4 along a plane $\mathbb{P}^2$, and prove both conjectures as well.

2. Preliminaries

In this section, we review some basic facts, and refer to [CoKa] and [GGI] for more details.

2.1. **Conjecture $\mathcal{O}$.** Let X be a Fano manifold, namely a compact complex manifold X with positive first Chern class $c_1(X)$. Let $\overline{\mathcal{M}}_{0,k}(X,\mathbf{d})$ denote the moduli stack of k-pointed genus 0 stable maps $(f : C \to X; p_1, p_2, \cdots, p_k)$ of class $\mathbf{d} \in H_2(X,\mathbb{Z})$, which has a coarse moduli space $\overline{M}_{0,k}(X,\mathbf{d})$. Its virtual fundamental class $[\overline{M}_{0,k}(X,\mathbf{d})]^{\mathrm{virt}}$ is of complex degree $\dim_{\mathbb{C}} X + \int_{\mathbf{d}} c_1(X) + k - 3$ in the Chow group $A_*(\overline{M}_{0,k}(X,\mathbf{d}))$. The k-pointed, genus 0 Gromov-Witten invariants of degree $\mathbf{d}$ for $\gamma_1, \gamma_2, \cdots, \gamma_k \in H^*(X) = H^*(X,\mathbb{Q})$ is defined by

$$\langle \gamma_1, \gamma_2, \cdots, \gamma_k \rangle_{\mathbf{d}}^X := \int_{[\overline{M}_{0,k}(X,\mathbf{d})]^{\mathrm{virt}}} ev_1^*(\gamma_i) \cup ev_2^*(\gamma_2) \cup \cdots \cup ev_k^*(\gamma_k).$$

Here ev_i denotes the i-th evaluation map. Set $m := \mathrm{rank} H_2(X,\mathbb{Z})$, take any homogeneous basis $\{\phi_i\}_{i=1}^N$ of $H^*(X)$, and let $\{\phi^i\}$ denote the dual basis of $H^*(X)$ that satisfy $(\phi_i, \phi^j)_X = \int_X \phi_i \cup \phi^j = \delta_{i,j}$ with respect to the Poincaré pairing. The (small) quantum cohomogy ring $QH^*(X) = (H^*(X) \otimes_{\mathbb{Q}} \mathbb{Q}[q_1, \cdots, q_m], \star)$ is a deformation of the classical cohomology $H^*(X)$. The quantum product of $\alpha, \beta \in H^*(X)$ is given by

$$\alpha \star \beta := \sum_{\mathbf{d} \in H_2(X,\mathbb{Z})} \sum_{i=1}^N \langle \alpha, \beta, \phi_i \rangle_{\mathbf{d}}^X \phi^i q^{\mathbf{d}}.$$

Here $q^{\mathbf{d}} = \prod_{j=1}^m q_j^{d_j}$ for $\mathbf{d} = (d_1, \cdots, d_m)$ with a basis of effective curve classes of $H_2(X,\mathbb{Z})$ being fixed a prior. The quantum product is a polynomial in $\mathbf{q}$, and is independent of choices of the basis $\{\phi_i\}_i$.

Consider the even part of the cohomology $H^\bullet(X) := H^{\mathrm{even}}(X)$ and let $QH^\bullet(X) := H^\bullet(X) \otimes \mathbb{Q}[\mathbf{q}]$. The first Chern class $c_1(X)$ induces a linear operator $\hat{c}_1 = \hat{c}_1(X)$ by the evaluation of the quantum product at $\mathbf{1} := (1, \cdots, 1)$, namely defined by

$$\hat{c}_1 : QH^\bullet(X) \longrightarrow QH^\bullet(X);\ \beta \mapsto c_1(X) \star \beta|_{\mathbf{q}=\mathbf{1}}.$$

Denote by $\rho = \rho(\hat{c}_1(X))$ the spectral radius of $\hat{c}_1$, namely

$$\rho := \max\{|\lambda| \,:\, \lambda \in \mathrm{Spec}(\hat{c}_1)\} \quad \text{where} \quad \mathrm{Spec}(\hat{c}_1) := \{\lambda \,:\, \lambda \in \mathbb{C} \text{ is an eigenvalue of } \hat{c}_1\}.$$

Conjecture $\mathcal{O}$ (Galkin-Golyshev-Iritani)**.** *Every Fano manifold X satisfies the following.*

1. $\rho \in$ *Spec*$(\hat{c}_1)$ *and it is of multiplicity one.*
2. *For any $\lambda \in$ Spec$(\hat{c}_1)$ with $|\lambda| = \rho$, we have $\lambda^s = \rho^s$, where s is the Fano index of X, namely $s = \max\{k \in \mathbb{Z} \,:\, \frac{c_1(x)}{k} \in H^2(X,\mathbb{Z})\}$.*

2.2. **Quantum cohomology of Q^4.** Let Q^4 be a four-dimensional smooth quadric in $\mathbb{P}^5$. The quadric Q^4 can be interpreted as the image of the complex Grassmannian $Gr(2,4) = \{V \leqslant \mathbb{C}^4 \mid \dim V = 2\}$ via the Plücker embedding $Pl : Gr(2,4) \hookrightarrow \mathbb{P}^5$. Therefore the integral cohomology $H^*(Q^4,\mathbb{Z}) \cong H^*(Gr(2,4),\mathbb{Z}) = \bigoplus_{2 \geq a \geq b \geq 0} \mathbb{Z}\sigma^{(a,b)}$ has a $\mathbb{Z}$-basis of Schubert classes $\sigma^{(a,b)}$.

Under the above identification, the hyperplane class $H := \sigma^{(1,0)} \in H^2(Q^4,\mathbb{Z})$ is the pullback of the hyperplane class in $H^2(\mathbb{P}^5,\mathbb{Z})$. Moreover, $\wp := \sigma^{(2,0)} - \sigma^{(1,1)} \in H^4(Q^4,\mathbb{Z})$ is a primitive class. Then the basis $\mathcal{B}$ of $H^*(Q^4,\mathbb{Z})$ of Schubert classes can be written as

$$\mathcal{B} = \{\mathbf{1}, H, \wp_+, \wp_-, \frac{1}{2}H^3, \frac{1}{2}H^4\},$$

in which $\wp_+ := \frac{1}{2}(H^2 + \wp) = \sigma^{(2,0)}$, $\wp_- := \frac{1}{2}(H^2 - \wp) = \sigma^{(1,1)}$, $\frac{1}{2}H^3 = \sigma^{(2,1)}$ and $\frac{1}{2}H^4 = \sigma^{(2,2)}$. Geometrically, Q^4 has two rulings by planes. Then $\wp_+$ and $\wp_-$ are respectively given by their cohomology classes, namely the Poincaré dual of their homology classes. The cohomology class of a line is $\frac{1}{2}H^3$, and the cohomology class of a point is $\frac{1}{2}H^4$. The dual basis $\mathcal{B}^\vee$ of $\mathcal{B}$ with respect to the Poincaré pairing is given by

$$\mathcal{B}^\vee = \{\frac{1}{2}H^4, \frac{1}{2}H^3, \wp_+, \wp_-, H, \mathbf{1}\}.$$

Let $\ell \in H_2(Q^4, \mathbb{Z})$ be the homology class of a line. The non-zero three-point Gromov-Witten invariants with insertions in $\mathcal{B}$ can be directly read off from a table of quantum product of Schubert classes (see e.g. [Miha, §8.2]), and are given as follows where $\theta \in \{\wp_+, \wp_-\}$.

$$\begin{aligned} \langle H^0, H^0, \frac{1}{2}H^4\rangle_0^{Q^4} &= 1, & \langle H, \frac{1}{2}H^3, \frac{1}{2}H^4\rangle_\ell^{Q^4} &= 1, \\ \langle H^0, H, \frac{1}{2}H^3\rangle_0^{Q^4} &= 1, & \langle \wp_+, \wp_-, \frac{1}{2}H^4\rangle_\ell^{Q^4} &= 1, \\ \langle H^0, \theta, \theta\rangle_0^{Q^4} &= 1, & \langle \theta, \frac{1}{2}H^3, \frac{1}{2}H^3\rangle_\ell^{Q^4} &= 1, \\ \langle H, H, \theta, \rangle_0^{Q^4} &= 1, & \langle \frac{1}{2}H^4, \frac{1}{2}H^4, \frac{1}{2}H^4\rangle_{2\ell}^{Q^4} &= 1. \end{aligned}$$

Closed formula for the spectral radius of the linear operators induced by quantum multiplication by Schubert classes on $QH^\bullet(Gr(k,n))$ has been given by Rietsch [Rie] (see also [ESSSW, LSYZ] for the formula). In particular for $Q^4 = Gr(2,4)$, we have $c_1(Q^4) = 4H = 4\sigma^{(1,0)}$ and the following property.

Proposition 2.1. *The spectral radius of $\hat{c}_1(Q^4)$ is equal to $4\frac{\sin\frac{2\pi}{4}}{\sin\frac{\pi}{4}} = 4\sqrt{2}$.*

Corollary 2.2. *If Conjecture 1.1 holds for a Fano blow-up $Bl_Z(Q^4)$ of Q^4 along a subvariety $Z \subset Q^4$, so does Galkin's lower bound conjecture.*

Proof. $\rho(\hat{c}_1(Bl_Z Q^4)) > \rho(\hat{c}_1(Q^4) = 4\sqrt{2} > 4+1 = \dim(Bl_Z Q^4) + 1.$ □

3. Quantum cohomology of $Bl_{\mathbb{P}^0}Q^4$

In this section, we will study the quantum cohomology of the blow-up $X_0 = Bl_{\mathbb{P}^0}Q^4$ of Q^4 at one point, and verify Conjecture $\mathcal{O}$ and Conjecture 1.1 for X_0.

3.1. **Geometric construction.** We will review a geometric construction of X_0 (see e.g. [Harr, §22]), which can be interpreted as both the blow-up of Q^4 at a point and the blow-up of $\mathbb{P}^4$ along a two-dimensional quadric Q_0^2 in a 3-plane of $\mathbb{P}^4$. Both interpretations will be used to compute the relevant 2-pointed genus zero Gromov-Witten invariants of X_0.

Fix $x \in Q^4$ and a 4-plane $\mathbb{P}_0^4$ in $\mathbb{P}^5$, such that $x \notin \mathbb{P}_0^4$. The rational map $f : \mathbb{P}^5 \dashrightarrow \mathbb{P}_0^4$, given by the projection from x to $\mathbb{P}_0^4$, defines a graph $\Gamma_f \subset \mathbb{P}^5 \times \mathbb{P}_0^4$ by the Zariski closure of $\{(y, f(y)) \mid y \in \mathbb{P}^5 \setminus \{x\}\}$. There are two natural projections $\Gamma_f \xrightarrow{p_1} \mathbb{P}^5$ and $\Gamma_f \xrightarrow{p_2} \mathbb{P}_0^4$. The morphism p_1 is the blow-up of $\mathbb{P}^5$ at x; the morphism p_2 endows the graph with the $\mathbb{P}^1$-bundle structure $\Gamma_f = \mathbb{P}_{\mathbb{P}_0^4}(\mathcal{O}_{\mathbb{P}_0^4}(1) \oplus \mathcal{O}_{\mathbb{P}_0^4})$, which can be viewed as the projective compactification of $\mathbb{P}^5 \setminus \{x\} = N_{\mathbb{P}_0^4|\mathbb{P}^5} \cong \mathcal{O}_{\mathbb{P}_0^4}(1)$.

Lemma 3.1. *For any smooth subvariety S in $\mathbb{P}_0^4$, we have*

$$p_2^{-1}(S) \cong \mathbb{P}_S(\mathcal{O}_S(1) \oplus \mathcal{O}_S),$$

where $\mathcal{O}_S(1)$ is the pullback of the hyperplane line bundle $\mathcal{O}_{\mathbb{P}_0^4}(1)$.

Proof. $p_1(p_2^{-1}(S))$ is the cone $\overline{xS}$ in $\mathbb{P}^5$ over S with vertex x. The statement follows by noting that the subvariety $p_2^{-1}(S)$ of Γ_f is the projective compactification of

$$\overline{xS}\setminus\{x\}=N_{\mathbb{P}^4_0|\mathbb{P}^5}|_S\cong\mathcal{O}_{\mathbb{P}^4_0}(1)|_S=\mathcal{O}_S(1).$$

□

Let $X_0\subset\mathbb{P}^5\times\mathbb{P}^1_0$ be the strict transform of Q^4, that is, X_0 is the Zariski closure of $p_1^{-1}(Q^4\setminus\{x\})$. Consider the restrictions

$$\pi_1=p_1|_{X_0}:X_0\to Q^4,\quad \pi_2=p_2|_{X_0}:X_0\to\mathbb{P}^4.$$

The morphism π_1 is the blow-up of Q^4 at x, and its exceptional divisor is given by

$$E_0:=X_0\cap D_\infty,$$

where $D_\infty:=\mathbb{P}_{\mathbb{P}^4_0}(\mathcal{O}_{\mathbb{P}^4_0}(1)\oplus\{0\})$ is the exceptional divisor of the blow-up p_1.

The tangent hyperplane H_x of Q^4 at x, consisting of the tangent lines in $T_x\mathbb{P}^5|_{Q^4}$, is naturally viewed as a 4-plane in $\mathbb{P}^5$. Noting that E_0 is a 3-plane in $D_\infty\cong\mathbb{P}^4$, we have

Lemma 3.2. *$\pi_2(E_0)$ is a 3-plane in $\mathbb{P}^4_0$. Moreover, $H_x\cap\mathbb{P}^4_0=\pi_2(E_0)$.*

Proof. The first statement follows directly from the construction of X_0. Indeed, for $w\in E_0$, let l_w be the tangent line of Q^4 at x with tangent direction given by w. Note that l_w is a line in $\mathbb{P}^5$ not contained in $\mathbb{P}^4_0$, and $\pi_2(w)$ is the unique intersection point of l_w and $\mathbb{P}^4_0$.

On one hand, for $y\in H_x\cap\mathbb{P}^4_0$, the line $\overline{xy}$ is a tangent line of Q^4 at x, and $\overline{xy}\cap\mathbb{P}^4_0=\{y\}$, which implies that $y\in\pi_2(E_0)$. On the other hand, for $w\in E_0$, we have $l_w\subset H_x$ and hence $\pi_2(w)\in H_x\cap\mathbb{P}^4_0$. Therefore the second statement follows. □

The quadric 3-fold $H_x\cap Q^4$ in H_x is a cone with vertex x over

$$Q^2_0:=H_x\cap Q^4\cap\mathbb{P}^4_0=Q^4\cap\pi_2(E_0).$$

The surface Q^2_0 is a smooth quadric in the 3-plane $\pi_2(E_0)$. By the definition of Q^2_0, for any $y\in Q^2_0$, the line $\overline{xy}$ is contained in Q^4. This implies $\pi_2^{-1}(Q^2_0)=p_2^{-1}(Q^2_0)$. Thus by Lemma 3.1, we have

$$E':=\pi_2^{-1}(Q^2_0)\cong\mathbb{P}_{Q^2_0}(\mathcal{O}_{Q^2_0}(1)\oplus\mathcal{O}_{Q^2_0}).$$

The morphism $\pi_2:X_0\to\mathbb{P}^4_0$ is actually the blow-up of $\mathbb{P}^4_0$ along Q^2_0 [Harr], with the exceptional divisor E'. Under the above identification, the intersection $E'\cap E_0$ is given by

$$Q^2_\infty:=\mathbb{P}_{Q^2_0}(\mathcal{O}_{Q^2_0}(1)\oplus\{0\}),$$

which is a smooth quadric in $E_0\cong\mathbb{P}^3$.

3.2. **Classical cohomology.** Arising from the blow-up $\pi_1:X_0\to Q^4$, the integral cohomology $H^*(X_0,\mathbb{Z})$ has a basis $\mathcal{B}_0$ together with its dual basis $\mathcal{B}_0^\vee$ with respect to the Poincaré pairing, coming from $\pi_1^*(H^*(Q^4,\mathbb{Z}))$ and the exceptional classes. Let P_0 be a plane in the exceptional divisor $E_0\cong\mathbb{P}^3$, and L_0 be a line in E_0. Precisely, we have

$$\mathcal{B}_0:=\{\mathbf{1},H,\wp_+,\wp_-,\frac{1}{2}H^3,\frac{1}{2}H^4,E_0,P_0,L_0\},$$
$$\mathcal{B}_0^\vee:=\{\frac{1}{2}H^4,\frac{1}{2}H^3,\wp_+,\wp_-,H,\mathbf{1},-L_0,-P_0,-E_0.\}.$$

Here by abuse of notation, we have simply denoted the pullback of a class $\alpha\in H^*(Q^4,\mathbb{Z})$ by α, and simply denote the cohomology class of a subvariety S of X_0 by S as well.

By direct calculations, the non-trivial products on $\mathcal{B}_0$ are given as follows, where $\theta \in \{\wp_+, \wp_-\}$.

$$\begin{aligned} H \cup H &= \wp_+ + \wp_-, & E_0 \cup E_0 &= -P_0, \\ H \cup \theta &= \frac{1}{2}H^3, & E_0 \cup P_0 &= -L_0, \\ H \cup \frac{1}{2}H^3 &= \frac{1}{2}H^4, & E_0 \cup L_0 &= -\frac{1}{2}H^4, \\ \theta \cup \theta &= \frac{1}{2}H^4, & P_0 \cup P_0 &= -\frac{1}{2}H^4. \end{aligned}$$

Note that $Q_0^2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ has two rulings by lines. We fix two lines $l_+, l_- \subset Q_0^2$ in different rulings, and denote by $S'_\pm$ the class of $\pi_2^{-1}(l_\pm)$. Let F' be the class of a fiber of $E' \cong \mathbb{P}_{Q_0^2}(\mathcal{O}_{Q_0^2}(1) \oplus \mathcal{O}_{Q_0^2})$. Let H' be the pullback of the hyperplane class in $\mathbb{P}_0^4$ of π_2^*. Arising from the blow-up $\pi_2 : X_0 \to \mathbb{P}_0^4$, $H^*(X_0, \mathbb{Z})$ has a basis $\mathcal{B}'_0$ together with its dual basis $\mathcal{B}_0'^\vee$, given by

$$\begin{aligned} \mathcal{B}'_0 &= \{\mathbf{1}, H', H'^2, H'^3, H'^4, E', S'_+, S'_-, F'\}, \\ \mathcal{B}_0'^\vee &= \{H'^4, H'^3, H'^2, H', \mathbf{1}, -F', -S'_-, -S'_+, -E'\}. \end{aligned}$$

3.3. **Comparison of two bases of $H^*(X_0)$.** In this subsection, we compare the two bases $\mathcal{B}_0$ and $\mathcal{B}'_0$ of $H^*(X_0, \mathbb{Z})$. We observe that $\frac{1}{2}H^4 = H'^4$.

3.3.1. *Comparison in $H^2(X_0)$ and $H^6(X_0)$.* Let $e \in H_2(X_0, \mathbb{Z})$ be the homology class of a line in $E_0 \cong \mathbb{P}^3$. Denote also by ℓ the Poincaré dual of $\frac{1}{2}H^3$, whose image via π_{1*} is the class of a line in Q^4. Then e and ℓ form a basis of $H_2(X_0, \mathbb{Z})$, and we have

$$\begin{cases} H.\ell &= 1, \\ H.e &= 0, \end{cases} \quad \text{and} \quad \begin{cases} E_0.\ell &= 0, \\ E_0.e &= -1. \end{cases}$$

Moreover, the classes e and $\ell - e$ generate the Mori cone of X_0, and $X_0 \xrightarrow{\pi_1} Q^4$ is the contraction associated to the extremal ray $\mathbb{R}_{\geq 0}e$.

Let $e' \in H_2(X_0, \mathbb{Z})$ be the homology class of a fiber of E', and $\ell' \in H_2(X_0, \mathbb{Z})$ be the Poincaré dual of H'^3. Then e' and ℓ' form another basis of $H_2(X_0, \mathbb{Z})$, and we have

$$\begin{cases} H'.\ell' &= 1, \\ H'.e' &= 0, \end{cases} \quad \text{and} \quad \begin{cases} E'.\ell &= 0, \\ E'.e &= -1. \end{cases}$$

Lemma 3.3. *$e' = \ell - e$.*

Proof. Note that the Mori cone of X_0 has only two extremal rays $\mathbb{R}_{\geq 0}e$ and $\mathbb{R}_{\geq 0}(\ell - e)$. The contraction $X_0 \xrightarrow{\pi_1} Q^4$ corresponds to the extremal ray $\mathbb{R}_{\geq 0}e$, and the contraction $X_0 \xrightarrow{\pi_2} \mathbb{P}_0^4$ corresponds to the extremal ray $\mathbb{R}_{\geq 0}e'$. So $e \neq e'$, and hence $e' = \ell - e$. □

Lemma 3.4. *$e = \ell' - 2e'$.*

Proof. Write $e = a\ell' + be'$ for some $a, b \in \mathbb{C}$. Since a line in $E_0 \cong \mathbb{P}^3$ is mapped via π_2 to a line in the 3-plane $\pi_2(E_0) \subset \mathbb{P}_0^4$, it follows that $a = 1$. Moreover, since a line in E_0 in general position meets $Q_\infty^2 = E' \cap E_0$ at two points, it follows that $E'.e = -2$, which implies that $b = -2$. □

So we see that

$$\begin{cases} \ell' &= 2\ell - e, \\ e' &= \ell - e, \end{cases} \quad \text{and} \quad \begin{cases} \ell &= \ell' - e', \\ e &= \ell' - 2e'. \end{cases}$$

By considering Poincaré duals, we obtain

$$\left\{\begin{array}{rcl} H'^3 & = & H^3 - L_0, \\ F' & = & \frac{1}{2}H^3 - L_0, \end{array}\right. \text{ and } \left\{\begin{array}{rcl} \frac{1}{2}H^3 & = & H'^3 - F', \\ L_0 & = & H'^3 - 2F'. \end{array}\right.$$

By considering Poincaré pairings, we obtain

$$\left\{\begin{array}{rcl} H' & = & H - E_0, \\ E' & = & H - 2E_0, \end{array}\right. \text{ and } \left\{\begin{array}{rcl} H & = & 2H' - E', \\ E_0 & = & H' - E'. \end{array}\right.$$

3.3.2. *Comparison in $H^4(X_0)$.* Note $\pi_2^{-1}(Q_0^2) = p_2^{-1}(Q_0^2)$. Since $l_\pm \subset Q_0^2$, it follows that

$$\pi_1(\pi_2^{-1}(l_\pm)) = \pi_1(p_2^{-1}(l_\pm)) = p_1(p_2^{-1}(l_\pm)).$$

So $\pi_1(\pi_2^{-1}(l_\pm))$ is the cone over the line $l_\pm$ with vertex x, and hence it is a 2-plane. Moreover, for any $y \in l_\pm$, the line $\overline{xy}$ is contained in Q^4. So the 2-plane $\pi_1(\pi_2^{-1}(l_\pm))$ is contained in Q^4, and the intersection

$$\pi_1(\pi_2^{-1}(l_+)) \cap \pi_1(\pi_2^{-1}(l_-))$$

is the line passing through x and $l_+ \cap l_-$. So we see that $\pi_1(\pi_2^{-1}(l_+))$ and $\pi_1(\pi_2^{-1}(l_-))$ belong to different rulings by 2-planes of Q^4. Rename l_+ and l_- if necessary so that $[\pi_1(\pi_2^{-1}(l_\pm))] = \wp_\pm$,

Lemma 3.5. $S'_\pm = \wp_\pm - P_0$.

Proof. We have

$$S'_\pm = \wp_\pm + a_\pm P_0, \text{ for some } a_\pm \in \mathbb{Q}.$$

Since $\pi_2^{-1}(l_\pm) \cap E_0$ is a line in $E_0 \cong \mathbb{P}^3$, and P_0 is the class of a 2-plane in E_0, it follows that

$$S'_\pm . P_0 = 1 \tag{1}$$

Since $P_0.P_0 = -1$, it follows that $a_\pm = -1$. □

Lemma 3.6. $P_0 = H'^2 - S'_+ - S'_-$.

Proof. Since the image under π_2 of a 2-plane in $E_0 \cong \mathbb{P}^3$ is a 2-plane in $\pi_2(E_0) \subset \mathbb{P}_0^4$, it follows that

$$P_0 = H'^2 + b_+ S'_+ + b_- S'_-, \text{ for some } b_\pm \in \mathbb{Q}.$$

Since $S'_\pm . S'_\pm = 0$ and $S'_\pm . S'_\mp = -1$, it follows from (1) that $b_\pm = -1$. □

So we have

$$\left\{\begin{array}{rcl} S'_\pm & = & \wp_\pm - P_0, \\ H'^2 & = & \wp_+ + \wp_- - P_0, \end{array}\right. \text{ and } \left\{\begin{array}{rcl} \wp_\pm & = & H'^2 - S'_\mp, \\ P_0 & = & H'^2 - S'_+ - S'_-. \end{array}\right.$$

3.4. **Two-point invariants.**

Lemma 3.7. *The non-zero, degree-ke' for $k \geq 1$, two-point invariants with insertions in $\mathcal{B}'_0$ are*

$$\langle S'_+, S'_- \rangle_{e'}^{X_0} = \langle E', F' \rangle_{e'}^{X_0} = 1.$$

Proof. Let $\iota : E' \hookrightarrow X_0$ be the natural embedding, and we also denote by ι the induced embedding of moduli spaces of stable maps:

$$\iota : \overline{M}_{0,2}(E', ke') \hookrightarrow \overline{M}_{0,2}(X_0, ke').$$

Note that the blow-down morphism $X_0 \xrightarrow{\pi_2} \mathbb{P}^4_0$ is the contraction corresponding to the extremal ray $\mathbb{R}_{\geq 0}e'$. So we have

$$\iota\Big(\overline{M}_{0,2}(E', ke')\Big) = \overline{M}_{0,2}(X_0, ke').$$

Consider the universal diagram

$$\begin{array}{ccc} \overline{M}_{0,3}(E', ke') & \xrightarrow{ev_3} & E' \\ \pi\big\downarrow & & \\ \overline{M}_{0,2}(E', ke') & & \end{array}$$

and let $R := R^1\pi_* ev_3^* N_{E'|X_0}$, where π denotes the natural morphism by forgetting the third marking point. From the construction of virtual fundamental classes, we have

$$\iota_*\big(\mathbf{e}(R) \cap [\overline{M}_{0,2}(E', ke')]^{\mathrm{virt}}\big) = [\overline{M}_{0,2}(X_0, ke')]^{\mathrm{virt}},$$

where $\mathbf{e}(R)$ is the Euler class of R. So for $B_1, B_2 \in \mathcal{B}'_0$, we see that

$$\langle B_1, B_2\rangle^{X_0}_{ke'} = \int_{[\overline{M}_{0,2}(E', ke')]^{\mathrm{virt}}} ev_1^*\iota^*B_1 \cup ev_2^*\iota^*B_2 \cup \mathbf{e}(R).$$

For $i = 1, 2$, consider the morphism $f_i : \overline{M}_{0,2}(E', ke') \to Q_0^2$ defined by the composition

$$f_i := \pi_2 \circ ev_i.$$

We see that $f_1 = f_2$, and we let $f := f_1 = f_2$. Then we have

$$\langle B_1, B_2\rangle^{X_0}_{ke'} = \int_{Q_0^2} PD\Big(f_*\big(ev_1^*\iota^*B_1 \cup ev_2^*\iota^*B_2 \cup \mathbf{e}(R) \cap [\overline{M}_{0,2}(E', ke')]^{\mathrm{virt}}\big)\Big).$$

Let $L_\pm \in H^2(Q_0^2)$ be the class of $l_\pm$, and $P \in H^4(Q_0^2)$ be the class of a point. Then $\mathbf{1}, L_+, L_-, P$ form a basis $\mathcal{B}''$ of $H^*(Q_0^2)$. Set $\xi := c_1(N_{E'|X_0}) \in H^2(E')$. We have

$$\begin{aligned} &\iota^*\mathbf{1} = \mathbf{1}, \quad \iota^*H' = \pi_2^*(L_+ + L_-), \quad \iota^*H'^2 = \pi_2^*(2P), \quad \iota^*H'^3 = 0, \\ &\iota^*H'^4 = 0, \quad \iota^*E' = \xi, \quad \iota^*S'_\pm = \pi_2^*L_\pm \cup \xi, \quad \iota^*F' = \pi_2^*P \cup \xi. \end{aligned}$$

We see that for $i = 1, 2$, ι^*B_i has the form

$$\iota^*B_i = \pi_2^*\underline{B_i} \cup \xi^{\alpha(i)}, \quad \underline{B_i} \in \mathcal{B}'', \alpha(i) \in \{0, 1\},$$

which implies that

$$ev_i^*\iota^*B_i = f^*\underline{B_i} \cup ev_i^*\xi^{\alpha(i)}.$$

So we use the projection formula to get

$$\langle B_1, B_2\rangle^{X_0}_{ke'} = \int_{Q_0^2} \underline{B_1} \cup \underline{B_2} \cup PD\Big(f_*\big(ev_1^*\xi^{\alpha(1)} \cup ev_2^*\xi^{\alpha(2)} \cup \mathbf{e}(R) \cap [\overline{M}_{0,2}(E', ke')]^{\mathrm{virt}}\big)\Big).$$

Assume that $\langle B_1, B_2\rangle^{X_0}_{ke'} \neq 0$. Then the above formula gives

$$\deg \underline{B_1} + \deg \underline{B_2} \leq 4 \text{ and } \underline{B_1} \cup \underline{B_2} \neq 0.$$

We use the dimension constraint to get

$$(4-3) + 3k + 2 = \Big(\deg \underline{B_1} + \deg \underline{B_2}\Big) + \Big(\alpha(1) + \alpha(2)\Big) \leq 4 + 2 = 6,$$

which implies that $k = 1$. We use the dimension constraint again to get

$$\deg \underline{B_1} + \deg \underline{B_2} = 4 \text{ and } \alpha(1) = \alpha(2) = 1.$$

From the condition $\underline{B_1} \cup \underline{B_2} \neq 0$, we only need to consider the cases

$$(B_1, B_2) = (E', F') \text{ and } (S'_+, S'_-).$$

Since $k = 1$, it follows from $H^1(\mathbb{P}^1, \mathcal{O}(-1)) = 0$ that $\mathbf{e}(R) = 0$. So we obtain

$$\langle E', F' \rangle^{X_0}_{e'} = \langle \xi, \pi_2^* P \cup \xi \rangle^{E'}_{e'} \text{ and } \langle S'_+, S'_- \rangle^{X_0}_{e'} = \langle \pi_2^* L_+ \cup \xi, \pi_2^* L_- \cup \xi \rangle^{E'}_{e'}.$$

Both of the Gromov-Witten invariants of E' are equal to one, since there is a unique line contained in a fiber of E'. □

Lemma 3.8. *A curve class in $H_2(X_0, \mathbb{Z})$ admits non-zero two-point invariants only if it belongs to $\{\ell - e, e, \ell, 2\ell - e\}$.*

Proof. An effective curve class has the form

$$a(\ell - e) + be, \quad a, b \in \mathbb{Z}_{\geq 0}.$$

Notice $c_1(X_0) = 4H - 3E_0$ (see e.g. [GrHa, Chapter 4, Section 6])). Using the dimension constraint, we see that $a(\ell - e) + be$ admits non-zero two-point invariants only if

$$a + 3b \leq 5.$$

Note that $(a, b) \neq (0, 0)$ since the space $\overline{M}_{0,2}(X_0, 0)$ is empty. Note that $\ell - e = e'$. So from Lemma 3.7, we can exclude the cases $k(\ell - e)$ for $2 \leq k \leq 5$. □

Lemma 3.9. *The non-zero, degree-$(\ell - e)$, two-point invariants with insertions in $\mathcal{B}_0$ are*

$$\langle P_0, P_0 \rangle^{X_0}_{\ell - e} = \langle H, L_0 \rangle^{X_0}_{\ell - e} = \langle E_0, L_0 \rangle^{X_0}_{\ell - e} = 2$$

and

$$\langle P_0, \wp_\pm \rangle^{X}_{\ell - e} = \langle \wp_+, \wp_- \rangle^{X}_{\ell - e} = \langle H, \frac{1}{2} H^3 \rangle^{X}_{\ell - e} = \langle E_0, \frac{1}{2} H^3 \rangle^{X}_{\ell - e} = 1.$$

Proof. From Lemma 3.7, the only non-zero, degree-e', two-point invariants with insertions in $\mathcal{B}'_0$ are

$$\langle S'_+, S'_- \rangle^{X_0}_{e'} = \langle E', F' \rangle^{X_0}_{e'} = 1.$$

Note that $e' = \ell - e$, and we can use this to determine invariants with insertions in $\mathcal{B}_0$. For example,

$$\langle \wp_+, \wp_- \rangle^{X_0}_{\ell - e} = \langle H'^2 - S'_-, H'^2 - S'_+ \rangle^{X_0}_{e'} = 1.$$

We leave the rest cases to interested readers. □

Lemma 3.10. *The only non-zero, degree-e, two-point invariant with insertions in $\mathcal{B}_0$ is*

$$\langle L_0, L_0 \rangle^{X_0}_{e} = 1.$$

Proof. The proof is similar to that of Lemma 3.7, and we leave it to interested readers. □

Lemma 3.11. *The only non-zero, degree-ℓ, two-point invariants with insertions in $\mathcal{B}_0$ is*

$$\langle \frac{1}{2} H^3, \frac{1}{2} H^4 \rangle^{X_0}_{\ell} = 1.$$

Proof. By the dimension constraint, we only need to consider the cases

$$\langle \frac{1}{2}H^3, \frac{1}{2}H^4\rangle_\ell^{X_0} \text{ and } \langle L_0, \frac{1}{2}H^4\rangle_\ell^{X_0}.$$

For the former case, we use the blow-up formula [Hu1, Theorem 1.2]. For the latter case, observe that for $y \in Q^4$, there is a line in Q^4 passing through x and y iff $y \in H_x \cap Q^4$. As a consequence, for $y \in Q^4$ in general position, there is no line in Q^4 passing through x and y. So we have $\langle L_0, \frac{1}{2}H^4\rangle_\ell^{X_0} = 0$. □

Lemma 3.12. *The only non-zero, degree-$(2\ell - e)$, two-point invariants with insertions in $\mathcal{B}_0$ is*

$$\langle \frac{1}{2}H^4, \frac{1}{2}H^4\rangle_{2\ell-e}^{X_0} = 1.$$

Proof. Use the dimension constraint and blow-up formula [Hu1, Theorem 1.4]. □

In general, there are very few tools to study the eigenvalues of linear operators on a vector space. One remarkable tool is the Frobenius-Perron theory on irreducible nonnegative matrices [Perr, Frob] (see e.g. [Minc]), provided that a *good* basis of the vector space could be found. Now we consider the operator $\hat{c}_1(X_0)$, where we recall $c_1(X_0) = 4H - 3E_0$.

Theorem 3.13. *Both Conjecture 1.1 and Conjecture $\mathcal{O}$ hold for the blow-up $X_0 = Bl_{\mathbb{P}^0}Q^4$.*

Proof. Instead of the bases $\mathcal{B}_0$ and $\mathcal{B}_0'$, we consider another basis of $H^*(X_0, \mathbb{Z})$ given by

$$\hat{\mathcal{B}}_0 := \{\mathbf{1}, H, H - E_0, \wp_+, \wp_-, \wp_+ + \wp_- - P_0, \frac{1}{2}H^3, \frac{1}{2}H^3 - L_0, \frac{1}{2}H^4\}.$$

By Lemmas 3.8, 3.9, 3.10, 3.11 and 3.12, we have

$$\begin{aligned} c_1(X_0) \star H|_{\mathbf{q}=\mathbf{1}} &= c_1(X_0) \cup H + \langle c_1(X_0), H, -L_0\rangle_{\ell-e}^{X_0} E_0 + \langle c_1(X_0), H, \frac{1}{2}H^3\rangle_{\ell-e}^{X_0} H \\ &= 4\wp_+ + 4\wp_- + (c_1(X_0).(\ell - e))(\langle H, -L_0\rangle_{\ell-e}^{X_0} E_0 + \langle H, \frac{1}{2}H^3\rangle_{\ell-e}^{X_0} H) \\ &= 4\wp_+ + 4\wp_- - H + 2(H - E). \end{aligned}$$

Here the second equality follows from the divisor axiom for Gromov-Witten invariants (see e.g. [CoKa]) and the computation $c_1(X_0).(\ell - e) = (4H - 3E_0).(\ell - e) = 1$. Similarly, we can calculate the product $c_1(X_0) \star \alpha|_{\mathbf{q}=\mathbf{1}}$ for all $\alpha \in \hat{\mathcal{B}}_0$. As a result, we obtain the associated matrix M in $\hat{c}_1(X_0)\hat{\mathcal{B}}_0 = \hat{\mathcal{B}}_0 M$ with

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 5 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 3 & 4 \\ 3 & 2 & 0 & 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 4 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 4 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 \end{pmatrix}.$$

By direct calculations, the power M^{17} is a positive matrix. Thus by Perron's theorem on positive matrices [Perr], the spectral radius $\rho(M^{17})$ is a simple eigenvalue of M^{17}, and the modulus of any other eigenvalue of M^{17} is strictly less than $\rho(M^{17})$. Since M is a real matrix, if λ is an eigenvalue of M, so is the conjugate $\bar{\lambda}$. It follows that $\rho(M) = (\rho(M^{17}))^{\frac{1}{17}}$ is a simple eigenvalue of M, and the modulus of any other eigenvalue of M

is strictly less than $(\rho(M^{17}))^{\frac{1}{17}}$. Hence, Conjecture $\mathcal{O}$ holds for X_0. (Here we notice the Fano index of X_0 is equal to 1, since $c_1(X_0).(\ell - e) = 1$.)

Notice that $\det(\lambda I_9 - M)|_{\lambda=4\sqrt{2}} < 0$. Hence, there exists a real eigenvalue λ_0 in the interval $(4\sqrt{2}, +\infty)$. Hence, by Proposition 2.1, we have

$$\rho(\hat{c}_1(X_0)) = \rho(M) \geq \lambda_0 > 4\sqrt{2} = \rho(\hat{c}_1(Q^4)).$$

That is, Conjecture 1.1 holds for $Y = Q^4$ and $Z = \mathbb{P}^0$. □

Remark 3.14. *To prove Conjecture $\mathcal{O}$, one can also apply the generalized Frobenius-Perron theorem* [HKLY, Theorem 3.2] *to M^{16}, which is the first power of M that becomes a positive matrix.*

4. Quantum cohomology of $Bl_{\mathbb{P}^2}Q^4$

In this section, we will study the quantum cohomology of the blow-up $X_2 = Bl_{\mathbb{P}^2}Q^4$ of Q^4 along a projective plane $\mathbb{P}^2$ inside it, and verify Conjecture $\mathcal{O}$ and Conjecture 1.1 for X_2.

4.1. **Classical cohomology.** A geometric construction of X_2 is similar to that of X_0. Precisely, we let $\mathbb{P}^2_+$ and $\mathbb{P}^2_-$ be two disjoint 2-planes in Q^4 such that $P.D.[\mathbb{P}^2_\pm] = \wp_\pm \in H^4(Q^4)$. Then they represent different rulings by 2-planes of Q^4, and

$$N_{\mathbb{P}^2_\pm|Q^4} \cong \Omega_{\mathbb{P}^2_\pm}(2),$$

where Ω denotes the cotangent sheaf.

Consider the rational map $f' : \mathbb{P}^5 \dashrightarrow \mathbb{P}^2_-$, given by the projection from $\mathbb{P}^2_+$, which defines a graph $\Gamma_{f'} \subset \mathbb{P}^5 \times \mathbb{P}^2_-$ by the Zariski closure of $\{(y, f'(y)) \mid y \in \mathbb{P}^5 \setminus \mathbb{P}^2_+\}$. By abuse of notation, we consider the natural projections $\Gamma_{f'} \xrightarrow{p_1} \mathbb{P}^5$ and $\Gamma_{f'} \xrightarrow{p_2} \mathbb{P}^2_-$ as well as their restrictions $\pi_i := p_i|_{X_2}$ with X_2 being the strict transform of Q^4 (i.e. the Zariski closure of $p_1^{-1}(Q^4 \setminus \mathbb{P}^2_+)$). Then $\pi_1 : X_2 \to Q^4$ is the blow-up of Q^4 along $\mathbb{P}^2_+$, and $\pi_2 : X_2 \to \mathbb{P}^2_-$ endows X_2 with the projective bundle structure $\mathbb{P}_{\mathbb{P}^2_-}(\Omega_{\mathbb{P}^2_-}(2) \oplus \mathcal{O}_{\mathbb{P}^2_-})$ [SzWi], which can be viewed as the projective compactification of $Q^4 \setminus \mathbb{P}^2_+ = N_{\mathbb{P}^2_-|Q^4} \cong \Omega_{\mathbb{P}^2_-}(2)$. The divisor $E_2 := \mathbb{P}_{\mathbb{P}^2_-}(\Omega_{\mathbb{P}^2_-}(2) \oplus \{0\}) \subset \mathbb{P}_{\mathbb{P}^2_-}(\Omega_{\mathbb{P}^2_-}(2) \oplus \mathcal{O}_{\mathbb{P}^2_-})$ is the exceptional divisor of the blow-up $X_2 \xrightarrow{\pi_1} Q^4$.

The integral cohomology $H^*(X_2, \mathbb{Z})$ has a basis $\mathcal{B}_2$ arising from the blow-up $X_2 \xrightarrow{\pi_1} Q^4$, given by

$$\mathcal{B}_2 = \{\mathbf{1}, H, \wp_+, \wp_-, \frac{1}{2}H^3, \frac{1}{2}H^4, E_2, S_2, L_2\};$$

its dual basis $\mathcal{B}_2^\vee$ of $H^*(X_2, \mathbb{Z})$ with respect to the Poincaré pairing is given by

$$\mathcal{B}_2^\vee = \{\frac{1}{2}H^4, \frac{1}{2}H^3, \wp_+, \wp_-, H, \mathbf{1}, -L_2, -S_2, -E_2\}.$$

Here $E_2 \in H^2(X_2, \mathbb{Z})$ denotes the (cohomology) class of the exceptional divisor E_2 by abuse of notation, L_2 is the class of a fiber in $E_2 = \mathbb{P}_{\mathbb{P}^2_-}(\Omega_{\mathbb{P}^2_-}(2) \oplus \{0\})$, and S_2 is the class of the preimage of a line in $\mathbb{P}^2_+$ of π_1. By direct calculations, the non-zero cup products in $\mathcal{B}_2 \setminus \{\mathbf{1}\}$ are given as follows.

$$H \cup H = \wp_+ + \wp_-, \qquad \wp_+ \cup E_2 = L_2,$$

$$H \cup \wp_\pm = \frac{1}{2}H^3, \qquad E_2 \cup E_2 = -\wp_+ + S_2,$$

$$H \cup \frac{1}{2}H^3 = \frac{1}{2}H^4, \qquad E_2 \cup S_2 = -\frac{1}{2}H^3 + L_2,$$
$$H \cup E_2 = S_2, \qquad E_2 \cup L_2 = -\frac{1}{2}H^4,$$
$$H \cup S_2 = L_2, \qquad S_2 \cup S_2 = -\frac{1}{2}H^4.$$

Note that X_2 admits the projective bundle structure $X_2 = \mathbb{P}_{\mathbb{P}^2_-}(\Omega_{\mathbb{P}^2_-}(2) \oplus \mathcal{O}_{\mathbb{P}^2_-}) \xrightarrow{\pi_2} \mathbb{P}^2_-$. Hence, $H^*(X_2, \mathbb{Z})$ is generated by $H^2(X_2, \mathbb{Z})$. By the Leray-Hirsch theorem, we see that $H^*(X_2, \mathbb{Z})$ has another $\mathbb{Z}$-basis $\mathcal{B}_2'$ given by

$$\mathcal{B}_2' = \{\mathbf{1}, H_-, H_-^2, E_2, E_2H_-, E_2H_-^2, E_2^2, E_2^2H_-, E_2^2H_-^2\}.$$

Here $H_- \in H^2(X_2, \mathbb{Z})$ is the pullback of the hyperplane class in $H^2(\mathbb{P}^2_-, \mathbb{Z})$ via π_2^*. Let $\tilde{f}$ be the homology class of a fiber of π_2. Then π_2 is the contraction corresponding to the extremal ray $\mathbb{R}_{\geq 0}\tilde{f}$, and $\tilde{f} = \ell - e$. So we have

$$H.\tilde{f} = E_2.\tilde{f} = 1.$$

Since $H_-.\tilde{f} = 0$, it follows that we can write $H_- = a(H - E_2)$ for some $a \in \mathbb{Z}$. Moreover, take a line in $\mathbb{P}^2_- \subset Q^4$. Then the intersection number of $(\pi_1)_*H_-$ and this line is one. So we see that $(\pi_1)_*H_-.\ell = 1$, which implies that $H_- = H - E_2$. In sum, we have

$$\begin{cases} \mathbf{1} &= \mathbf{1}, \\ H_- &= H - E_2, \\ H_-^2 &= \wp_- - S_2, \\ E_2 &= E_2, \\ E_2H_- &= \wp_+, \\ E_2H_-^2 &= \frac{1}{2}H^3 - L_2, \\ E_2^2 &= S_2 - \wp_+, \\ E_2^2H_- &= L_2, \\ E_2^2H_-^2 &= \frac{1}{2}H^4, \end{cases} \quad \text{and} \quad \begin{cases} \mathbf{1} &= \mathbf{1}, \\ H &= H_- + E_2, \\ \wp_+ &= E_2H_-, \\ \wp_- &= H_-^2 + E_2H_- + E_2^2, \\ \frac{1}{2}H^3 &= E_2H_-^2 + E_2^2H_-, \\ \frac{1}{2}H^4 &= E_2^2H_-^2, \\ E_2 &= E_2, \\ S_2 &= E_2H_- + E_2^2, \\ L_2 &= E_2^2H_-. \end{cases}$$

4.2. **Two-point invariants.** Similar to the case of X_0, we study genus zero, two-point Gromov-Witten invariants of X_2, and obtain the following lemmas.

Lemma 4.1. *The only non-zero, degree-ke with $k \geq 1$, two-point invariants with insertions in $\mathcal{B}_2$ are*

$$\langle E_2, L_2\rangle_e^{X_2} = \langle S_2, S_2\rangle_e^{X_2} = 1.$$

Proof. The argument is completely similar to that of Lemma 3.7. □

Lemma 4.2. *A curve class in $H_2(X_2, \mathbb{Z})$ admits non-zero two-point invariants only if it belongs to $\{e, \ell - e, \ell, \ell + e\}$.*

Proof. An effective curve class has the form

$$a(\ell - e) + be, \quad a, b \in \mathbb{Z}_{\geq 0}.$$

Notice $c_1(X_2) = 4H - E_2$ (see e.g. [GrHa, Chapter 4, Section 6]). Using the dimension constraint, we see that $a(\ell - e) + be$ admits non-zero two-point invariants only if

$$3a + b \leq 5 \text{ and } (a, b) \neq (0, 0).$$

Note that $(a, b) \neq (0, 0)$ since the space $\overline{M}_{0,2}(X_0, 0)$ is empty. From Lemma 4.1, we exclude the cases be for $2 \leq b \leq 5$. □

Lemma 4.3. *The non-zero, degree-($\ell - e$), two-point invariant with insertions in $\mathcal{B}_2$ are*

$$\langle \wp_-, \frac{1}{2}H^4\rangle^{X_2}_{\ell-e} = \langle E_2^2, \frac{1}{2}H^4\rangle^{X_2}_{\ell-e} = 1,$$
$$\langle \frac{1}{2}H^3, \frac{1}{2}H^3\rangle^{X_2}_{\ell-e} = \langle \frac{1}{2}H^3, L_2\rangle^{X_2}_{\ell-e} = \langle L_2, L_2\rangle^{X_2}_{\ell-e} = 1.$$

Proof. Note that X_2 admits a projective bundle structure $X_2 = \mathbb{P}_{\mathbb{P}^2_-}(\Omega_{\mathbb{P}^2_-}(2) \oplus \mathcal{O}_{\mathbb{P}^2_-}) \xrightarrow{\pi_2} \mathbb{P}^2_-$, and $\ell - e$ is the homology class of a line in a fiber of π_2. By using similar arguments as in the proof of Lemma 3.7, one can prove that the only non-zero, degree-($\ell - e$), two-point invariant with insertions in $\mathcal{B}'_2$ are

$$\langle E_2^2, E_2^2H_-^2\rangle^{X_2}_{\ell-e} = \langle E_2^2H_-, E_2^2H_-\rangle^{X_2}_{\ell-e} = 1.$$

So we can use this to determine invariants with insertions in $\mathcal{B}_2$. For example,

$$\langle \wp_-, \frac{1}{2}H^4\rangle^{X_2}_{\ell-e} = \langle H_-^2 + E_2H_- + E_2^2, E_2^2H_-^2\rangle^{X_2}_{\ell-e} = 1.$$

We leave the rest cases to interested readers. □

Lemma 4.4. *The only non-zero, degree-ℓ, two-point invariant with insertions in $\mathcal{B}_2$ is*

$$\langle \frac{1}{2}H^3, \frac{1}{2}H^4\rangle^{X_2}_{\ell} = 1.$$

Proof. By the dimension constraint, we only need to consider the case

$$\langle \frac{1}{2}H^3, \frac{1}{2}H^4\rangle^{X_2}_{\ell} \text{ and } \langle L_2, \frac{1}{2}H^4\rangle^{X_2}_{\ell}.$$

For the former case, we use the blow-up formula [Hu2]. For the latter case, observe that for $x \in \mathbb{P}^2_+$ and $y \in Q^4$, there is a line in Q^4 passing through x and y iff $y \in H_x \cap Q^4$. As a consequence, for $x \in \mathbb{P}^2_+$ and $y \in Q^4$ both in general position, there is no line in Q^4 passing through x and y. So we have $\langle L_2, \frac{1}{2}H^4\rangle^{X_2}_{\ell} = 0$. □

Lemma 4.5. *There is no non-zero, degree-($\ell + e$), two-point invariant with insertions in $\mathcal{B}_2$.*

Proof. By the dimension constraint, we only need to consider the case

$$\langle \frac{1}{2}H^4, \frac{1}{2}H^4\rangle^{X_2}_{\ell+e}.$$

Observe that for $x, y \in Q^4$, there is a line in Q^4 passing through x and y iff $y \in H_x \cap Q^4$. As a consequence, for $x, y \in Q^4$ in general position, there is no line in Q^4 passing through x and y. So we have $\langle \frac{1}{2}H^4, \frac{1}{2}H^4\rangle^{X_2}_{\ell+e} = 0$. □

Now we consider the operator $\hat{c}_1(X_2)$, where we recall $c_1(X_2) = 4H - E_2$.

Theorem 4.6. *Both Conjecture 1.1 and Conjecture $\mathcal{O}$ hold for the blow-up $X_2 = Bl_{\mathbb{P}^2}Q^4$.*

Proof. Instead of the bases $\mathcal{B}_2$ and $\mathcal{B}'_2$, we consider another basis of $H^*(X_2, \mathbb{Z})$ given by

$$\hat{\mathcal{B}}_2 := \{\mathbf{1}, H, H - E_2, \wp_+, \wp_-, \wp_- - S_2, \frac{1}{2}H^3, \frac{1}{2}H^3 - L_2, \frac{1}{2}H^4\}.$$

The associated matrix M of the operator $\hat{c}_1(X_2)$ is given by $\hat{c}_1(X_2)\hat{\mathcal{B}}_2 = \hat{\mathcal{B}}_2 M$ with

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 3 & 0 & 4 & 4 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 4 \\ 1 & 0 & -1 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 4 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 0 & -1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 3 & 0 \end{pmatrix}.$$

Here the matrix M is obtained by combining all the lemmas in this subsection, the divisor axiom for Gromov-Witten invariants, and the definition of the quantum product.

By direct calculations, the power M^{13} is a positive matrix. Thus by Perron's theorem on positive matrices [Perr], the spectral radius $\rho(M^{13})$ is a simple eigenvalue of M^{13}, and the modulus of any other eigenvalue of M^{13} is strictly less than $\rho(M^{13})$. Since M is a real matrix and 13 is odd, it follows that $\rho(M) = (\rho(M^{13}))^{\frac{1}{13}}$ is a simple eigenvalue of M, and the modulus of any other eigenvalue of M is strictly less than $(\rho(M^{13}))^{\frac{1}{13}}$. Hence, Conjecture $\mathcal{O}$ holds for X_2. (Here we notice the Fano index of X_2 is equal to 1, since $c_1(X_2).e = (4H - E_2).e = 1$.)

Notice that $\det(\lambda I_9 - M)|_{\lambda=4\sqrt{2}} < 0$. Hence, there exists a real eigenvalue λ_0 in the interval $(4\sqrt{2}, +\infty)$. Hence, by Proposition 2.1, we have

$$\rho(\hat{c}_1(X_2)) = \rho(M) \geq \lambda_0 > 4\sqrt{2} = \rho(\hat{c}_1(Q^4)).$$

That is, Conjecture 1.1 holds for $Y = Q^4$ and $Z = \mathbb{P}^2$. □

Remark 4.7. *M^{12} is the first power of M that becomes a positive matrix.*

Acknowledgements

The authors would like to thank Kwokwai Chan and Heng Xie for useful discussions. Hu is partially supported by NSFC Grants 11890662 and 11831017. Ke is partially supported by NSFC Grants 12271532 and 11831017. Li is partially supported by NSFC Grant 11831017 and Guangdong Introducing Innovative and Enterpreneurial Teams No. 2017ZT07X355. Song is partially supported by Guangdong Basic and Applied Basic Research Foundation No. 2020A1515010876.

References

[BaMa] A. Bayer and Y.I. Manin, *(Semi)simple exercises in quantum cohomology*, The Fano Conference, 143–173, Univ. Torino, Turin, 2004.

[ChHa] D. Cheong, M. Han, *Galkin's lower bound conjecture for Lagrangian and orthogonal Grassmannians*, Bull. Korean Math. Soc. 57 (2020), no. 4, 933–943.

[CoKa] D.A. Cox and S. Katz, *Mirror symmetry and algebraic geometry*, Mathematical Surveys and Monographs, 68. American Mathematical Society, Providence, RI, 1999.

[ESSSW] L. Evans, L. Schneider, R. Shifler, L. Short and S. Warman, *Galkin's lower bound conjecture holds for the Grassmannian*, Comm. Algebra 48 (2020), no. 2, 857–865.

[Frob] G. Frobenius, *Über Matrizen aus nicht negativen Elementen*, S.-B. Presuss. Akad. Wiss. Berlin (1912) 456–477.

[Ga1] S. Galkin, *The conifold point*, preprint at arXiv: math.AG/1404.7388.

[GGI] S. Galkin, V. Golyshev and H. Iritani, *Gamma classes and quantum cohomology of Fano manifolds: gamma conjectures*, Duke Math. J. 165 (2016), no. 11, 2005–2077.

[GaIr] S. Galkin and H. Iritani, *Gamma conjecture via mirror symmetry*, Primitive Forms and Related Subjects, Advanced Studies in Pure Mathematics Vol. 83, 55–115 (2019).

[Gath] A. Gathmann, *Gromov-Witten invariants of blow-ups* , J. Algebraic Geom. 10(3)(2001), 399–432.

[GrHa] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994.

[Harr] J. Harris, *Algebraic geometry. A first course*, Graduate Texts in Mathematics, 133. Springer-Verlag, New York, 1995.

[Hu1] J. Hu, *Gromov-Witten invariants of blow-ups along points and curves*, Math. Z. 233(2000), 709–739.

[Hu2] J. Hu, *Gromov-Witten invariants of blow-ups along surfaces*, Compositio Math. 125 (2001), no. 3, 345–352.

[HKLY] J. Hu, H. Ke, C. Li and T. Yang, *Gamma conjecture I for del Pezzo surfaces*, Adv. Math. 386 (2021), 107797.

[HLR] J. Hu, T.-J. Li and Y. Ruan, *Birational cobordism invariance of uniruled symplectic manifolds*, Invent. Math. 172(2008). 231–275.

[Ke] H.-Z. Ke, *On Conjecture $\mathcal{O}$ for projective complete intersections*, preprint at arxiv: math.AG/1809.10869.

[Lai] H.-H. Lai, *Gromov-Witten invariants of blow-ups along submanifolds with convex normal bundles*, Geom. Topology, 13(2009), 1–48.

[LSYZ] C. Li, R.M. Shifler, M. Yang and C. Zhang, *On Frobenius-Perron dimension*, Proc. Amer. Math. Soc. 150 (2022), no. 12, 5035–5045.

[MaPa] D. Maulik and R. Pandharipande, *A topological view of Gromov-Witten theory*, Topology 45(2006), 887–918.

[Miha] L.C. Mihalcea, Equivariant quantum Schubert calculus, Adv. Math. 203 (2006), no. 1, 1–33.

[Minc] H. Minc, *Nonnegative Matrices*, John Willy & Sons, 1988.

[Perr] O. Perron, *Zur Theorie der Matrices*, Math. Ann. 64 (1907), no. 2, 248–263.

[Rie] K. Rietsch, *Quantum cohomology rings of Grassmannians and total positivity*, Duke Math. J. (2001) 110, no. 3, 523–553.

[SzWi] M. Szurek and J.A. Wiśniewski, *On Fano manifolds, which are $\mathbf{P}^k$-bundles over $\mathbf{P}^2$*, Nagoya Math. J. 120 (1990), 89–101.

[Yang] Z. Yang, *Gamma conjecture I for blowing up $\mathbb{P}^n$ along $\mathbb{P}^r$*, preprint at arxiv: math.AG/2022.04234.

SCHOOL OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, GUANGZHOU 510275, P.R. CHINA
E-mail address: `stsjxhu@mail.sysu.edu.cn`

SCHOOL OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, GUANGZHOU 510275, P.R. CHINA
E-mail address: `kehuazh@mail.sysu.edu.cn`

SCHOOL OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, GUANGZHOU 510275, P.R. CHINA
E-mail address: `lichangzh@mail.sysu.edu.cn`

SCHOOL OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, GUANGZHOU 510275, P.R. CHINA
E-mail address: `songlei3@mail.sysu.edu.cn`