

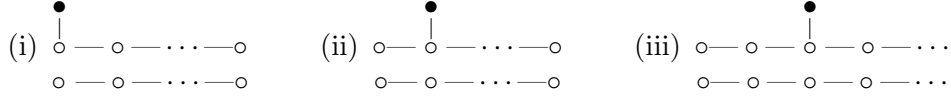
# A-D-E DIAGRAMS, HODGE–TATE HYPERPLANE SECTIONS AND SEMISIMPLE QUANTUM COHOMOLOGY

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WITH AN APPENDIX BY PIETER BELMANS AND MARKUS REINEKE

ABSTRACT. It is known that the semisimplicity of quantum cohomology implies the vanishing of off-diagonal Hodge numbers (Hodge–Tateness). We investigate which hyperplane sections of homogeneous varieties possess either of the two properties. We provide a new efficient criterion for non-semisimplicity of the small quantum cohomology ring of Fano manifolds that depends only on the Fano index and Betti numbers. We construct a bijection between Dynkin diagrams of types  $A$ ,  $D$  or  $E$ , and complex Grassmannians with Hodge–Tate smooth hyperplane sections. By applying our criteria and using monodromy action, we completely characterize the semisimplicity of the small quantum cohomology of smooth hyperplane sections in the case of complex Grassmannians, and verify a conjecture of Benedetti and Perrin in the case of (co) adjoint Grassmannians.

## HIGHLIGHTS

- Hyperplane sections of complex Grassmannians  $Gr(k, n)$  are Hodge–Tate if and only if there is a Dynkin diagram of type  $A_n$  ( $k = 1$ ),  $D_n$  ( $k = 2$ ) or  $E_n$  ( $k = 3$  and  $n \leq 8$ ) obtained by adding a node adjacent to the Dynkin diagram of type  $A_{n-1}$ . Namely,



- Quite often knowledge of Betti numbers and Fano index is sufficient to witness non-semisimplicity of small quantum cohomology, even for Hodge–Tate Fano manifolds. Complete characterizations for the case of hyperplane sections of generalized Grassmannians are proposed in Conjecture 1.7, with partial cases confirmed.
- Quantum Lefschetz hyperplane principle admits user-friendly formulations, namely Proposition 1.10 after [GI], directly applicable by non-specialists in Gromov–Witten theory. We illustrate this advantage in the case of  $Gr(3, 8)$ , an avatar of  $E_8$ .

## 1. INTRODUCTION

The big quantum cohomology ring  $BQH^*(X)$  of a Fano manifold  $X$  encodes genus-zero Gromov–Witten invariants, and is canonically equipped with a Frobenius manifold structure. It is important to investigate the semisimplicity of  $BQH^*(X)$ . For instance by Givental’s conjecture [Giv01] proved by Teleman [Te12], all higher genus Gromov–Witten invariants of  $X$  are determined by the genus-zero ones, provided that  $BQH^*(X)$  is generically semisimple. The remarkable Dubrovin’s conjecture [Du98] with clarification [GMS15] on part 1, together with the refinement of its part 3 known as Gamma conjecture II [GGI16] and independently formulated in [Du13, CDG], is also concerned with the semisimplicity of the big quantum cohomology.

The (small) quantum cohomology ring  $QH^*(X) = (H^*(X) \otimes \mathbb{C}[\mathbf{q}], \star)$  is a deformation of the classical cohomology  $H^*(X) = H^*(X, \mathbb{C})$  by incorporating genus-zero, three-point Gromov–Witten invariants. It is relatively more accessible than  $BQH^*(X)$ , and if  $QH^*(X)$  is semisimple at some specialization of the quantum variables  $\mathbf{q}$ , then  $BQH^*(X)$  is generically semisimple. Some important classes of Fano manifolds have generically semisimple small quantum cohomology, including toric Fano manifolds [Ba93, OsTy09], complete flag manifolds [Ko96] and (co)minusculi Grassmannians [CMP10]. Nevertheless, there are some examples with nice geometry such as most of (co)adjoint Grassmannians [CP11, PS21] that have non-semisimple small quantum cohomology, whilst having its big quantum cohomology generically semisimple. It is known that semisimplicity of either big or small quantum cohomology  $QH^*(X)$  puts some constraints on the classical cohomology  $H^*(X)$ . The first one was given by Bayer and Manin, and was strengthened by Hertling, Manin and Teleman.

**Proposition 1.1** ([BM04, HMT09]). *The even part of big quantum cohomology  $BQH^{\text{ev}}(X)$  is generically semisimple only if  $H^*(X)$  is of Hodge–Tate type, i.e.  $H^*(X) = \bigoplus_p H^{p,p}(X)$ .*

There is another criterion for generic semi-simplicity of small quantum cohomology given by Chaput and Perrin [CP11, Theorem 4] in terms of the degree of defining equations for a presentation of  $H^*(X)$  when  $X$  is of Picard number one.

Recall that the Fano index of  $X$  is defined by

$$r = r_X := \max\{k \in \mathbb{Z} \mid \frac{c_1(X)}{k} \in H^2(X, \mathbb{Z})\}.$$

Define *index-periodic even Betti numbers*

$$\tilde{b}_{\bar{i}} = \tilde{b}_{\bar{i}}(X) := \sum_{j \equiv i \pmod{r_X}} b_{2j}(X),$$

with indices  $\bar{i}$  running over residue classes modulo  $r_X$ . As a very simple observation, we obtain the following theorem.

**Theorem 1.2.** *The even quantum cohomology  $QH^{\text{ev}}(X)$  is generically semisimple only if both of the following hold.*

- (1)  $\tilde{b}_{\bar{i}}(X) \leq \tilde{b}_{\overline{d\bar{i}}}(X)$  for all integer  $i$  and  $d$ ,
- (2)  $\tilde{b}_{\bar{i}}(X) = \tilde{b}_{-\bar{i}}(X)$  for all integer  $i$ .

Here part (2) is a direct consequence of part (1) by taking  $d = r - 1$ . We specify this property as it is already very useful in many cases. For instance, we consider a generalized Grassmannian  $G/P_k$ , where  $G$  is a connected complex simple Lie group and  $P_k$  is the maximal parabolic subgroup that corresponds to the complement of the  $k$ -th simple root in the base of simple roots. Here we follow the label of the Dynkin diagram for  $G$  as in [Bou], see also [Bel]. Therein we can see that 9 (resp. 14) among all the 27 Grassmannians of exceptional Lie type have semisimple (resp. non-semisimple) small quantum cohomology. As a first application, we will reprove in **Theorem 2.3** that the non-semisimplicity of all the 14 known cases but  $E_8/P_4$  follows directly from Theorem 1.2 (2).

A second application is given in Appendix A, by Pieter Belmans and Markus Reineke, where it is shown in Theorem A.2 that Theorem 1.2 is also strong enough to deduce the non-semisimplicity of a certain infinite family of Kronecker moduli. These are instances of moduli spaces of stable quiver representations, which for other choices of the parameters include the usual Grassmannians, which have semisimple quantum cohomology.

As one main aim of this paper, we investigate the semisimplicity of  $QH^*(Y)$  for a smooth hyperplane section  $Y$  of a generalized Grassmannian  $G/P_k$  (with respect to its minimal embedding). It is natural to start with  $G$  of type  $A$ , i.e. the complex Grassmannian  $A_{n-1}/P_k =$

$Gr(k, n) = \{V \leq \mathbb{C}^n \mid \dim V = k\}$ . When  $k = 1$ ,  $Y = \mathbb{P}^{n-2}$  and  $QH^*(Y)$  is generically semisimple. When  $k = 2$  and  $n = 2m$  with  $m > 2$ ,  $Y \simeq C_m/P_2$  and  $QH^*(Y)$  is non-semisimple [CP11]. When  $k = 2$  and  $n = 2m + 1$  with  $m > 1$ ,  $Y$  is a quasi-homogeneous variety and  $QH^*(Y)$  is generically semisimple [Pec13, Per14]. When  $k > 2$  and  $n > 3k$ ,  $H^*(Y)$  is not of Hodge–Tate type by [BeFaM21, Theorem 3]. As one main result of this paper, we provide the following complete characterization, which is a combination of **Theorem 3.8** and **Theorem 3.22**.

**Theorem 1.3.** *Let  $Y$  be a smooth hyperplane section of  $Gr(k, n) \simeq Gr(n - k, n)$  where  $n \geq 2k$ .*

(1)  *$H^*(Y)$  is of Hodge–Tate type if and only if one of the following holds:*

(i)  $k = 1$ ; (ii)  $k = 2$ ; (iii)  $k = 3$  and  $n \in \{6, 7, 8\}$ .

(2)  *$QH^*(Y)$  is generically semisimple if and only if one of the following holds:*

(i)  $k = 1$ ; (ii)  $k = 2$  and  $n \in \{4\} \cup \{2m + 1 \mid m \in \mathbb{Z}_{\geq 2}\}$ ; (iii)  $k = 3$  and  $n \in \{7, 8\}$ .

**Remark 1.4.** The classification in Theorem 1.3 (1) is in bijection with ADE Dynkin diagrams in the sense that the diagram obtained by adding a node adjacent to the  $k$ -th node of type  $A_{n-1}$  is a simply-laced Dynkin diagram. Moreover, by [BaFuM20, SK77, Sc06], the following are all equivalent:

- (1)  $H^*(Y)$  is of Hodge–Tate type.
- (2)  $Y$  is locally rigid.
- (3) The fundamental representation  $\Lambda^k \mathbb{C}^n$  of  $GL_n$  has a Zariski open orbit.
- (4) The cluster algebra of  $A_{n-1}/P_k$  is of finite type.

However, we do not have a conceptual explanation for the equivalence.

Besides the cases in Theorem 1.3 (1), (co)adjoint Grassmannians  $G/P_k$  provide another family of generalized Grassmannians such that the cohomology of a smooth hyperplane section  $Y$  is of Hodge–Tate type, as shown by Benedetti and Perrin [BP22]. Such hyperplane sections  $Y$  admit a uniform characterization. Following the terminology in [BP22] and restricting to the minimal-embedding, we list all the (co)adjoint Grassmannians in Table 1. Therein we further assume  $n \geq 3$  for types  $B_n, C_n$  and  $n \geq 4$  for type  $D_n$ ; the lower-rank cases are projective spaces or quadrics and are treated separately.

| type      | $B$       | $C$       | $D$       | $E_6$     | $E_7$     | $E_8$     | $F_4$     | $G_2$     |
|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| adjoint   | $B_n/P_2$ | –         | $D_n/P_2$ | $E_6/P_2$ | $E_7/P_1$ | $E_8/P_8$ | $F_4/P_1$ | $G_2/P_2$ |
| coadjoint | –         | $C_n/P_2$ |           |           |           |           | $F_4/P_4$ | $G_2/P_1$ |

TABLE 1. Minimally embedded (Co)Adjoint Grassmannians  $G/P_k$

By Conjecture 1.10 (2) in [BP22],  $QH^*(Y)$  is generically semisimple if and only if so is  $QH^*(G/P_k)$ . Now this conjecture holds true by combining the study of the semisimplicity of  $QH^*(G/P_k)$  in [CP11, PS21] and the following theorem.

**Theorem 1.5.** *Let  $Y$  be a smooth hyperplane section of  $G/P_k$  of (co)adjoint type. Then  $QH^*(Y)$  is generically semi-simple if and only if  $G/P_k$  is adjoint and not coadjoint or is isomorphic to a smooth quadric, i.e.  $G/P_k \in \{B_n/P_2, C_n/P_1, F_4/P_1, G_2/P_2, G_2/P_1\}$ .*

The semisimplicity of  $QH^*(Y)$  has been precisely determined except for  $Y$  in  $C_n/P_2$  or  $D_n/P_2$  (see [BP22, Theorem 1.11]). As another main result of this paper, we obtain the following, which is a combination of **Theorem 4.1** and **Theorem 4.5** and fills in the last piece of the proof of Theorem 1.5.

**Theorem 1.6.** *Let  $Y$  be a smooth hyperplane section of  $C_n/P_2$  or  $D_n/P_2$ . Then  $QH^*(Y)$  is not semi-simple.*

For  $Y$  in  $Gr(3, 6) = A_5/P_3$  or the coadjoint Grassmannians  $C_n/P_2, F_4/P_4$ , we achieve the non-semisimplicity of  $QH^*(Y)$  by a direct application of Theorem 1.2 (2).

The monodromy action plays a surprising role in the proof (resp. disproof) of the semisimplicity of quantum cohomology for  $Y$  in  $A_7/P_3$  (resp.  $D_n/P_2$ ). More precisely, a hyperplane section depends on a choice of global sections, and thus can be viewed as a fiber of a universal family (see Sections 3.3 and 4.2 for precise descriptions). The fundamental group of the smooth locus acts on  $H^*(Y)$  by monodromy, and preserves the quantum product. We obtain the semisimplicity of (the radical part of)  $QH^*(Y)$  in case  $A_7/P_3$  by using Deligne's invariant cycle theorem. We obtain the non-semisimplicity in case  $D_n/P_2$  by carrying out a perverse sheaf-theoretic study of the universal family and applying Springer theory in geometric representation theory. This will lead to an extra  $\mathbb{Z}_2$ -graded algebra structure of the quantum cohomology, so that Theorem 1.2 (1) (strictly speaking, Lemma 2.1) can be applied. We remark that our proof works straightforwardly for  $Y$  in a coadjoint Grassmannian of type  $E$  as well. We also notice that the monodromy action method has been used to study Gromov–Witten invariants [Hu15, Mi19] before.

In our proof of Theorem 1.3 (1), the key ingredient is the combinatorial characterization of nonvanishing cohomology by Snow [Sn86] (see Proposition 3.1) from the parabolic Borel–Weil Theorem by Bott [Bo57]. The hypotheses on  $(k, n)$  in Theorem 1.3 (1) are equivalent to the following condition (see Lemma 3.5)

$$\dim Gr(k, n) = k(n - k) < 2n = 2r_{Gr(k, n)}.$$

For any  $G/P_k$  of (co)adjoint type, the inequality  $\dim G/P_k < 2r_{G/P_k}$  holds by direct calculations, and  $H^*(Y)$  is always of Hodge–Tate type [BP22]. These observations, together with our computations in examples of general Lie type, lead us to the following conjecture<sup>1</sup>. We remark that the inequality  $\dim X < 2r_X$  also appeared in the early study of Fano complete intersections in projective spaces [Be95, TX97]. Although the cases for  $G/P_k$  of type  $A$  or (co)adjoint type have been classified in Theorems 1.3 and 1.5, we include them in the statement below for completeness.

**Conjecture 1.7.** *Let  $Y$  be a smooth hyperplane section of  $G/P_k$ . Then  $H^*(Y)$  is of Hodge–Tate type if and only if  $\dim G/P_k < 2r_{G/P_k}$ , namely  $G/P_k$  is given by one of the cases:*

- (a) a complex Grassmannian in Theorem 1.3 (1);
- (b) a (co)adjoint Grassmannian in Table 1,  $C_n/P_1$ ,  $B_n/P_1$  or  $D_n/P_1$ ;
- (c) (i)  $C_3/P_3$ , (ii)  $B_{n-1}/P_{n-1} \simeq D_n/P_n \simeq D_n/P_{n-1}$  for  $n \in \{4, 5, 6, 7\}$ , (iii)  $E_6/P_1 \simeq E_6/P_6$  or (iv)  $E_7/P_7$ .

**Proposition 1.8.** *Let  $Y$  be a smooth hyperplane section of  $G/P_k$ . Assume Conjecture 1.7 holds. Then  $QH^*(Y)$  is generically semisimple if and only if  $G/P_k$  is given by one of the cases:*

- (a) a complex Grassmannian in Theorem 1.3 (2);
- (b)  $B_n/P_1$ ,  $B_n/P_2$ ,  $C_n/P_1$ ,  $D_n/P_1$ ,  $F_4/P_1$ ,  $G_2/P_1$  or  $G_2/P_2$ ;
- (c) (i)  $C_3/P_3$ , (ii)  $B_{n-1}/P_{n-1} \simeq D_n/P_n \simeq D_n/P_{n-1}$  for  $n \in \{4, 5\}$ .

**Remark 1.9.** There are in total 5 generalized Grassmannians  $G/P_k$  with  $\dim G/P_k = 2r_{G/P_k}$ , given by  $A_7/P_4$ ,  $A_8/P_3 = A_8/P_6$ ,  $B_7/P_7 = D_8/P_7 = D_8/P_8$ ,  $C_4/P_4$  and  $C_4/P_3$ . They are not  $N$ -Calabi–Yau in the sense of Bernardara, Fatighenti and Manivel [BeFaM21]. For a general

<sup>1</sup>Conjecture 1.7 can be established as a theorem via an AI-assisted extension of our proof of Theorem 1.3 (1). We have chosen not to include this proof in the present paper, keeping all the mathematical results the entire work of humans.

hyperplane section  $Y$  in these cases, the Hodge number  $h^{r_Y+1, r_Y}(Y)$  is nonzero. As pointed out by Laurent Manivel, such a nonzero Hodge number in the first 4 cases is the genus of an algebraic curve that can be attached to  $Y$ . See [GSW13, BBFM25] and the references therein for the incorporation of higher genus curves with representations in Vinberg theory.

Finally, let us describe a convenient and powerful variation [GI] of a quantum Lefschetz hyperplane principle that originally organized our investigation for the cases  $A_6/P_3$  and  $A_7/P_3$ , and helped to lead it through. Information about quantum cohomology of homogeneous varieties is relatively easy to obtain and organize, in particular tables of [GG05], based on Peterson's quantum Chevalley formula often turn out to be sufficient. In turn, hyperplane sections of homogeneous varieties usually lack sufficient homogeneity, with a notable exception of (co)adjoint varieties, as observed in [BP22]. So ways to transfer information about quantum cohomology from an ambient space to its hypersurface or the other way around are valuable. Quantum Lefschetz Hyperplane Principles provide a variety of such ways. Some of them more obviously can be related to classical Lefschetz hyperplane theorems, and others (e.g. some spectral formulations below) may look very different from classical counterparts, but more useful in practice. One feature that may look surprising in contrast to the classical case, is that in many non-trivial situations the passage from ambient space to hyperplane section is invertible, up to some minor constants fitting.

Now we let  $Y$  be a smooth ample hypersurface of  $X$  with natural inclusion  $j : Y \hookrightarrow X$ . The genus-zero Gromov–Witten theory of  $Y$  can be related with the (twisted) Gromov–Witten theory of  $X$  by the quantum Lefschetz principle theorems of [Ki99, Le01, CG07]. In these works it is mainly phrased as a relation either between the virtual fundamental classes of moduli spaces of stable maps or between Givental's  $J$ -functions when passing from an ambient space  $X$  to its hypersurface  $Y$ . There have been various extensions to other situations, such as [CCIT09, Ts10, IMM16], and we refer to [OhTh24] and references therein for more progress in recent years. Note that the induced ring homomorphism  $j^* : H^*(X) \rightarrow H^*(Y) = H_{\text{amb}}^*(Y) \oplus H_{\text{prim}}^*(Y)$  has image  $j^*(H^*(X)) = H^*(Y)_{\text{amb}}$ . All the various versions of the aforementioned quantum Lefschetz principle relate the information of  $X$  to the ambient part of the corresponding information of  $Y$ . By the classical Lefschetz hyperplane theorem, the restriction  $j^*|_{H^m(X)} : H^m(X) \rightarrow H^m(Y)$  is an isomorphism of abelian groups for  $0 \leq m < \dim Y$ .

These original quoted versions of quantum Lefschetz principle turn out to be cumbersome for some applications in practice, especially when important information is encoded differently. See [Go07, Section 6.6] for a formulation that relates quantum  $D$ -modules, [GM11, Section 7.3] for a relevant example that shows its convenience. A version of a quantum Lefschetz principle on the level of quantum  $D$ -modules was in [IMM16, Corollary 1.2].

A statement of a quantum Lefschetz hyperplane principle that directly relates  $QH^*(X)$  with  $QH^*(Y)$  on the level of algebras is being developed by Galkin and Iritani. For the purposes of this article the preliminary form available in [GI] would suffice.

For simplicity, here we restrict to Fano manifolds  $X$  of Picard number 1. In this case,  $QH^*(X) = H^*(X) \otimes \mathbb{C}[q_X]$  is a  $\mathbb{C}[q_X]$ -module, and we consider the linear operator  $\hat{\alpha}$  on  $QH^\bullet(X) := QH^*(X)|_{q_X=1}$  induced by the quantum multiplication:  $\beta \mapsto (\alpha \star \beta)|_{q_X=1}$ . Make  $H^*(X) = \bigoplus_{\bar{i}} \mathcal{A}^{\bar{i}}$  into a  $\mathbb{Z}_r$ -graded vector space, where

$$\mathcal{A}^{\bar{i}} := \bigoplus_{j \equiv i \pmod r} H^{2j}(X).$$

Note, in particular, that periodic Betti numbers  $\tilde{b}_i = \dim \mathcal{A}^{\bar{i}}$  are dimensions of graded pieces. Define  $\text{Rad}(X) := \{\alpha \in H^*(X) \mid (c_1(X))^{*m} \star \alpha = 0 \text{ for some } m > 0\}$ , and consider its

orthogonal complement

$$\mathcal{A}_\perp^0(X) := \{\gamma \in \mathcal{A}^0(X) \mid \int_X \alpha \cup \gamma = 0 \text{ for any } \alpha \in \text{Rad}(X)\}.$$

For  $\dim Y > 2$ , by Lefschetz hyperplane theorem, the induced map  $\text{Pic}(X) \rightarrow \text{Pic}(Y)$  is an isomorphism of free cyclic groups  $\text{Pic}(X) = \mathbb{Z}\mathcal{O}_X(1)$  and  $\text{Pic}(Y) = \mathbb{Z}\mathcal{O}_Y(1)$  that sends an ample generator  $\mathcal{O}_X(1)$  to an ample generator  $\mathcal{O}_Y(1)$ . Denote by  $h_X = c_1(\mathcal{O}_X(1))$  and  $h_Y = j^*h_X = c_1(\mathcal{O}_Y(1))$ , so that  $c_1(X) = r_X h_X$  and  $c_1(Y) = r_Y h_Y$ .

**Proposition 1.10** (Quantum Lefschetz hyperplane; Galkin–Iritani [GI]). *Let  $j : Y \hookrightarrow X$  be the natural inclusion of a smooth Fano hypersurface  $Y$  in a Fano manifold  $X$  of Picard number 1. Assume  $\dim Y > 2$ .*

- (1) *There is a ring homomorphism  $j_q^* : QH^*(X) \rightarrow QH^*(Y)$  that fits into a commutative diagram*

$$\begin{array}{ccc} QH^*(X) & \xrightarrow{j_q^*} & QH^*(Y) \\ \pi_X \downarrow & & \downarrow \pi_Y \\ QH^*(X)/(q_X) = H^*(X) & \xrightarrow{j^*} & H^*(Y) = QH^*(Y)/(q_Y) \end{array}$$

*and  $j_q^*(q_X) \neq 0$ . Moreover,  $j_q^*(q_X) = q_Y h_Y$  if both  $r_X - 1 = r_Y$  and  $r_Y > 1$  hold.*

- (2) *When  $r_Y > 1$ , the set of nonzero eigenvalues of  $\widehat{h}_X^{r_X}$  coincides with that of  $\widehat{h}_Y^{r_Y}$ , up to dilation by a constant if  $r_X - r_Y > 1$ .*

**Remark 1.11.** The stability of spectra in part (2) of the above proposition could have been observed independently more than once. We know it was observed and used in early 2000s by mirror symmetry research group in Moscow, that included Golyshev, Galkin, and Przyjalkowski, not later than 2004, and is implicitly used in supplementary materials [GG05] for [GG06]. Some of these developments were announced in Golyshev’s report [Go08] on spectra and strains, including various definitions of spectra, and a claim of spectral stability. We note that the case  $r_Y = 1$  will further require a shift of  $\text{Spec}(\widehat{h}_Y)$  by a constant.

**Remark 1.12.** In [GI], Galkin and Iritani studied the relationship between  $QH^*(X)$  and the Euler-twisted quantum cohomology  $QH_{\text{tw}}^*(X, L)$  (twisted by a nef line bundle  $L$ ), and showed a ring homomorphism  $QH_{\text{tw}}^*(X, L) \rightarrow QH^*(Y)$ . As a consequence, they can obtain the current Proposition 1.10. In specific cases, Proposition 1.10 can be verified more directly. Indeed, the cases of Fano complete intersections in projective spaces follow immediately from [Giv96, Corollaries 9.3 and 10.9] and [Ke24, Lemmas 3.2 and 4.2]. In Theorem 3.19, we also provide a direct verification for the cases of hyperplane sections in  $A_6/P_3$  or  $A_7/P_3$ .

We may further compare  $\mathcal{A}_\perp^0(X)$  and  $\mathcal{A}_\perp^0(Y)$  as follows.

**Conjecture 1.13.** *With the same notation in Proposition 1.10, there is a natural injective morphism of algebras  $\mathcal{A}_\perp^0(X) \rightarrow \mathcal{A}_\perp^0(Y)$  that intertwines operators  $\widehat{h}_X^{r_X}$  and  $\widehat{h}_Y^{r_Y}$ , up to the normalization described in Proposition 1.10(2) and Remark 1.11 when  $r_X - r_Y > 1$  or  $r_Y = 1$ .*

Conjecture 1.13 tells us that the semisimplicity of  $QH^*(X)$  could be related to the semisimplicity of the subalgebra of  $QH^*(Y)$  generated by  $\mathcal{A}_\perp^0(Y)$  and  $c_1(Y)$ . This could be useful when  $\text{Rad}(Y)$  is of small dimension. Indeed, we were guided by this philosophy when investigating  $QH^*(Y)$  for  $Y$  in  $A_6/P_3$  or  $A_7/P_3$ . We succeeded in verifying all the expected properties, providing evidence for Conjecture 1.13 and achieving the semisimplicity of  $QH^*(Y)$ .

This paper is organized as follows. In Section 2, we provide necessary conditions for the semisimplicity of small quantum cohomology. In Section 3, we completely characterize the

semisimplicity of  $QH^*(Y)$  for smooth hyperplane sections  $Y$  in  $A_{n-1}/P_k$ . Finally in Section 4, we show the non-semisimplicity of  $QH^*(Y)$  for  $Y$  in  $C_n/P_2$  or  $D_n/P_2$ .

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## 2. CRITERIA FOR THE SEMISIMPLICITY

**2.1. Semisimple commutative algebra.** Let  $\mathcal{A}$  be a finite-dimensional commutative (unital)  $\mathbb{C}$ -algebra. The algebra  $\mathcal{A}$  is called semisimple if for any  $\alpha \in \mathcal{A}$ , the induced linear operator  $\hat{\alpha} : \mathcal{A} \rightarrow \mathcal{A}; \beta \mapsto \alpha \cdot \beta$  is semisimple; or equivalently, every nonzero  $\alpha$  is not nilpotent. Suppose that  $\mathcal{A} = \bigoplus_{\chi \in M} \mathcal{A}^\chi$  is equipped with an  $M$ -graded algebra structure, where  $M$  is an abelian group.

**Lemma 2.1.** *Suppose that there exists  $(d, \chi) \in \mathbb{Z}_{>0} \times M$  such that  $\dim \mathcal{A}^\chi > \dim \mathcal{A}^{d\chi}$ . Then there exists  $\varepsilon \in \mathcal{A} \setminus \{0\}$  such that  $\varepsilon^d = 0$ .*

*Proof.* Take a basis  $\{e_i\}_{i=1}^n$  (resp.  $\{\hat{e}_j\}_{j=1}^m$ ) of  $\mathcal{A}^\chi$  (resp.  $\mathcal{A}^{d\chi}$ ). Then  $0 = \varepsilon^d = (\sum_{i=1}^n a_i e_i)^d = \sum_{j=1}^m f_j(\mathbf{a}) \hat{e}_j$  holds if and only if  $\mathbf{a} \in \mathbb{C}^n$  is a common root of the  $m$  homogeneous polynomials  $f_j(y_1, \dots, y_n)$  of degree  $d > 0$ . Since  $n > m$ , there exists a nonzero common root.  $\square$

**Lemma 2.2.** *Take any  $m > 0$  and any  $\alpha \in \mathcal{A}$  with  $\hat{\alpha}$  invertible. Then  $\mathcal{A}$  is semisimple if and only if  $\mathcal{A}[y]/(y^m - \alpha)$  is semisimple.*

*Proof.* If  $\mathcal{A}$  is semisimple, we have the decomposition  $\mathcal{A} = \bigoplus_i A_i$  into one-dimensional subalgebras  $A_i$ . The map  $\alpha \mapsto \alpha_i$  by taking the  $A_i$ -component of  $\alpha$  induces a ring homomorphism  $\mathcal{A}[y] \rightarrow A_i[y]$ , and it further induces a ring isomorphism  $\mathcal{A}[y]/(y^m - \alpha) \cong \bigoplus_i A_i[y]/(y^m - \alpha_i)$ . As each  $A_i$  is isomorphic to the complex field, the algebra  $A_i[y]/(y^m - \alpha_i)$  is semisimple if and only if every root of  $y^m - \alpha_i$  is of multiplicity one. Note that  $\alpha$  is invertible if and only if every component  $\alpha_i$  is invertible. Hence,  $y^m - \alpha_i$  does not have multiple roots.

Conversely, the subalgebra  $\mathcal{A}$  of the semisimple algebra  $\mathcal{A}[y]/(y^m - \alpha)$  is semisimple.  $\square$

**2.2. Quantum cohomology.** We refer to [CoKa] for more details of Gromov–Witten theory. Throughout the paper, we assume  $X$  to be a Fano manifold with even cohomology only. Let  $\overline{\mathcal{M}}_{0,m}(X, \mathbf{d})$  denote the moduli space of stable maps to  $X$  of degree  $\mathbf{d} \in H_2(X, \mathbb{Z})$ , and  $\text{ev}_i : \overline{\mathcal{M}}_{0,m}(X, \mathbf{d}) \rightarrow X$  denote the  $i$ th evaluation map. For  $\gamma_1, \dots, \gamma_m \in H^*(X)$ , we consider the genus-zero,  $m$ -point Gromov–Witten invariant defined by

$$\langle \gamma_1, \dots, \gamma_m \rangle_{\mathbf{d}}^X := \int_{[\overline{\mathcal{M}}_{0,m}(X, \mathbf{d})]^{\text{vir}}} \text{ev}_1^*(\gamma_1) \cup \dots \cup \text{ev}_m^*(\gamma_m). \quad (1)$$

Here the virtual fundamental class  $[\overline{\mathcal{M}}_{0,m}(X, \mathbf{d})]^{\text{vir}}$  lies in  $H_{2\text{expdim}}(\overline{\mathcal{M}}_{0,m}(X, \mathbf{d}), \mathbb{Q})$ , where

$$\text{expdim} = \dim X + \int_{\mathbf{d}} c_1(X) + m - 3. \quad (2)$$

Take a basis  $\{[C_1], \dots, [C_b]\}$  of the Mori cone  $\overline{\text{NE}}(X)$  of effective curve classes. (Here we have assumed that the effective curve semigroup is free for notational simplicity; the proof of

Theorem 1.2 is unchanged in general.) Each generator  $[C_i]$  associates with an indeterminate  $q_i$ . For  $\mathbf{d} = \sum_j d_j [C_j]$ , we denote  $\mathbf{q}^{\mathbf{d}} := \prod_j q_j^{d_j}$ . The (small) quantum cohomology  $QH^*(X) = (H^*(X) \otimes \mathbb{C}[q_1, \dots, q_b], \star)$  is an associative commutative algebra with unit  $1 \in H^0(X) \otimes \mathbb{C}$ , with the quantum product defined by

$$\alpha_i \star \alpha_j := \sum_k \sum_{\mathbf{d} \in \overline{\text{NE}}(X)} \langle \alpha_i, \alpha_j, \alpha_k^\vee \rangle_{\mathbf{d}}^X \alpha_k \mathbf{q}^{\mathbf{d}};$$

Here  $\{\alpha_i\}_i$  denotes a basis of  $H^*(X)$ , and  $\{\alpha_i^\vee\}_i$  denotes its dual basis with respect to Poincaré pairing:  $(\alpha_i, \alpha_j^\vee)_X = \int_{[X]} \alpha_i \cup \alpha_j^\vee = \delta_{i,j}$ . The quantum cohomology  $QH^*(X)$  is naturally a  $\mathbb{Z}$ -graded algebra with respect to the grading

$$\deg q_i := \int_{[C_i]} c_1(X), \quad \deg \alpha := j, \quad \forall \alpha \in H^{2j}(X) \setminus \{0\}.$$

*Proof of Theorem 1.2.* Note that  $r$  divides  $\deg q_1, \dots, \deg q_b$ . Thus for any specialization  $\mathbf{q} = \eta \in \mathbb{C}^b$ , the  $\mathbb{Z}$ -graded algebra  $QH^*(X)$  naturally induces an  $\mathbb{Z}_r$ -graded algebra  $(H^*(X) = QH^*(X)|_{\mathbf{q}=\eta} = \bigoplus_{\bar{i} \in \mathbb{Z}_r} \mathcal{A}^{\bar{i}}, \star_\eta)$ , where  $\mathcal{A}^{\bar{i}} = \bigoplus_{\bar{j}=\bar{i}} H^{2j}(X)$ .

If  $QH^*(X)$  is semisimple at some  $\mathbf{q} = \eta$ , then  $(H^*(X), \star_\eta)$  has no nonzero nilpotent element, and hence statement (1) follows from Lemma 2.1.

Consequently, for any  $i$ , we have  $\dim \mathcal{A}^{\bar{i}} \leq \dim \mathcal{A}^{(r-1)\bar{i}} = \dim \mathcal{A}^{-\bar{i}}$ . Applying it to  $-i$ , we obtain  $\dim \mathcal{A}^{-\bar{i}} \leq \dim \mathcal{A}^{\bar{i}}$ . Hence, we have  $\dim \mathcal{A}^{\bar{i}} = \dim \mathcal{A}^{-\bar{i}}$ . That is, statement (2) holds.  $\square$

**2.3. First applications.** Given a Lie type  $\mathcal{D}_n$  and an integer  $1 \leq k \leq n$ , we denote by  $\mathcal{D}_n/P_k$  the quotient of a simply-connected, complex simple Lie group  $G$  of Lie type  $\mathcal{D}_n$  by the maximal parabolic subgroup  $P_k$  of  $G$  that corresponds to the subset  $\Delta \setminus \{\alpha_k\}$ . Here  $\Delta = \{\alpha_1, \dots, \alpha_n\}$  is a base of simple roots for  $G$  with the same ordering as in [Bel], and  $\mathcal{D}_n/P_k$  is called a generalized Grassmannian (of type  $\mathcal{D}_n$ ). In particular, the Grassmannian of type  $A_{n-1}$ ,

$$A_{n-1}/P_k = \{V \leq \mathbb{C}^n \mid \dim V = k\} =: Gr(k, n),$$

is known as a complex Grassmannian. There are in total 27 Grassmannians of exceptional Lie type: 9 of which are known to have semisimple quantum cohomology, 14 are known to have non-semisimple quantum cohomology, and the other 4 cases are unknown [Bel]. Here we provide a proof for 13 non-semisimple cases by Theorem 1.2. Unfortunately, the remaining non-semisimple case  $E_8/P_4$  cannot be checked by Theorem 1.2, nor does it give any obstruction for the 4 unknown cases  $E_7/P_2$ ,  $E_7/P_4$ ,  $E_7/P_5$ ,  $E_8/P_6$ .

**Theorem 2.3.** *For any  $X \in \{E_6/P_2, E_6/P_4, E_7/P_1, E_7/P_3, E_7/P_6, E_8/P_1, E_8/P_2, E_8/P_3, E_8/P_5, E_8/P_7, E_8/P_8, F_4/P_3, F_4/P_4\}$ , the quantum cohomology  $QH^*(X)$  is not semisimple.*

*Proof.* Simply denote  $r = r_X$  and  $b_{2j} = b_{2j}(X)$ . Then we can read off the data of  $\dim X, r, b_{2j}$  of each  $X$  from [Bel] directly, where  $b_{2j} = 0$  if  $j > \dim X$ . Then we can calculate  $\dim \mathcal{A}^{\pm \bar{4}}$  for  $F_4/P_4$  and  $\dim \mathcal{A}^{\pm \bar{1}}$  for the other 12 cases. For instance for  $E_7/P_6$ , we have  $\dim \mathcal{A}^{\bar{1}} = b_2 + b_{2 \cdot 14} + b_{2 \cdot 27} + b_{2 \cdot 40} = 1 + 26 + 29 + 2 = 58$ , and  $\dim \mathcal{A}^{-\bar{1}} = b_{2 \cdot 12} + b_{2 \cdot 25} + b_{2 \cdot 38} + b_{2 \cdot 51} = 21 + 34 + 4 + 0 = 59$ .

| $X$       | $\dim X$ | $r$ | $i$ | $m$ | $\dim \mathcal{A}^{\bar{i}} = \sum_{j=0}^m b_{2(i+jr)}$ | $\dim \mathcal{A}^{-\bar{i}} = \sum_{j=0}^m b_{2(r-i+jr)}$ |
|-----------|----------|-----|-----|-----|---------------------------------------------------------|------------------------------------------------------------|
| $E_6/P_2$ | 21       | 11  | 1   | 1   | 6                                                       | 7                                                          |
| $E_6/P_4$ | 29       | 7   |     | 4   | 102                                                     | 104                                                        |
| $E_7/P_1$ | 33       | 17  |     | 1   | 7                                                       | 8                                                          |
| $E_7/P_3$ | 47       | 11  |     | 4   | 183                                                     | 184                                                        |
| $E_7/P_6$ | 42       | 13  |     | 3   | 58                                                      | 59                                                         |
| $E_8/P_1$ | 78       | 23  |     | 3   | 94                                                      | 95                                                         |
| $E_8/P_2$ | 92       | 17  |     | 5   | 1016                                                    | 1017                                                       |
| $E_8/P_3$ | 98       | 13  |     | 7   | 5317                                                    | 5318                                                       |
| $E_8/P_5$ | 104      | 11  |     | 10  | 21993                                                   | 21992                                                      |
| $E_8/P_7$ | 83       | 19  |     | 4   | 354                                                     | 355                                                        |
| $E_8/P_8$ | 57       | 29  |     | 1   | 8                                                       | 9                                                          |
| $F_4/P_3$ | 20       | 7   |     | 2   | 13                                                      | 14                                                         |
| $F_4/P_4$ | 15       | 11  | 4   | 2   | 3                                                       | 2                                                          |

As from the table, none of the 13 cases satisfies  $\dim \mathcal{A}^{\bar{i}} = \dim \mathcal{A}^{-\bar{i}}$  for the given  $i$ . Hence, none of them has semisimple quantum cohomology by Theorem 1.2 (2).  $\square$

### 3. SMOOTH HYPERPLANE SECTIONS OF $Gr(k, n)$

In this section, we let  $X = Gr(k, n) = \{V \leq \mathbb{C}^n \mid \dim V = k\}$ , which is a closed subvariety in  $\mathbb{P}^{\binom{n}{k}-1}$  via the Plücker embedding. The intersection of  $X$  with a general hyperplane of  $\mathbb{P}^{\binom{n}{k}-1}$  gives a smooth hyperplane section  $Y$  of the complex Grassmannian  $X$ . We will investigate the semisimplicity of  $QH^*(Y)$ . Since  $Gr(k, n) \cong Gr(n-k, n)$ , we can always assume  $n \geq 2k$ .

**3.1. Characterization of  $H^*(Y)$  of Hodge–Tate type.** Due to Proposition 1.1, we start with the study of the Hodge diamond of  $Y$ , which has its own interest in classical algebraic geometry. One key ingredient is the combinatorial characterization of nonvanishing cohomology by Snow [Sn86] from the parabolic Borel–Weil Theorem by Bott [Bo57].

Denote by  $\mathcal{P}_{k,n} := \{\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{Z}^k \mid n-k \geq \lambda_1 \geq \dots \geq \lambda_k \geq 0\}$ . Denote  $|\lambda| := \sum_i \lambda_i$ . The hook length of a cell in the Young diagram of a partition  $\lambda$  is defined to be the total number of cells which are either directly to the right or directly below the cell together with the cell.

**Proposition 3.1** ([Sn86, Section 3.1 (2)]). *For  $\ell \geq 0$ , we have*

$$H^j(Gr(k, n), \Omega^p(\ell)) \neq 0$$

*if and only if there exists  $\lambda \in \mathcal{P}_{k,n}$  of  $p$  cells with no cell of hook length  $\ell$  such that  $j$  equals the number of cells in  $\lambda$  of hook length larger than  $\ell$ .*

**Remark 3.2.** Recall a smooth projective variety  $X$  is said to satisfy Bott vanishing if

$$H^j(X, \Omega_X^p \otimes L) = 0 \quad j > 0, p \geq 0$$

for any ample line bundle  $L$ . However, Bott vanishing fails for complex Grassmannians other than projective spaces, see [BTLM97, Section 4.3]. We refer to [Be25, Fo25] for the failure of Bott vanishing for certain generalized Grassmannians of general Lie type.

We call a partition  $\lambda$  a  $j$ -core partition, if  $j$  does not appear as the hook length of cells in  $\lambda$ . We first assume Lemma 3.3, which will be proved in Section 3.4 in a purely combinatorial way.

**Lemma 3.3.** *For  $3 \leq k \leq n/2$ , there exists  $(\lambda, i) \in \mathcal{P}_{k,n} \times [1, n-1]$  such that  $|\lambda| \geq k(n-k) - i$  and  $\lambda$  is an  $(n-i)$ -core partition if and only if  $(k, n) \in \{(3, 6), (4, 8), (3, 9)\}$ .*

**Example 3.4.** Below are partitions  $(3, 2, 1)$ ,  $(4, 3, 2, 1)$ ,  $(6, 4, 2)$ ,  $(4, 4, 2, 2)$ , whose cells are labeled by their hook length. For example, the partition  $(6, 4, 2)$  has 12 cells with no cell of hook length 3, and has 6 cells of hook length larger than 3. Thus  $H^6(Gr(3, 9), \Omega^{12}(3)) \neq 0$  by Proposition 3.1.

|             |                                                                                                                                       |             |             |             |   |   |  |   |  |  |                                                                                                                                                                                                                    |   |   |   |   |   |   |   |  |   |   |  |  |   |  |  |  |                                                                                                                                                                                                                              |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |  |  |                                                                                                                                                                                                                      |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |
|-------------|---------------------------------------------------------------------------------------------------------------------------------------|-------------|-------------|-------------|---|---|--|---|--|--|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---|---|---|---|---|---|---|--|---|---|--|--|---|--|--|--|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---|---|---|---|---|---|---|---|---|---|--|--|---|---|--|--|--|--|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---|---|---|---|---|---|---|---|---|---|--|--|---|---|--|--|
| $\lambda$   | <table> <tr><td>5</td><td>3</td><td>1</td></tr> <tr><td>3</td><td>1</td><td></td></tr> <tr><td>1</td><td></td><td></td></tr> </table> | 5           | 3           | 1           | 3 | 1 |  | 1 |  |  | <table> <tr><td>7</td><td>5</td><td>3</td><td>1</td></tr> <tr><td>5</td><td>3</td><td>1</td><td></td></tr> <tr><td>3</td><td>1</td><td></td><td></td></tr> <tr><td>1</td><td></td><td></td><td></td></tr> </table> | 7 | 5 | 3 | 1 | 5 | 3 | 1 |  | 3 | 1 |  |  | 1 |  |  |  | <table> <tr><td>8</td><td>7</td><td>5</td><td>4</td><td>2</td><td>1</td></tr> <tr><td>5</td><td>4</td><td>2</td><td>1</td><td></td><td></td></tr> <tr><td>2</td><td>1</td><td></td><td></td><td></td><td></td></tr> </table> | 8 | 7 | 5 | 4 | 2 | 1 | 5 | 4 | 2 | 1 |  |  | 2 | 1 |  |  |  |  | <table> <tr><td>7</td><td>6</td><td>3</td><td>2</td></tr> <tr><td>6</td><td>5</td><td>2</td><td>1</td></tr> <tr><td>3</td><td>2</td><td></td><td></td></tr> <tr><td>2</td><td>1</td><td></td><td></td></tr> </table> | 7 | 6 | 3 | 2 | 6 | 5 | 2 | 1 | 3 | 2 |  |  | 2 | 1 |  |  |
| 5           | 3                                                                                                                                     | 1           |             |             |   |   |  |   |  |  |                                                                                                                                                                                                                    |   |   |   |   |   |   |   |  |   |   |  |  |   |  |  |  |                                                                                                                                                                                                                              |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |  |  |                                                                                                                                                                                                                      |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |
| 3           | 1                                                                                                                                     |             |             |             |   |   |  |   |  |  |                                                                                                                                                                                                                    |   |   |   |   |   |   |   |  |   |   |  |  |   |  |  |  |                                                                                                                                                                                                                              |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |  |  |                                                                                                                                                                                                                      |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |
| 1           |                                                                                                                                       |             |             |             |   |   |  |   |  |  |                                                                                                                                                                                                                    |   |   |   |   |   |   |   |  |   |   |  |  |   |  |  |  |                                                                                                                                                                                                                              |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |  |  |                                                                                                                                                                                                                      |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |
| 7           | 5                                                                                                                                     | 3           | 1           |             |   |   |  |   |  |  |                                                                                                                                                                                                                    |   |   |   |   |   |   |   |  |   |   |  |  |   |  |  |  |                                                                                                                                                                                                                              |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |  |  |                                                                                                                                                                                                                      |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |
| 5           | 3                                                                                                                                     | 1           |             |             |   |   |  |   |  |  |                                                                                                                                                                                                                    |   |   |   |   |   |   |   |  |   |   |  |  |   |  |  |  |                                                                                                                                                                                                                              |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |  |  |                                                                                                                                                                                                                      |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |
| 3           | 1                                                                                                                                     |             |             |             |   |   |  |   |  |  |                                                                                                                                                                                                                    |   |   |   |   |   |   |   |  |   |   |  |  |   |  |  |  |                                                                                                                                                                                                                              |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |  |  |                                                                                                                                                                                                                      |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |
| 1           |                                                                                                                                       |             |             |             |   |   |  |   |  |  |                                                                                                                                                                                                                    |   |   |   |   |   |   |   |  |   |   |  |  |   |  |  |  |                                                                                                                                                                                                                              |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |  |  |                                                                                                                                                                                                                      |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |
| 8           | 7                                                                                                                                     | 5           | 4           | 2           | 1 |   |  |   |  |  |                                                                                                                                                                                                                    |   |   |   |   |   |   |   |  |   |   |  |  |   |  |  |  |                                                                                                                                                                                                                              |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |  |  |                                                                                                                                                                                                                      |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |
| 5           | 4                                                                                                                                     | 2           | 1           |             |   |   |  |   |  |  |                                                                                                                                                                                                                    |   |   |   |   |   |   |   |  |   |   |  |  |   |  |  |  |                                                                                                                                                                                                                              |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |  |  |                                                                                                                                                                                                                      |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |
| 2           | 1                                                                                                                                     |             |             |             |   |   |  |   |  |  |                                                                                                                                                                                                                    |   |   |   |   |   |   |   |  |   |   |  |  |   |  |  |  |                                                                                                                                                                                                                              |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |  |  |                                                                                                                                                                                                                      |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |
| 7           | 6                                                                                                                                     | 3           | 2           |             |   |   |  |   |  |  |                                                                                                                                                                                                                    |   |   |   |   |   |   |   |  |   |   |  |  |   |  |  |  |                                                                                                                                                                                                                              |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |  |  |                                                                                                                                                                                                                      |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |
| 6           | 5                                                                                                                                     | 2           | 1           |             |   |   |  |   |  |  |                                                                                                                                                                                                                    |   |   |   |   |   |   |   |  |   |   |  |  |   |  |  |  |                                                                                                                                                                                                                              |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |  |  |                                                                                                                                                                                                                      |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |
| 3           | 2                                                                                                                                     |             |             |             |   |   |  |   |  |  |                                                                                                                                                                                                                    |   |   |   |   |   |   |   |  |   |   |  |  |   |  |  |  |                                                                                                                                                                                                                              |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |  |  |                                                                                                                                                                                                                      |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |
| 2           | 1                                                                                                                                     |             |             |             |   |   |  |   |  |  |                                                                                                                                                                                                                    |   |   |   |   |   |   |   |  |   |   |  |  |   |  |  |  |                                                                                                                                                                                                                              |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |  |  |                                                                                                                                                                                                                      |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |
| $(k, n, i)$ | $(3, 6, 4)$                                                                                                                           | $(4, 8, 6)$ | $(3, 9, 6)$ | $(4, 8, 4)$ |   |   |  |   |  |  |                                                                                                                                                                                                                    |   |   |   |   |   |   |   |  |   |   |  |  |   |  |  |  |                                                                                                                                                                                                                              |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |  |  |                                                                                                                                                                                                                      |   |   |   |   |   |   |   |   |   |   |  |  |   |   |  |  |

As we will see from the proof of Lemma 3.3, they are the only  $(n - i)$ -core partitions.

**Lemma 3.5.** *For  $n \geq 2k$ ,  $k(n - k) > 2n$  holds if and only if one of the following holds:*

- (i)  $k \geq 5$ ;      (ii)  $k = 4$  and  $n \geq 9$ ;      (iii)  $k = 3$  and  $n \geq 10$ .

*Proof.* For  $k \geq 5$ ,  $k(n - k) - 2n = (k - 2)n - k^2 \geq (k - 2)2k - k^2 = k^2 - 4k > 0$ . For  $k \leq 4$ , by direct calculation,  $k(n - k) > 2n$  if and only if  $(k = 3, n \geq 10)$  or  $(k = 4, n \geq 9)$  holds.  $\square$

**Lemma 3.6.** *Assume  $\dim X > 2n$ . Then for any  $0 \leq p \leq n$ , we have*

$$H^s(X, \Omega_X^{\dim X - j}(p - j)) = 0 \quad \text{for all } s \text{ and } 1 \leq j < p.$$

*Proof.* Assume  $H^s(X, \Omega_X^{\dim X - j}(p - j)) \neq 0$  for some  $(s, j, p)$ . Then by Proposition 3.1, there exists  $\lambda \in \mathcal{P}_{k, n}$  with  $|\lambda| = k(n - k) - j$ , having no cell of hook length  $p - j$  and having  $s$  cells of hook length larger than  $p - j$ . Since  $1 \leq p - j < n$ , then for  $i := n - p + j \in [j, n - 1]$ ,  $|\lambda| \geq k(n - k) - i$  and  $\lambda$  is an  $(n - i)$ -core partition. Thus by Lemma 3.3,  $(k, n) \in \{(3, 6), (3, 9), (4, 8)\}$ . This contradicts the assumption  $\dim X = k(n - k) > 2n$ .  $\square$

We have the following proposition, whose proof is a refined version of Griffiths' theory ([Gr69], see also [Voi, Section 6.1.2]) with more precise control of cohomology vanishing. A similar argument appears in [DV10, Theorem 2.2].

**Proposition 3.7.** *Assume  $k(n - k) > 2n$ . Then*

$$h^{k(n-k)-p, p-1}(Y) = \begin{cases} 0, & p < n, \\ 1, & p = n. \end{cases}$$

*In particular,  $H^*(Y)$  is not of Hodge-Tate type.*

*Proof.* Let  $U = X \setminus Y$  be the complement of  $Y$ . Let  $\Omega_X^p(\log Y)$  be the sheaf of logarithmic  $p$ -forms, i.e. meromorphic differential  $p$ -forms  $\phi$  such that  $\phi$  and  $d\phi$  both have a pole of order at most 1 along  $Y$ . By the proof of [Voi, Theorem 6.5], the Hodge filtration of  $H^{\dim X}(U, \mathbb{C})$  is given by

$$F^{\dim X - p + 1} H^{\dim X}(U, \mathbb{C}) = H^{p-1}(X, \Omega_X^{\dim X - p + 1}(\log Y)_{\text{closed}}).$$

Note that by definition,  $\Omega_X^{\dim X - p + 1}(\log Y)_{\text{closed}} = \Omega_X^{\dim X - p + 1}(Y)_{\text{closed}}$ . We have the following exact sequence for  $p < \dim X$

$$0 \rightarrow \Omega_X^{\dim X - p + 1}(Y)_{\text{closed}} \xrightarrow{\subseteq} \Omega_X^{\dim X - p + 1}(Y) \xrightarrow{d} \Omega_X^{\dim X - p + 2}(2Y) \xrightarrow{d} \dots \xrightarrow{d} K_X(pY) \rightarrow 0.$$

By the vanishing in Lemma 3.6, we can shift the degree, and hence for  $1 \leq p \leq n$ ,

$$F^{\dim X - p + 1} H^{\dim X}(U) \cong H^0(X, K_X(pY)).$$

Then by the assumption  $\dim X > 2n$ , we have

$$F^{\dim X - p} H^{\dim X - 1}(Y) \cong H^0(X, K_X(pY)).$$

It has dimension 0 when  $p < n$  and 1 when  $p = n$ . This gives the Hodge number.  $\square$

**Theorem 3.8.** *Let  $Y$  be a smooth hyperplane section of  $Gr(k, n)$  where  $n \geq 2k$ . Then  $H^*(Y)$  is of Hodge–Tate type if and only if either (i)  $k \in \{1, 2\}$  or (ii)  $k = 3$  and  $n \in \{6, 7, 8\}$  holds.*

*Proof.* By Proposition 3.7, it remains to check the case when  $k(n - k) \leq 2n$ . For  $k = 1, 2$ , the hyperplane sections are known to be Hodge–Tate. Thus by Lemma 3.5, it remains to check the cases when  $(k, n)$  belongs to  $\{(3, 6), (3, 7), (3, 8), (3, 9), (4, 8)\}$ .

To compute the Hodge numbers, we consider the  $\lambda$ -class  $\lambda_y(Y) = \sum_{p \geq 0} y^p [\Omega_Y^p] \in K(Y)[y]$  in the polynomial ring of  $y$  with coefficients in the  $K$ -theory of  $Y$  [Hir]. We have

$$(1 + y\mathcal{O}(-Y)) \cdot \lambda_y(Y) = \lambda_y(X)|_Y.$$

So

$$\chi_y(Y) := \sum_{p \geq 0} y^p \chi(Y, \Omega_Y^p) = \chi(Y, \lambda_y(Y)) = \chi\left(X, \lambda_y(X) \frac{1 - \mathcal{O}(-Y)}{1 + y\mathcal{O}(-Y)}\right) \in \mathbb{Z}[y].$$

This can be obtained by using the Bott–Lefschetz localization formula and taking the non-equivariant limit. For  $X = Gr(3, 9)$ , we have

$$\begin{aligned} \chi_y(Y) = & -y^{17} + y^{16} - 2y^{15} + 3y^{14} - 4y^{13} + 5y^{12} - 7y^{11} + 7y^{10} - 6y^9 \\ & + 6y^8 - 7y^7 + 7y^6 - 5y^5 + 4y^4 - 3y^3 + 2y^2 - y + 1. \end{aligned}$$

That is,

| $p$                   | 0 | 1  | 2 | 3  | 4 | 5  | 6 | 7  | 8 | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|-----------------------|---|----|---|----|---|----|---|----|---|----|----|----|----|----|----|----|----|----|
| $\chi(Y, \Omega_Y^p)$ | 1 | -1 | 2 | -3 | 4 | -5 | 7 | -7 | 6 | -6 | 7  | -7 | 5  | -4 | 3  | -2 | 1  | -1 |

By Lefschetz theorem and Poincaré duality, we have

- (1)  $h^{i,j}(Y) = 0$  unless  $i = j$  or  $i + j = \dim Y = 17$ ;
- (2) when  $i + j < \dim Y = 17$ ,  $h^{i,j}(Y) = h^{i,j}(X)$ .

Comparing them with the Poincaré polynomial of  $Gr(3, 9)$ , we conclude when  $i + j = 17$ ,

$$h^{8,9} = h^{9,8} = 2, \quad h^{i,j} = 0 \text{ otherwise.}$$

In particular,  $H^*(Y)$  is not of Hodge–Tate type. By the same methods, we can work out the other cases. The following diagrams list their Hodge diamonds.

| $Y \subset Gr(3, 6)$                                                          | $Y \subset Gr(3, 7)$                                                               | $Y \subset Gr(3, 8)$                                                                                   | $Y \subset Gr(3, 9)$                                                                                                                 | $Y \subset Gr(4, 8)$                                                                                                       |
|-------------------------------------------------------------------------------|------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------|
| $\begin{array}{c} 1 \\ 1 \\ 2 \\ 3 \\ 4 \\ 3 \\ 3 \\ 2 \\ 1 \\ 1 \end{array}$ | $\begin{array}{c} 1 \\ 1 \\ 2 \\ 3 \\ 4 \\ 4 \\ 4 \\ 3 \\ 2 \\ 1 \\ 1 \end{array}$ | $\begin{array}{c} 1 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 6 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{array}$ | $\begin{array}{c} 1 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 7 \\ 7 \\ 7 \\ 8 \\ 8 \\ 8 \\ 7 \\ 7 \\ 5 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \\ 1 \end{array}$ | $\begin{array}{c} 1 \\ 1 \\ 2 \\ 3 \\ 3 \\ 5 \\ 5 \\ 7 \\ 7 \\ 8 \\ 7 \\ 7 \\ 5 \\ 5 \\ 5 \\ 3 \\ 2 \\ 1 \\ 1 \end{array}$ |

Hence, for  $k \geq 3$ ,  $H^*(Y)$  is of Hodge–Tate type if and only if  $k = 3$  and  $n \in \{6, 7, 8\}$ .  $\square$

**Remark 3.9.** The method in the proof is well-known and works for any  $Gr(k, n)$  in principle, while the computation of  $\chi_y(Y)$  is not efficient in practice. The polynomial  $\chi_y(Y)$  can be efficiently computed via the Pieri rule of motivic Chern classes of Schubert cells over Grassmannian [FGSX24]. The readers can try it online: <https://cubicbear.github.io/PluckerHodge.html>.

### 3.2. Quantum Pieri rule for Hodge–Tate hyperplane sections.

3.2.1. *Quantum Pieri rule for  $X$ .* Here we review some facts on  $QH^*(X)$  (see e.g. [Bu03]).

For  $\lambda \in \mathcal{P}_{k,n}$ , the Schubert subvariety  $X_\lambda \subset X$  of codimension  $|\lambda|$ , associated to a fixed complete flag  $E_\bullet$  of  $\mathbb{C}^n$ , is defined by  $X_\lambda = \{V \in X \mid \dim V \cap E_{n-k+i-\lambda_i} \geq i, \quad i = 1, \dots, k\}$ . The Schubert classes  $\sigma_\lambda := P.D.([X_\lambda]) \in H^{2|\lambda|}(X, \mathbb{Z})$  form an additive basis of  $H^*(X, \mathbb{Z})$ . The dual basis is given by  $\{(\sigma_\lambda)^\vee = \sigma_{\lambda^\vee}\}_\lambda$ , where  $\lambda^\vee = (n-k-\lambda_k, \dots, \dots, n-k-\lambda_1)$  is the dual partition. We simply denote the special partitions  $p = (p, 0, \dots, 0)$  and  $1^p = (1, \dots, 1, 0, \dots, 0)$  where there are  $p$  copies of 1. There is an exact sequence

$$0 \rightarrow \mathcal{S} \rightarrow \mathbb{C}^n \rightarrow \mathcal{Q} \rightarrow 0$$

of tautological vector bundles over  $X$ . The fiber of the tautological subbundle  $\mathcal{S}$  at a point  $V \in X$  is given by the vector space  $V$ . The Schubert classes  $\sigma_p$  (resp.  $\sigma_{1^p}$ ) coincide with the  $p$ -th Chern classes  $c_p(\mathcal{Q})$  (resp.  $(-1)^p c_p(\mathcal{S})$ ). Hence, they are related by  $c(\mathcal{S}) \cup c(\mathcal{Q}) = 1$ , i.e.

$$\sum_{i=0}^m (-1)^i \sigma_{1^i} \sigma_{m-i} = 0 \quad (3)$$

for all  $m \geq 1$ . Here we take the convention  $\sigma_\lambda = 0$ , whenever  $\lambda \notin \mathcal{P}_{k,n}$ . There is a canonical ring isomorphism, where  $\sigma_a$ 's are polynomials in  $\sigma_{1^b}$ 's read off from (3),

$$QH^*(X) \cong \mathbb{C}[\sigma_1, \dots, \sigma_{1^k}, q_X] / (\sigma_{n-k+1}, \dots, \sigma_{n-1}, \sigma_n + (-1)^k q_X). \quad (4)$$

The quantum multiplications by  $\sigma_p$ 's (or  $\sigma_{1^p}$ 's) are known as the quantum Pieri rule, and were first provided by Bertram. We refer to [BCFF99, Proposition 4.2] for the following form.

**Proposition 3.10** (Quantum Pieri rule). *Let  $1 \leq p \leq k$  and  $\lambda \in \mathcal{P}_{k,n}$ . In  $QH^*(X)$ , we have*

$$\sigma_{1^p} \star \sigma_\lambda = \sum_{\mu} \sigma_\mu + q_X \sum_{\nu} \sigma_\nu,$$

*the first sum over  $\mu$  obtained by adding  $p$  cells to  $\lambda$  with no two in the same row, and the second sum over  $\nu$  with  $|\nu| = |\lambda| + p - n$  such that  $\tilde{\lambda}_1 - 1 \geq \tilde{\nu}_1 \geq \tilde{\lambda}_2 - 1 \geq \tilde{\nu}_2 \geq \dots \geq \tilde{\lambda}_{n-k} - 1 \geq \tilde{\nu}_{n-k} \geq 0$ , which occurs only if  $\lambda_1 = n - k$  and where  $\tilde{\lambda} \in \mathcal{P}_{n-k,n}$  denotes the transpose of  $\lambda$ .*

3.2.2. *Quantum Pieri rule for  $Y$ .* Denote by  $j : Y \hookrightarrow X$  the natural inclusion. It induces an algebra homomorphism  $j^* : H^*(X) \rightarrow H^*(Y)$ , with  $j^*|_{H^{2i}(X)}$  an isomorphism of vector spaces for  $0 \leq 2i < \dim Y$ . Taking a line  $\mathbb{P}^1 \subset Y$ , we have  $H_2(Y, \mathbb{Z}) = \mathbb{Z}[\mathbb{P}^1]$  and  $H_2(X, \mathbb{Z}) = \mathbb{Z}j_*[\mathbb{P}^1]$ , and simply denote by  $d \in \mathbb{Z}$  a curve class under this identification.

**Proposition 3.11** ([BP22, Proposition 5.13 and Theorem 5.11 (1)]). *Assume  $n \geq 2k > 3$ .*

- (1) *For  $d \in \{1, 2\}$ ,  $\overline{\mathcal{M}}_{0,n}(Y, d)$  is irreducible of expected dimension.*
- (2) *For any  $\alpha, \alpha' \in H^*(Y)$  and  $\gamma \in H^{2i}(X)$  with  $i < n - 1$ , we have*

$$\langle \alpha, \alpha', j^* \gamma \rangle_1^Y = \langle j_* \alpha, j_* \alpha', \gamma \rangle_1^X.$$

**Remark 3.12.** Part (1) with  $d = 1$  is a consequence of the result in [LM03] as explained in [BP22, Corollary 5.8]. Then part (2) is obtained by a canonical argument with projection formula as in [BP22, Lemma 5.12]. Part (1) with  $d = 2$  was proved in [BP22, Theorem 5.11

(1)] for adjoint or quasi-minuscule Grassmannians (excluding type  $G_2$ ), and also holds in our situation by exactly the same arguments therein.

Extend the morphism  $j^* : H^*(X) \rightarrow H^*(Y)$  to a  $\mathbb{C}$ -linear map  $j^* : QH^*(X) \rightarrow QH^*(Y)$  by defining  $j^*(q_X^m) = q_Y^m$ , which is distinct from the lifting  $j_q^*$  in Proposition 1.10.

**Proposition 3.13.** *Let  $\lambda \in \mathcal{P}_{k,n}$ ,  $1 \leq p \leq k$  and  $\beta \in H_{\text{prim}}(Y)$ . In  $QH^*(Y)$ , we have*

$$\begin{aligned} j^* \sigma_{1^p} \star j^* \sigma_\lambda &\equiv j^*(\sigma_{1^p} \cup \sigma_\lambda) + j^*(\sigma_{1^p} \star (\sigma_\lambda \cup \sigma_1) - \sigma_{1^p} \cup (\sigma_\lambda \cup \sigma_1)) \mod q_Y^2, \\ j^* \sigma_{1^p} \star \beta &\equiv 0 \mod q_Y^2. \end{aligned}$$

*Proof.* Denote  $m := p + |\lambda| - (n - 1)$ . Take an ordering  $\{\sigma_{\mu^{(1)}}, \dots, \sigma_{\mu^{(b)}}\}$  of the Schubert basis of  $H^{2m}(X)$  such that  $H^{2m}(Y)$  has a basis  $\{j^* \sigma_{\mu^{(i)}}\}_{1 \leq i \leq a} \cup \{\beta_i\}_{1 \leq i \leq c}$  for some  $1 \leq a \leq b$  and primitive classes  $\beta_i$ 's which appear only if  $\dim Y = 2m$ . Write

$$j^* \sigma_{1^p} \star j^* \sigma_\lambda \equiv j^* \sigma_{1^p} \cup j^* \sigma_\lambda + q_Y \sum_{i=1}^a m_i j^* \sigma_{\mu^{(i)}} + q_Y \sum_{i=1}^c \tilde{m}_i \beta_i \mod q_Y^2.$$

Then for any  $1 \leq i \leq c$ ,  $\tilde{m}_i = \langle j^* \sigma_{1^p}, j^* \sigma_\lambda, \beta_i^\vee \rangle_1^Y = \langle \sigma_{1^p}, j_*(j^* \sigma_\lambda), j_*(\beta_i^\vee) \rangle_1^X = 0$  by using Proposition 3.11 and noting  $j_*(\beta_i^\vee) = 0$  (since  $\beta_i^\vee$  is again a primitive class). For  $1 \leq i \leq a$ ,

$$m_i = \langle j^* \sigma_{1^p}, j^* \sigma_\lambda, (j^* \sigma_{\mu^{(i)}})^\vee \rangle_1^Y = \langle \sigma_{1^p}, j_*(j^* \sigma_\lambda), j_*((j^* \sigma_{\mu^{(i)}})^\vee) \rangle_1^X.$$

Since  $Y$  is a hyperplane section of  $X$ , by projection formula we have  $j_*(j^* \sigma_\lambda) = j_*(j^* \sigma_\lambda \cup 1) = \sigma_\lambda \cup j_* 1 = \sigma_\lambda \cup \sigma_1$ . For any  $\nu \in \mathcal{P}_{k,n} \setminus \{\mu^{(s)}\}_{a < s \leq b}$ , we have

$$\int_{[X]} j_*((j^* \sigma_{\mu^{(i)}})^\vee) \cup \sigma_\nu = \int_{[Y]} (j^* \sigma_{\mu^{(i)}})^\vee \cup j^* \sigma_\nu = \delta_{\mu^{(i)}, \nu}$$

For  $a < s \leq b$ , writing  $j^* \sigma_{\mu^{(s)}} = \sum_{t=1}^a c_{st} j^* \sigma_{\mu^{(t)}}$ , we have  $\int_{[X]} j_*((j^* \sigma_{\mu^{(i)}})^\vee) \cup \sigma_{\mu^{(s)}} = \int_{[Y]} (j^* \sigma_{\mu^{(i)}})^\vee \cup (\sum_t c_{st} j^* \sigma_{\mu^{(t)}}) = c_{si}$ . It follows that  $j_*((j^* \sigma_{\mu^{(i)}})^\vee) = \sigma_{\mu^{(i)}}^\vee + \sum_s c_{si} \sigma_{\mu^{(s)}}^\vee$ . Hence,

$$\begin{aligned} q_Y \sum_{i=1}^a m_i j^* \sigma_{\mu^{(i)}} &= q_Y \sum_{i=1}^a (\langle \sigma_{1^p}, \sigma_\lambda \cup \sigma_1, \sigma_{\mu^{(i)}}^\vee \rangle_1^X j^* \sigma_{\mu^{(i)}} + \sum_{s=a+1}^b \langle \sigma_{1^p}, \sigma_\lambda \cup \sigma_1, \sigma_{\mu^{(s)}}^\vee \rangle_1^X c_{si} j^* \sigma_{\mu^{(i)}}) \\ &= q_Y \sum_{i=1}^b \langle \sigma_{1^p}, \sigma_\lambda \cup \sigma_1, \sigma_{\mu^{(i)}}^\vee \rangle_1^X j^* \sigma_{\mu^{(i)}} \\ &= j^*(\sigma_{1^p} \star (\sigma_\lambda \cup \sigma_1) - \sigma_{1^p} \cup (\sigma_\lambda \cup \sigma_1)). \end{aligned}$$

Here the last equality follows from the quantum Pieri rule for  $X$ , which shows that the quantum part of  $\sigma_{1^p} \star (\sigma_\lambda \cup \sigma_1)$  consists of the degree-one part of the quantum product.  $\square$

The *span*  $\text{Span}(Z)$  of a subvariety  $Z \subset X$  is the smallest vector subspace of  $\mathbb{C}^n$  that contains all the  $k$ -dimensional spaces given by points of  $Z$ . It has been used to study  $QH^*(X)$  in [Bu03].

**Proposition 3.14.** *Let  $k \geq 3$  and  $Z$  be a closed subvariety of  $Y$  of dimension  $m$ . If  $n > 2k + m$ , then for any  $\alpha \in H^*(Y)$ , we have*

$$\langle P.D.[\text{pt}], P.D.[Z], \alpha \rangle_2^Y = 0.$$

*Proof.* It suffices to show the case when  $\alpha$  is represented by a closed subvariety of dimension  $\tilde{m} = 2 \dim Y - m - 2(n - 1)$ . Assume  $\langle [\text{pt}], [Z], \alpha \rangle_2^Y \neq 0$ , then for any point  $P \in Y$  and any closed subvariety  $Z'$  of dimension  $\tilde{m}$ , there exists a conic  $C$  passing through  $P$ ,  $Z$  and  $Z'$  by Proposition 3.11 (1). Say  $\hat{P} \in C \cap Z$ , then we have  $\dim \text{Span}(P) = \dim(\hat{P}) = k$ . Hence,  $\text{Span}(P) \cap \text{Span}(\hat{P}) \neq 0$ , by noting that they are both vector subspaces of  $\text{Span}(C)$  while  $\dim \text{Span}(C) \leq k + 2$  by [Bu03, Lemma 1].

Consider the configuration space

$$B_Z := \{(V_1, V_k, \bar{V}_k) \mid V_1 \leq \bar{V}_k, V_1 \leq V_k \in Z, \dim V_1 = 1, \dim V_k = \dim \bar{V}_k = k\}$$

as well as the natural projections  $\pi_i$  by sending  $(V_1, V_k, \bar{V}_k)$  to the  $i$ th vector space. Note that  $B_Z$  is a closed variety, which is a fibration over  $Z$  via  $\pi_2$  with fiber at  $V_k$  being a  $Gr(k-1, n-1)$ -bundle over  $\mathbb{P}(V_k)$ . Thus  $\dim B_Z = \dim Z + (k-1) + (k-1)(n-k)$ . Hence,

$$\text{codim}_X \pi_3(B_Z) = \dim X - \dim \pi_3(B_Z) \geq \dim X - \dim B_Z = n - 2k - m + 1.$$

Since  $Y$  is codimension 1 in  $X$ ,  $\text{codim}_Y(Y \cap \pi_3(B_Z)) \geq n - 2k - m > 0$  by the hypothesis. Therefore, there exists  $\bar{V}_k \in Y \setminus \pi_3(B_Z)$ . Then for any  $V_k \in Z$  and any 1-dimensional vector subspace  $V_1$ , either  $V_1 \not\leq \bar{V}_k$  or  $V_1 \not\leq V_k$  holds. That is,  $\bar{V}_k \cap V_k = 0$ , saying that the span of the point  $\bar{V}_k$  in  $Y$  and the span of the point  $V_k \in Z \subset Y$  has zero intersection and resulting in a contradiction.  $\square$

**Definition 3.15.** We call  $Y$  a Hodge–Tate hyperplane section, if  $H^*(Y)$  is of Hodge–Tate type.

**Proposition 3.16** (Quantum Pieri rule for  $Y$ ). *Let  $Y$  be a Hodge–Tate hyperplane section. Take any  $1 \leq p \leq k$ ,  $\lambda \in \mathcal{P}_{k,n}$  and  $\beta \in H_{\text{prim}}(Y)$ . In  $QH^*(Y)$ , we have*

$$j^* \sigma_{1^p} \star j^* \sigma_\lambda = j^*(\sigma_{1^p} \cup \sigma_\lambda) + j^*(\sigma_{1^p} \star (\sigma_\lambda \cup \sigma_1) - \sigma_{1^p} \cup (\sigma_\lambda \cup \sigma_1)); \quad j^* \sigma_{1^p} \star \beta = 0.$$

*Proof.* Denote  $q := q_Y$ . For  $k \in \{1, 2\}$ , we have  $H_{\text{prim}}(Y) = 0$  and  $p + |\lambda| \leq k + \dim Y = k + k(n-k) - 1 < 2(n-1) = 2 \deg q$ . Thus there are no  $q^d$ -terms with  $d \geq 2$  in  $j^* \sigma_{1^p} \star j^* \sigma_\lambda$ . By Theorem 3.8, it remains to consider the case when  $k = 3$  and  $n \in \{6, 7, 8\}$ .

By degree counting again, there are no  $q^d$ -terms with  $d \geq 3$  (resp.  $d \geq 2$ ) in  $j^* \sigma_{1^p} \star j^* \sigma_\lambda$  (resp.  $j^* \sigma_{1^p} \star \beta$ ). Thus by Proposition 3.13, we have  $j^* \sigma_{1^p} \star \beta = 0$ .

For  $p = 1$ , we further conclude that there are no  $q^2$ -terms in  $j^* \sigma_1 \star j^* \sigma_\lambda$ . Indeed, when  $n = 6$ , this holds by degree counting. When  $n = 7$ , a  $q^2$ -term occurs only if  $12 = 2 \deg q \leq 1 + |\lambda| \leq 1 + \dim Y = 12$ , implying  $\lambda = (4, 4, 3)$ . The coefficient of  $q^2 \cdot 1$  in  $j^* \sigma_1 \star j^* \sigma_{(4,4,3)}$  equals  $\langle j^* \sigma_1, P.D.[\text{pt}], P.D.[\text{pt}] \rangle_2^Y$ , and hence equals 0 by Proposition 3.14. When  $n = 8$ , there are at most two possibilities by degree counting, being the coefficient of  $q^2 \cdot 1$  (resp.  $q^2 j^* \sigma_1$ ) in  $j^* \sigma_1 \star j^* \sigma_{(5,5,3)}$  (resp.  $j^* \sigma_1 \star j^* \sigma_{(5,5,4)}$ ). They both equal  $\langle j^* \sigma_1, P.D.[\text{pt}], P.D.[\mathbb{P}^1] \rangle_2^Y$ , and hence equal 0 by Proposition 3.14 again. Hence, the first equality in the statement holds for  $p = 1$ .

For  $p \in \{2, 3\}$ , it follows from the associativity of quantum products that  $q^2$ -terms do not occur. For instance for  $p = 2$  and  $n = 7$ ,  $q^2$ -terms occur at most in  $j^* \sigma_{(1,1,0)} \star j^* \sigma_{(4,4,2)}$  and  $j^* \sigma_{(1,1,0)} \star j^* \sigma_{(4,4,3)}$ , by degree counting. By direct calculations, we have

$$\begin{aligned} j^* \sigma_1 \star j^* \sigma_{(4,4,1)} &= j^* \sigma_{(4,4,2)} + q j^* \sigma_{(3,1,0)}, \\ j^* \sigma_{(1,1,0)} \star j^* \sigma_{(4,4,1)} \star j^* \sigma_1 &= q(j^* \sigma_{(3,2,0)} + j^* \sigma_{(4,1,0)}) \star j^* \sigma_1 = q(j^* \sigma_{(3,2,0)} + j^* \sigma_{(4,1,0)}) \cup j^* \sigma_1 + q \cdot q, \\ q j^* \sigma_{(3,1,0)} \star j^* \sigma_{(1,1,0)} &= q j^* \sigma_{(3,1,0)} \cup j^* \sigma_{(1,1,0)} + q \cdot q. \end{aligned}$$

Hence,  $j^* \sigma_{(1,1,0)} \star j^* \sigma_{(4,4,2)} = q(j^* \sigma_{(3,2,0)} + j^* \sigma_{(4,1,0)}) \cup j^* \sigma_1 - q j^* \sigma_{(3,1,0)} \cup j^* \sigma_{(1,1,0)}$ , which does not contain  $q^2$ -term. The arguments for the remaining few cases are similar.  $\square$

**3.2.3. Applications.** Denote by  $p_{\alpha,V}(x)$  the characteristic polynomial of the induced operator  $\hat{\alpha}$  on an algebra  $V$  containing  $\alpha$ . Using the quantum Pieri rules, we have the following.

**Lemma 3.17.** *Let  $Y$  be a smooth hyperplane section of  $X = Gr(3, n)$ , where  $n \in \{7, 8\}$ .*

$$(1) \text{ For } n = 7, p_{\sigma_1^7, A^0(X)}(x) = p_{(j^* \sigma_1)^6, A^0(Y)}(x) = (128 - 13x + x^2)(1 - 57x - 289x^2 + x^3).$$

(2) For  $n = 8$ , we have  $p_{\sigma_1^8, \mathcal{A}^0(X)}(x) = (x-1)^3(1-1154x+x^2)(6561-34x+x^2)$  and  $p_{\sigma_1^6 \star \sigma_{1^2}, \mathcal{A}^0(X)}(x) = (1-x)(1+478x-x^2)(1+x^2)(2187+6x+x^2)$ . Moreover,

$$p_{(j^* \sigma_1)^7, \mathcal{A}^0(Y)}(x) = x^2 p_{\sigma_1^8, \mathcal{A}^0(X)}(x), \quad p_{(j^* \sigma_1)^5 \star j^* \sigma_{1^2}, \mathcal{A}^0(Y)}(x) = x^2 p_{\sigma_1^6 \star \sigma_{1^2}, \mathcal{A}^0(X)}(x)$$

*Proof.* For  $n = 7$ ,  $\mathcal{A}^0(X) = \mathbb{C}\{1, \sigma_{(4,3,0)}, \sigma_{(4,2,1)}, \sigma_{(3,3,1)}, \sigma_{(3,2,2)}\}$  and  $\mathcal{A}^0(Y) = \mathbb{C}\{1, j^* \sigma_{(4,1,1)}, j^* \sigma_{(3,3,0)}, j^* \sigma_{(3,2,1)}, j^* \sigma_{(2,2,2)}\}$  with  $j^* \sigma_{(4,2,0)} = 2j^* \sigma_{(4,1,1)} + j^* \sigma_{(3,3,0)} - j^* \sigma_{(3,2,1)} + j^* \sigma_{(2,2,2)}$ .

For  $n = 8$ ,  $\mathcal{A}^0(X) = \mathbb{C}\{1, \sigma_{(5,3,0)}, \sigma_{(5,2,1)}, \sigma_{(4,4,0)}, \sigma_{(4,3,1)}, \sigma_{(4,2,2)}, \sigma_{(3,3,2)}\}$  and  $\mathcal{A}^0(Y) = \mathbb{C}\{1, j^* \sigma_{(5,2,0)}, j^* \sigma_{(5,1,1)}, j^* \sigma_{(4,2,1)}, j^* \sigma_{(4,3,0)}, j^* \sigma_{(3,2,2)}, j^* \sigma_{(3,3,1)}, j^* \sigma_{(5,5,4)}, \beta\}$ , where  $\beta$  is a primitive class and we have  $j^* \sigma_1 \star \beta = 0$ .

Then the statements follow from direct calculations by using Proposition 3.16.  $\square$

**Proposition 3.18.** *Let  $Y$  be a smooth hyperplane section of  $X = \text{Gr}(3, n)$  with  $n \in \{7, 8\}$ . There is an isomorphism of algebras*

$$\mathcal{A}^0(X) \xrightarrow{\cong} \mathcal{A}_\perp^0(Y)$$

with  $\sigma_1^{r_X} \mapsto ((j^* \sigma_1)^{r_Y})_\perp$ .

*Proof.* Denote  $\alpha_X := \sigma_1^{r_X}$  and  $\alpha_Y := (j^* \sigma_1)^{r_Y}$ . For  $n = 7$ , by direct calculations,  $\{1, \alpha_X, \dots, \alpha_X^4\}$  (resp.  $\{1, \alpha_Y, \dots, \alpha_Y^4\}$ ) form an additive basis of  $\mathcal{A}^0(X)$  (resp.  $\mathcal{A}^0(Y)$ ). Therefore the minimal polynomial of  $\alpha_X$  (resp.  $\alpha_Y$ ) is given by the characteristic polynomial of it. Therefore, by Lemma 3.17,  $\mathcal{A}^0(X)$  (resp.  $\mathcal{A}^0(Y)$ ) is isomorphic to  $\mathbb{C}[x]/(p_{\alpha_X, \mathcal{A}^0(X)}(x))$  by sending  $\alpha_X$  (resp.  $\alpha_Y$ ) to  $x$ .

Now we consider  $n = 8$  and denote  $\tilde{\alpha}_X := \sigma_1^{r_X-2} \star \sigma_{1^2}$ ,  $\tilde{\alpha}_Y := (j^* \sigma_1)^{r_Y-2} \star j^* \sigma_{1^2}$ . By exactly the same argument for  $n = 7$ , we conclude that  $\mathcal{A}^0(X)$  (resp.  $\mathcal{A}_\perp^0(Y)$ ) is isomorphic to  $\mathbb{C}[x]/(p_{\tilde{\alpha}_X, \mathcal{A}^0(X)}(x))$  by sending  $\tilde{\alpha}_X$  (resp.  $(\tilde{\alpha}_Y)_\perp$ ) to  $x$ . By direct calculations, we see that  $\alpha_X$  and  $(\alpha_Y)_\perp$  have exactly the same linear expansion in terms of  $\{\tilde{\alpha}_X^i\}_i$  and  $\{(\tilde{\alpha}_Y^i)_\perp\}_i$  respectively. Hence  $\alpha_X$  is sent to  $(\alpha_Y)_\perp$  under the isomorphism.  $\square$

**Theorem 3.19.** *Let  $Y$  be a smooth hyperplane section of  $X = \text{Gr}(3, n)$  with  $n \in \{7, 8\}$ . Then both Proposition 1.10 and Conjecture 1.13 hold for  $Y$ .*

*Proof.* Conjecture 1.13 and Part (2) of Proposition 1.10 follow directly from Proposition 3.18 and Lemma 3.17 respectively.

Recall  $QH^*(\text{Gr}(3, n)) \cong \mathbb{C}[\sigma_1, \sigma_{1^2}, \sigma_{1^3}, q_X]/(\sigma_{n-2}, \sigma_{n-1}, \sigma_n - q_X)$ , where  $\sigma_j$  can be obtained by the relation (3). Denote by  $e_1 = j^* \sigma_1, e_2 = j^* \sigma_{1^2}$  and  $e_3 = j^* \sigma_{1^3}$ . Note  $j_q^*(\sigma_{n-2}) = 0$ , as  $n-2 < r_Y$ . To verify part (1) of Proposition 1.10, it suffices to show that

$$j_q^*(\sigma_{n-1}) = 0, \quad j_q^*(\sigma_n) = j_q^*(q_X) = q_Y e_1,$$

where  $j_q^*(\sigma_{n-1}) = h_6, j_q^*(\sigma_n) = h_7$  for  $n = 7$  (resp.  $j_q^*(\sigma_{n-1}) = h_7, j_q^*(\sigma_n) = h_8$  for  $n = 8$ ) with

$$\begin{aligned} h_6 &= e_1^6 - 5e_1^4 e_2 + 6e_1^2 e_2^2 + 4e_1^3 e_3 - e_2^3 - 6e_1 e_2 e_3 + e_3^2, \\ h_7 &= e_1^7 - 6e_1^5 e_2 + 10e_1^3 e_2^2 + 5e_1^4 e_3 - 4e_1 e_2^3 - 12e_1^2 e_2 e_3 + 3e_2^2 e_3 + 3e_1 e_3^2, \\ h_8 &= e_1^8 - 7e_1^6 e_2 + 15e_1^4 e_2^2 + 6e_1^5 e_3 - 10e_1^2 e_2^3 - 20e_1^3 e_2 e_3 + e_2^4 + 12e_1 e_2^2 e_3 + 6e_1^2 e_3^2 - 3e_2 e_3^2. \end{aligned}$$

Then we are done by direct calculations using Proposition 3.16.  $\square$

**3.3. Semisimplicity of  $QH^*(Y)$ .** Here we investigate the semisimplicity of  $QH^*(Y)$  for a smooth hyperplane section  $Y$  of  $X = Gr(k, n)$ . We will need to apply the following proposition in the case  $(k, n) = (3, 8)$ .

Let  $\pi : \mathcal{Y} \rightarrow U$  be a projective smooth morphism. Let  $p \in U$  be any point and denote  $\mathcal{Y}_p := \pi^{-1}(p)$  the fiber of  $\pi$  at  $p$ . There is a *monodromy action* (see e.g. [Voi, Chapter 3]) of the fundamental group  $\pi_1(U, p)$  on the classical cohomology  $H^*(\mathcal{Y}_p)$ .

**Proposition 3.20** ([LT98, Theorem 4.3], see also [Hu15, Corollary 3.2]). *The monodromy action of  $\pi_1(U, p)$  on  $H^*(\mathcal{Y}_p)$  naturally extends to an action on  $QH^*(\mathcal{Y}_p)$  and preserves the quantum product.*

For  $Y$  in  $Gr(3, 8)$ , we consider the family  $\pi|_U : \mathcal{Y}^\circ \rightarrow U$  obtained by restricting the natural projection

$$\pi : \mathcal{Y} = \{(V, \tau) \in Gr(3, 8) \times \Lambda^3(\mathbb{C}^8)^* \mid \tau|_V = 0\} \longrightarrow \Lambda^3(\mathbb{C}^8)^*.$$

to the unique Zariski open  $GL_8$ -orbit  $U$  of  $\Lambda^3(\mathbb{C}^8)^*$ . More precisely, the first projection  $\mathcal{Y} \rightarrow Gr(3, 8)$  is a vector bundle over  $Gr(3, 8)$  whose fiber at  $V$  is the space  $\ker[\Lambda^3(\mathbb{C}^8)^* \rightarrow \Lambda^3 V^*]$ . Therefore, the smooth hyperplane section  $Y$  of  $Gr(3, 8)$  can be realized as a fiber over any  $p \in U$ . Notice  $H_{\text{prim}}(Y) = \mathbb{C}\{\beta\}$ .

**Lemma 3.21.** *For  $Y$  in  $Gr(3, 8)$ , there exists  $\sigma \in \pi_1(U)$  such that  $\sigma\beta = -\beta$  under monodromy action.*

*Proof.* By Deligne invariant cycle theorem [Voi, Theorem 4.24], the image of the restriction

$$H^*(X, \mathbb{Q}) \cong H^*(\mathcal{Y}, \mathbb{Q}) \rightarrow H^*(Y, \mathbb{Q})$$

is the monodromy invariant subalgebra. In particular, the restriction of  $\sigma_1 \in H^2(X, \mathbb{Q})$  is invariant, so the primitive space  $\mathbb{Q}\beta$  is preserved. Since  $\beta$  is not in the image, there exists  $\sigma \in \pi_1(U)$  such that  $\beta \neq \sigma\beta \in \mathbb{Q}\beta$ . Note that  $\int_Y \beta^2 = \int_Y \sigma\beta^2 \neq 0$ , the only possibility is  $\sigma\beta = -\beta$ .  $\square$

**Theorem 3.22.** *Let  $Y$  be a smooth hyperplane section of  $Gr(k, n)$  where  $n \geq 2k$ . Then  $QH^*(Y)$  is generically semisimple if and only if one of the following holds:*

- (a)  $k = 1$ ; (b)  $k = 2$  and either  $n = 4$  or  $n$  is odd; (c)  $k = 3$  and  $n \in \{7, 8\}$ .

*Proof.* Let  $X = Gr(k, n)$  with  $n \geq 2k$ . For  $k = 1$ ,  $Y$  is a projective space and hence  $QH^*(Y)$  is generically semisimple. For  $X = Gr(2, 4)$ ,  $X$  is a quadric in  $\mathbb{P}^5$ , and hence  $Y$  is a quadric in  $\mathbb{P}^4$  with generically semisimple quantum cohomology. For  $k = 2$  and  $n = 2m$  with  $m > 2$ ,  $Y \cong SG(2, 2m)$  and  $QH^*(Y)$  is known to be non-semisimple [CP11]. For  $k = 2$  and  $n = 2m + 1$  with  $m > 1$ ,  $Y \cong SG(2, 2m + 1)$  is a quasi-homogeneous variety with generically semisimple quantum cohomology [Pec13, Per14]. For  $k > 3$  or ( $k = 3$  and  $n > 8$ ),  $QH^*(Y)$  is not semisimple by Theorem 3.8 and Proposition 1.1.

Now we assume  $k = 3$ . It remains to investigate the cases  $n \in \{6, 7, 8\}$ . For  $n = 6$ , we notice that  $\dim \mathcal{A}^{\bar{1}} = b_2(Y) + b_{12}(Y) = 1 + 2 \neq 4 = b_8(Y) = \dim \mathcal{A}^{-\bar{1}}$ . Hence,  $QH^*(Y)$  is not semisimple in this case.

Now we consider  $n \in \{7, 8\}$ . Notice  $\mathcal{A}_\perp^0(Y) = \mathcal{A}^0(Y)$  and  $Rad(Y) = 0$  if  $n = 7$ . It is well known that  $QH^*(X)$  is semisimple, so is the subalgebra  $\mathcal{A}^0(X)$ . By Proposition 3.18,  $\mathcal{A}_\perp^0(Y) \cong \mathcal{A}^0(X)$  is semisimple. By Lemma 3.17, the restriction of  $(j^*\sigma_1)^{r_Y}$  to the complement of  $Rad(Y)$  in  $QH^*(Y)$  is invertible, so is  $j^*\sigma_1$ . Hence,  $\mathcal{A}_\perp^0(Y)[j^*\sigma_1] \cong \mathcal{A}_\perp^0(Y)[y]/(y^{r_Y} - (j^*\sigma_1)^{r_Y})$  is semisimple by Lemma 2.2. This settles the case  $n = 7$ . Now we consider  $n = 8$ , for which  $\dim \mathcal{A}_\perp^0(Y)[j^*\sigma_1] = 49$ . We have  $QH^*(Y) = Rad(Y) \oplus \mathcal{A}_\perp^0(Y)[j^*\sigma_1]$ , where  $Rad(Y) = \mathbb{C}\{\beta, \gamma\}$  with

$$\gamma := 2\sigma_{(5,5,4)} - \sigma_{(3,2,2)} + \sigma_{(3,3,1)} + \sigma_{(4,2,1)} - 2\sigma_{(4,3,0)} - 3\sigma_{(5,1,1)} + 2\sigma_{(5,2,0)} \in H^*(Gr(3, 8)).$$

Then we are done by showing that  $Rad(Y)$  is semisimple. Indeed, since  $Rad(Y)$  is a two-dimensional unital algebra factor, the nilpotent elements of  $Rad(Y)$  form a subspace of dimension at most 1 and any nilpotent element has nilpotent index no more than 2. Thus  $j^*\gamma$  is not nilpotent, by noting  $\int_Y(j^*\gamma) \cup (j^*\gamma) = 6 \neq 0$ . If  $a\beta + bj^*\gamma$  is nilpotent, then so is  $-a\beta + bj^*\gamma$  by applying the monodromy action by Proposition 3.20 and Lemma 3.21. This forces  $b = 0$ . Then it follows from  $\int_Y \beta^2 \neq 0$  that  $a = 0$ .  $\square$

**3.4. Proof of Lemma 3.3.** We discuss all the possible  $n - i$ .

**Case I:**  $n - i \in \{1, 2\}$ . If  $n - i = 1$ , only the empty partition is a 1-core partition, but our assumption implies  $k(n - k) \geq k^2 > k \geq i$ . Thus it is impossible.

If  $n - i = 2$ , then the 2-core partition must be a staircase. The maximal 2-core partition in  $\mathcal{P}_{k,n}$  is  $(k, \dots, 2, 1)$  and has  $\frac{k(k+1)}{2}$  cells. By the assumption, we have

$$\frac{k(k+1)}{2} \geq k(n - k) - i.$$

Hence,  $\frac{k(k+1)}{2} > k(n - k) - n$ , implying  $\frac{k(k+1)}{2} + k^2 > (k - 1)n \geq 2(k - 1)k$  and consequently  $k < 5$ . Combining the above inequality with  $3 \leq k \leq \frac{n}{2}$ , we obtain  $(k, n, i) \in \{(3, 6, 4), (4, 8, 6)\}$ , see the first two diagrams in Example 3.4.

To proceed in the remaining cases, we use the bijection  $\mathcal{P}_{k,n} \rightarrow \binom{[n]}{k}$  defined by  $\lambda \mapsto A = \{a_1, \dots, a_k\}$  with  $a_i = \lambda_i + k - i + 1$  for all  $i$ . Under this identification, we have

- (i)  $|\lambda| \geq k(n - k) - i \iff \sum_{i=1}^k a_i \geq n + (n - 1) + \dots + (n - k + 1) - i$ ;
- (ii)  $\lambda$  is  $(n - i)$ -core  $\iff$  if  $a \in A$  and  $a - (n - i) \geq 1$  then  $a - (n - i) \in A$ .

Here the equivalence in (ii) follows from the fact that the multiset of hook lengths of  $\lambda$  is given by  $\bigcup_{i=1}^k (\{1, \dots, a_i - 1\} \setminus \bigcup_{j>i} \{a_i - a_j\})$ , see [St99, proof of Lemma 7.21.1].

**Case II:**  $n - i \geq 3$  and  $i < n - i$ . Let us consider

$$B = \{a \in A \mid n - i < a \leq n\}.$$

Since  $n - i < n < 2(n - i)$ , we have the disjoint union

$$A = B \cup B' \cup C$$

where  $B' = \{b - (n - i) \mid b \in B\}$  and  $C = A \setminus (B \cup B')$ . Assume  $|B| = b$ . Note that

$$k = |B| + |B'| + |C| \geq 2b, \quad k - b = |B'| + |C| \leq n - i$$

and

$$\#(A \cap \{1, \dots, i\}) \geq b, \quad \#(A \cap \{1, \dots, n - i\}) \geq k - b.$$

That is,

$$\begin{aligned} a_b &< \dots < a_2 < a_1 \leq n \\ a_{k-b} &< \dots < a_{b+2} < a_{b+1} \leq n - i \\ a_k &< \dots < a_{k-b+2} < a_{k-b+1} \leq i. \end{aligned}$$

This implies

$$\begin{aligned} a_1 + \dots + a_k &\leq n + (n - 1) + \dots + (n - b + 1) \\ &\quad + (n - i) + (n - i - 1) + \dots + (n - i + 1 - k + 2b) \\ &\quad + i + (i - 1) + \dots + (i + 1 - b) \\ &= n + (n - 1) + \dots + (n - k + 1) - (k - 2b)(i - b) - b(n - k + b - i). \end{aligned}$$

By the assumption (i), we have

$$0 \geq (k - 2b)(i - b) + b(n - k + b - i) - i, \quad (5)$$

implying  $b > 0$  (otherwise we would have  $0 \geq ki - i > 0$ ). If  $n - k + b - i \leq 1$ , then we have  $k > k - b \geq n - i - 1 > \frac{n}{2} - 1$ , contradicting  $k \leq \frac{n}{2}$ . Thus we can assume  $n - k + b - i \geq 2$ .

If  $k - 2b > 0$ , then it follows from (5) that  $0 \geq (i - b) + 2b - i = b > 0$ , a contradiction. It follows that  $k = 2b$ . Since  $k \geq 3$ , we have  $b \geq 2$  and  $n \geq 4b$ . Using  $i < n/2$ , we obtain

$$b(n - b) - (b + 1)i > b(n - b) - \frac{(b + 1)n}{2} \geq 2b(b - 1) - b^2 = b(b - 2) \geq 0.$$

This contradicts the inequality (5), where  $k = 2b$  in this case.

**Case III:**  $n - i \geq 3$  and  $i \geq n - i$ . Let us consider

$$M = \{a \in A \mid i < a \leq n\}, \quad m = |M|.$$

By (ii), we have an injective map  $\phi : M \rightarrow \{1, \dots, n - i\} \cap A$  such that  $\phi(a) = a - \ell(n - i)$  for some  $\ell \geq 0$ . Moreover since  $i \geq n - i$ , we have  $a - (n - i) \geq a - i > 0$ , hence  $\ell \geq 1$  and in particular  $a \neq \phi(a)$ . Now  $M \cap \phi(M) = \emptyset$ , we can conclude that

$$k \geq 2m, \quad \#(A \cap \{1, \dots, n - i\}) \geq m.$$

We first exclude  $m = 0$ . If  $M = \emptyset$ , then  $a_1 \leq i$  and we have  $a_k \leq n - i$  (otherwise, condition (ii) would imply  $a_k - (n - i) \in A$ , contradicting the minimality of  $a_k$ ). Since the elements of  $A$  are distinct, we obtain

$$\sum_{j=1}^k a_j \leq (k - 1)i - \frac{(k - 1)(k - 2)}{2} + (n - i) < \sum_{j=1}^k (n - j + 1) - i,$$

where the second inequality follows from  $k \geq 3$  and  $n - i \geq 3$ . This contradicts condition (i). Hence  $m \geq 1$ .

**Subcase III.1.** When  $|M| = 1$ , i.e.  $M = \{a_1\}$ , we have

$$a_k \leq n - i, \quad a_{k-1} < \dots < a_3 < a_2 \leq i, \quad a_1 \leq n.$$

Then

$$\begin{aligned} a_1 + \dots + a_k &\leq n + i + (i - 1) + \dots + (i - k + 3) + (n - i) \\ &= n + (n - 1) + (n - 2) + \dots + (n - k + 1) - (k - 2)(n - i - 1) - (i - k + 1). \end{aligned}$$

By assumption (i) and noting  $n - i \geq 3$ , we have

$$0 \geq (k - 2)(n - i - 1) - k + 1 \geq 2(k - 2) - k + 1 = k - 3 \geq 0.$$

So  $k = 3$ , and all equalities should be achieved at each step, i.e.

$$n - i = 3, \quad a_3 = n - i, \quad a_2 = i, \quad a_1 = n.$$

Then  $a_3 = 3$ . Since  $a_2 > a_3 = n - i$ , by (ii), this forces  $a_2 - (n - i) = a_3$ , i.e.  $i = 6$  and  $n = 9$ . So the only case is  $(k, n, i) = (3, 9, 6)$ , see the third diagram in Example 3.4.

**Subcase III.2.** When  $|M| = m \geq 2$ , we have

$$a_k < a_{k-1} < \cdots < a_{k-m+1} \leq n-i \quad \text{and} \quad a_{k-m} < \cdots < a_{m+2} < a_{m+1} \leq i.$$

Then

$$\begin{aligned} a_1 + \cdots + a_k &\leq n + (n-1) + \cdots + (n-m+1) \\ &\quad + i + (i-1) + \cdots + (i-(k-2m)+1) \\ &\quad + (n-i) + (n-i-1) + \cdots + (n-i-m+1) \\ &= n + (n-1) + \cdots + (n-k+1) - (k-2m)(n-m-i) - m(i+m-k). \end{aligned}$$

By (i), we have

$$0 \geq (k-2m)(n-m-i) + m(i+m-k) - i \quad (6)$$

If  $n-m-i=0$ , then it follows from (6) that

$$0 = m(n-k) - i \geq 2(n-k) - i \geq n-i > 0,$$

a contradiction. So  $n-m-i > 0$ , and by (6) we have

$$0 \geq (k-2m) + 2(i+m-k) - i = i-k \geq n/2 - n/2 \geq 0.$$

This shows  $i=k$  and all equalities should be achieved at each step. Then  $i=k=n/2$ ,  $m=2$  (since  $i+m-k=m>0$ ) and  $k-2m=0$  (since if  $n-m-i=i-2=1$ , we can solve  $(k,n,i)=(3,6,3)$  contradicting  $k \geq 2m$ ). So the only possibility is  $(k,n,i)=(4,8,4)$  with  $a_1=n=8$ ,  $a_2=a_1-1=7$ ,  $a_3=n-i=4$  and  $a_4=a_3-1=3$ , see the last diagram in Example 3.4. Now the proof of Lemma 3.3 is completed.

#### 4. SMOOTH HYPERPLANE SECTIONS OF GENERALIZED GRASSMANNIANS

In this section, we consider a generalized Grassmannian  $X = G/P_k$ . There is an embedding  $X \hookrightarrow \mathbb{P}(V_{\omega_k})$ , where  $\omega_k$  is the  $k$ -th fundamental weight and  $V_{\omega_k}$  denotes the corresponding fundamental representation of  $G$ . Let  $Y$  be a smooth hyperplane section of  $X$ , given by the intersection of a general hyperplane in  $\mathbb{P}(V_{\omega_k})$  with  $X$ . We begin with some (co)adjoint Grassmannians in Table 1.

**4.1. Hyperplane sections of  $C_n/P_2$ .** The coadjoint Grassmannian  $G/P_2$  of type  $C_n$  is the symplectic Grassmannian  $SG(2, 2n) = \{V \leq \mathbb{C}^{2n} \mid \dim V = 2, \omega(V, V) = 0\}$ , where  $\omega$  is a symplectic form on  $\mathbb{C}^{2n}$  and  $n \geq 3$ . It can be realized as a smooth hyperplane section of  $Gr(2, 2n)$ . In particular,  $r_X = 2n-1$  and  $\dim X = 4n-5$ . The following theorem verifies [BP22, Conjecture 1.10 (2)] for type  $C$ , and provides a new proof of the non-semisimplicity of  $QH^*(SG(2, 2n))$  as well.

**Theorem 4.1.** *Let  $n \geq 3$ . For a smooth hyperplane section  $Y$  of  $X = SG(2, 2n)$ , both  $QH^*(X)$  and  $QH^*(Y)$  are non-semisimple.*

*Proof.* Note that the odd Betti numbers  $b_{2j+1}(X) = \dim H^{2j+1}(X)$  all vanish. Since the two-step flag variety  $SFl(1, 2; 2n)$  of type  $C$  is a  $\mathbb{P}^1$ -bundle over  $SG(2, 2n)$ , and also a  $\mathbb{P}^{2n-3}$ -bundle over  $\mathbb{P}^{2n-1}$ , we have the Poincaré polynomial (with respect to complex degree)

$$\begin{aligned} p_X(t) &= \frac{1-t^{2n}}{1-t} \frac{1-t^{2n-2}}{1-t} \Big/ (1+t) \\ &= \frac{(1-t^{2n})(1-t^{2n-2})}{(1-t)(1-t^2)} = (1+t+\cdots+t^{2n-1})(1+t^2+\cdots+t^{2n-4}). \end{aligned}$$

Thus for  $0 \leq i \leq 2n-3$ , we have  $b_{2i}(X) = \begin{cases} \frac{i+2}{2}, & \text{if } i \text{ is even,} \\ \frac{i+1}{2}, & \text{if } i \text{ is odd.} \end{cases}$

Since  $r_X = 2n - 1$  and  $\dim X = 4n - 5$ , we have  $\dim \mathcal{A}^{-\bar{2}}(X) = b_{2(2n-3)}(X) = \frac{2n-3+1}{2} = n - 1$  and  $\dim \mathcal{A}^{\bar{2}}(X) = b_{2,2}(X) + b_{2(2n+1)}(X) = b_{2,2}(X) + b_{2(2n-6)}(X) = \frac{2+2}{2} + \frac{2n-6+2}{2} = n \neq \dim \mathcal{A}^{-\bar{2}}$ . Hence,  $QH^*(X)$  is not semisimple by Theorem 1.2 (2).

Note  $r_Y = 2n - 2$  and  $\dim Y = 4n - 6$ . By [BP22, Theorem 1.2 and Lemma 3.1],  $Y$  is invariant under the natural action of a maximal torus of  $G$ , and its torus-fixed loci coincide with the torus-fixed loci of  $X$ . Therefore  $X$  and  $Y$  have the same Euler number. It follows that  $b_{2(2n-3)}(Y) = b_{2(2n-3)}(X) + b_{2(2n-2)}(X) = 2n - 2$  and  $b_{2i}(Y) = b_{2i}(X)$  for  $0 \leq i \leq 2n - 4$ . Thus  $\dim \mathcal{A}^1(Y) = b_2(Y) + b_{2(2n-1)}(Y) = b_2(Y) + b_{2(2n-5)}(Y) = 1 + \frac{2n-5+1}{2} = n - 1 \neq b_{2(2n-3)}(Y) = \dim \mathcal{A}^{-\bar{1}}$ . Hence,  $QH^*(Y)$  is not semisimple by Theorem 1.2 (2).  $\square$

**Remark 4.2.** The same argument also works for  $Y$  in  $F_4/P_4$ .

**4.2. Hyperplane sections of  $D_n/P_2$ .** For  $n \geq 4$ , the (co)adjoint variety  $D_n/P_2$  is the Grassmannian of isotropic planes in  $\mathbb{C}^{2n}$  with respect to a non-degenerate symmetric bilinear form. The argument for the type  $C$  case does not work for the type  $D$  case. Here we use the monodromy technique, which works not only in the type  $D$  case.

Let  $G/P$  denote a coadjoint Grassmannian of type  $D$  or  $E$ . Consider the natural projection

$$\pi : \mathcal{Y} := \{(gP, v) \in G/P \times \mathfrak{g} \mid v \in gL\} \longrightarrow \mathfrak{g}$$

where  $L \subset \mathfrak{g}$  is the span of root vectors and Cartan subalgebra except the lowest root space. Denote by  $\kappa$  the Killing form of  $\mathfrak{g}$ . Then the fiber  $\mathcal{Y}_x$  at  $x \in \mathfrak{g}$  can be identified with the hyperplane section defined by the linear equation  $\kappa(x, -)$  over  $\mathfrak{g}$  under the Plücker embedding  $G/P \hookrightarrow \mathbb{P}(\mathfrak{g})$ . Moreover, for any  $x$  in the set  $U = \mathfrak{g}^{rs}$  of regular semisimple elements of  $\mathfrak{g}$ , the fiber  $\mathcal{Y}_x$  is smooth, being a smooth hyperplane section  $Y$  of  $X$ . It is well-known that the fundamental group of  $\mathfrak{g}^{rs}$  is the braid group of the Weyl group  $W$  of  $G$ .

**Lemma 4.3.** *The monodromy action factors through the Weyl group  $W$ , and as a  $W$ -representation,*

$$H^*(Y) = H^*(Y)_{\text{inv}} \oplus H^*(Y)_{\text{std}}$$

with  $H^*(Y)_{\text{std}} \cong \chi_{\text{std}}$  the standard (i.e. reflection) representation of  $W$ .

*Proof.* We will use Springer correspondence in terms of perverse sheaves and Fourier transformation (see e.g. [Gin98]). Let  $\mathbb{C}_Z$  be the constant sheaf of a variety  $Z$ . Denote by  $IC(Z, \chi)$  the intersection complex of the local system  $\chi$  over  $Z$  and we omit  $\chi$  if  $\chi = \mathbb{C}_Z$ . By abuse of notation, throughout this proof we write  $f_*$  for  $Rf_*$ , suppress cohomological shifts, and regard multiplicity spaces as ungraded, with multiplicities summed over all shifts. By the Decomposition Theorem [BBD], we have

$$\pi_* \mathbb{C}_{\mathcal{Y}} = \bigoplus_{\chi} IC(\mathfrak{g}^{rs}, \chi) \otimes V_{\chi} \oplus (\text{perverse sheaves with smaller support}) \quad (7)$$

where the sum goes over irreducible representations of  $\pi_1(\mathfrak{g}^{rs})$ , and  $V_{\chi}$  is the multiplicity space of  $\chi$  in  $H^*(Y)$ . Let us consider the analogue  $\rho$  of Springer resolutions

$$\rho : \mathcal{Y}^{\perp} := \{(gP, v) \in G/P \times \mathfrak{g} \mid v \in \text{span}(ge_{\theta})\} \longrightarrow \mathfrak{g}$$

where  $\theta$  is the highest root and  $e_{\theta}$  is the corresponding root vector. The image of  $\rho$  contains only two nilpotent orbits, the zero orbit  $\mathbb{O}_0 = \{0\}$  and the minimal  $G$ -orbit  $\mathbb{O}_{\min} = Ge_{\theta}$ . Since  $\rho$  is one-to-one over  $\mathbb{O}_{\min}$ , by the Decomposition Theorem [BBD] again, we have

$$\rho_* \mathbb{C}_{\mathcal{Y}^{\perp}} = IC(\mathbb{O}_{\min}) \oplus IC(\mathbb{O}_0) \otimes V$$

for some coefficient space  $V$ . Let us denote by  $\mathcal{F}$  the Fourier transformation between  $\mathfrak{g}$  and  $\mathfrak{g}^* \simeq \mathfrak{g}$ , where the isomorphism is given by the Killing form. Since  $\mathfrak{g}$  is simply-laced, by the

Springer correspondence, we have

$$\mathcal{F}(IC(\mathbb{O}_0)) = IC(\mathfrak{g}^{rs}), \quad \mathcal{F}(IC(\mathbb{O}_{\min})) = IC(\mathfrak{g}^{rs}, \chi_{\text{std}}),$$

see for example [Ju08]. Since Fourier transformation is functorial, similarly to [Gin98, Claim 8.4], we have  $\mathcal{F}(\rho_* \mathbb{C}_{Y^\perp}) = \pi_* \mathbb{C}_Y$ . This implies

$$\pi_* \mathbb{C}_Y = IC(\mathfrak{g}^{rs}, \chi_{\text{std}}) \oplus IC(\mathfrak{g}^{rs}) \otimes V.$$

Comparing with (7), the multiplicity space  $V_\chi = 0$  unless  $\chi = \chi_{\text{std}}$  or  $\chi_{\text{inv}}$  and  $\dim V_\chi = 1$  for  $\chi = \chi_{\text{std}}$ . Hence, the decomposition in the statement follows.  $\square$

**Lemma 4.4.** *For any finite irreducible Coxeter group  $W$  not of type  $A$ , we have*

$$\dim(\text{Sym}^2 \chi_{\text{std}})^W = 1, \quad \dim(\text{Sym}^3 \chi_{\text{std}})^W = 0.$$

*Proof.* By a theorem of Chevalley, for example [Hum, Section 3.5], we have an algebra isomorphism to the polynomial ring  $\bigoplus_{d \geq 0} (\text{Sym}^d \chi_{\text{std}})^W \cong \mathbb{C}[p_1, p_2, \dots, p_n]$ , with  $p_i$  homogeneous of degree  $d_i$ . The numbers  $d_1 \leq d_2 \leq \dots \leq d_n$  are classified, and the statement follows from the fact  $d_1 = 2$  and  $d_2 \geq 4$  except type  $A$ , see [Hum, Section 3.7].  $\square$

**Theorem 4.5.** *For a smooth hyperplane section  $Y$  of a coadjoint Grassmannian  $X$  of type  $D$ , the quantum cohomology  $QH^*(Y)$  is non-semisimple.*

*Proof.* By Proposition 3.20, the symmetric cubic form

$$(\gamma_1 \star_{q_Y=1} \gamma_2, \gamma_3)_Y \in \mathbb{C}, \quad \gamma_1, \gamma_2, \gamma_3 \in H^\bullet(Y)$$

is  $W$ -invariant. By Lemma 4.4, when  $\gamma_1, \gamma_2, \gamma_3 \in H^*(Y)_{\text{std}}$ , we have  $(\gamma_1 \star_{q_Y=1} \gamma_2, \gamma_3)_Y = 0$ . Since the Poincaré pairing is non-degenerate and  $W$ -invariant, so is its restriction to  $H^*(Y)_{\text{std}}$  by Lemma 4.4. So we have  $\gamma_1 \star_{q_Y=1} \gamma_2 \in H^*(Y)_{\text{inv}}$ .

If  $\gamma_1, \gamma_2 \in H^*(Y)_{\text{inv}}$  and  $\gamma_3 \in H^*(Y)_{\text{std}}$ , then we have  $(\gamma_1 \star_{q_Y=1} \gamma_3, \gamma_2)_Y = (\gamma_1 \star_{q_Y=1} \gamma_2, \gamma_3)_Y = 0$ . Again, since the Poincaré pairing is non-degenerate and  $W$ -invariant, so is its restriction to  $H^*(Y)_{\text{inv}}$ . So we have  $\gamma_1 \star_{q_Y=1} \gamma_3 \in H^*(Y)_{\text{std}}$ .

Notice that  $\dim Y = 2r_Y$ , and  $r_Y$  is even. Moreover,

$$H^{2i}(Y) = H^{2i}(Y)_{\text{inv}} \text{ unless } i = r_Y.$$

By the discussion above, we can construct a subalgebra  $\mathcal{B}$  of  $\mathcal{A}$  with a  $\mathbb{Z}_2$ -grading by

$$\mathcal{B}^{\bar{0}} = H^0(Y) \oplus H^{2r_Y}(Y)_{\text{inv}} \oplus H^{4r_Y}(Y),$$

$$\mathcal{B}^{\bar{1}} = H^{r_Y}(Y) \oplus H^{3r_Y}(Y) \oplus H^{2r_Y}(Y)_{\text{std}}.$$

Since

$$\dim H^{2r_Y}(Y)_{\text{std}} = \dim H^{2r_Y}(Y)_{\text{inv}} = \text{rank } G,$$

$$\dim H^0(Y) = \dim H^{4r_Y}(Y) = 1,$$

$$\dim H^{r_Y}(Y) = \dim H^{3r_Y}(Y) > 1,$$

we have  $\dim \mathcal{B}^{\bar{0}} < \dim \mathcal{B}^{\bar{1}}$ . Hence,  $\mathcal{B}$  contains a nonzero nilpotent element by Lemma 2.1.  $\square$

**4.3. Proof of Proposition 1.8.** By Proposition 1.1, it suffices to determine the semisimplicity for those smooth hyperplane sections  $Y$  whose cohomology is of Hodge-Tate type. By the hypothesis, we have assumed that Conjecture 1.7 holds. Each of its cases has been distinguished accordingly in the statement to prove. Indeed, case (a) follows from Theorem 1.3 (2). Case (b) is a combination of Theorem 1.5 with the fact that  $C_n/P_1 \cong \mathbb{P}^{2n-1}$  and  $B_n/P_1$  (resp.  $D_n/P_1$ ) is an odd (resp. even) dimensional smooth quadric, so that their smooth hyperplane sections

have generic semisimple quantum cohomology, for being projective spaces or smooth quadrics as well.

It remains to discuss case (c) in Conjecture 1.7. By direct computations using the Betti numbers  $b_j(X)$  of  $X$ , we obtain the following table. For instance for  $X = D_6/P_6$  with  $i = 4$ , we have  $\dim Y = \dim X - 1 = 14$ ,  $r_Y = r_X - 1 = 9$  and  $\dim \mathcal{A}^i(Y) = b_8(Y) + b_{26}(Y) = b_8(Y) + b_2(Y) = b_8(X) + b_2(X) = 2 + 1 = 3$ , where the second equality uses the Poincaré duality and the third equality uses the Lefschetz theorem. Here  $\dim X, r_X$  and  $b_j(X)$  are known and for instance can be directly read off from [Bel]. Thus by Theorem 1.2 (2), all these four cases have non-semisimple small quantum cohomology.

| $X$       | $\dim Y$ | $r_Y$ | $i$ | $\dim \mathcal{A}^i(Y)$  | $\dim \mathcal{A}^{-i}(Y)$ |
|-----------|----------|-------|-----|--------------------------|----------------------------|
| $D_6/P_6$ | 14       | 9     | 4   | $b_8(X) + b_2(X) = 3$    | $b_{10}(X) + b_0(X) = 4$   |
| $D_7/P_7$ | 20       | 11    | 2   | $b_4(X) + b_{14}(X) = 5$ | $b_{18}(X) + b_0(X) = 6$   |
| $E_6/P_1$ | 15       | 11    | 4   | $b_8(X) + b_0(X) = 3$    | $b_{14}(X) = 2$            |
| $E_7/P_7$ | 26       | 17    | 8   | $b_{16}(X) + b_2(X) = 3$ | $b_{18}(X) + b_0(X) = 4$   |

Note that  $D_4/P_4$  is a smooth 6-dimensional smooth quadric. Hence its general hyperplane section  $Y$  is isomorphic to a 5-dimensional smooth quadric and has generic semisimple small quantum cohomology. The Schubert classes for  $C_3/P_3$  and  $D_5/P_5 \cong B_4/P_4$  can both be indexed by strict partitions. Their general hyperplane sections  $Y$  satisfy  $\dim Y = 5, r_Y = 3$  and  $\dim Y = 9, r_Y = 7$  respectively. Consequently, the algebras  $\mathcal{A}^0(Y)$  are both 2-dimensional, generated by  $\{1, j^* \sigma_3\}$  and  $\{1, j^* \sigma_{43}\}$  respectively. Proposition 3.13 applies to the quantum multiplication by  $h_Y$  in both cases, with the same proof.

For  $X = C_3/P_3$ , Proposition 3.14 also gives  $\langle h_Y, P.D.[pt], P.D.[pt] \rangle_2^Y = 0$ , by similar span arguments, which is the only possible quantum term of degree  $> 1$  in the multiplication by  $h_Y$ . By direct calculations using the quantum Chevalley formula [FW04], we obtain

$$h_Y^3 = 8j^* \sigma_3 + 4q_Y, \quad h_Y^3 \star j^* \sigma_3 = 20q_Y j^* \sigma_3 + 8q_Y^2.$$

For  $X = D_5/P_5 \simeq B_4/P_4$ , there are no quantum terms of degree  $> 1$  in the multiplication by  $h_Y$ , simply by the dimension constraint. By direct calculations, we have

$$h_Y^7 = 12j^* \sigma_{43} + 5q_Y, \quad h_Y^7 \star j^* \sigma_{43} = 29q_Y j^* \sigma_{43} + 12q_Y^2.$$

At  $q_Y = 1$ , these identities show that  $h_Y^{r_Y}$  generates  $\mathcal{A}^0(Y)$ , with respective minimal polynomials

$$t^2 - 24t + 16, \quad t^2 - 34t + 1.$$

Both have distinct nonzero roots, so  $\mathcal{A}^0(Y) \simeq \mathbb{C} \oplus \mathbb{C}$  and  $h_Y$  is invertible. Therefore  $QH^*(Y)$  is generically semisimple by Lemma 2.2. This completes the proof of Proposition 1.8.

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## APPENDIX A. GENERIC NON-SEMISIMPLICITY OF SMALL QUANTUM COHOMOLOGY OF KRONECKER MODULI

BY PIETER BELMANS AND MARKUS REINEKE

In this appendix, we explain how the semisimplicity obstruction from Theorem 1.2 is strong enough to show that certain Kronecker quiver moduli have non-semisimple small quantum cohomology. Quiver moduli are moduli spaces of (semi)stable representations of a quiver, and they admit many strong results, see [9] for a survey. We will focus on the case where the quiver is the  $m$ -Kronecker quiver with  $m$  arrows:

$$\begin{array}{c} \circ \quad \quad \quad \circ \\ \quad \quad \quad \vdots \\ \quad \quad \quad \circ \end{array} \quad (8)$$

where we will throughout assume that  $m \geq 3$ . We will write  $(d, e)$  for the dimension vector, where  $d, e \geq 1$ . We will assume that  $\gcd(d, e) = 1$ . As there are only 2 vertices, there is a unique non-trivial stability function, for which stability coincides with semistability. We denote the moduli space of (semi)stable representations of the  $m$ -Kronecker quiver with dimension vector  $(d, e)$  by  $M_{(d,e)}^m$ .

The important properties for us are that

- (1)  $M_{(1,e)}^m \cong \text{Gr}(e, m)$  and  $M_{(d,1)}^m \cong \text{Gr}(d, m)$ ;
- (2)  $M_{(d,e)}^m$  is Hodge–Tate [7, Theorem 3];
- (3)  $M_{(d,e)}^m$  is a smooth projective rational Fano variety of dimension  $mde - d^2 - e^2 + 1$ , Picard rank 1, and index  $m$ , see [5, Corollary 5.2].

In addition to generic semisimplicity of small quantum cohomology of Kronecker moduli which happen to be Grassmannians, we have the following important result [8, Theorem 5.23], which follows from an explicit presentation of the small quantum cohomology.

**Proposition A.1** (Meng). The small quantum cohomology  $\text{QH}(M_{(2,3)}^3)$  is generically semisimple.

Given that a version of Schofield’s conjecture predicts that the derived category of a quiver moduli space (and thus  $M_{(d,e)}^m$  in particular) admits a full exceptional collection [2], and Dubrovin’s conjecture then predicts that its big quantum cohomology is generically semisimple, one might (optimistically) wonder whether the small quantum cohomology is already semisimple, as is the case in all known examples. However, the following shows that this is *not* the case.

**Theorem A.2.** Assume that  $m \geq 5$  is odd. Then the small quantum cohomology  $\text{QH}(M_{(2,m)}^m)$  is not generically semisimple.

We will obtain this as a consequence of Theorem 1.2, and thus we can conclude that this result is also strong enough to show that certain Kronecker moduli spaces have generically non-semisimple small quantum cohomology, similar to the application in Theorem 2.3 for exceptional generalized Grassmannians.

**Two lemmas.** Given a polynomial with integer coefficients

$$P(q) = \sum_{i=0}^N a_i q^i \in \mathbb{Z}[q] \quad (9)$$

and an integer  $n \geq 1$ , we define

$$\tilde{b}_j := \sum_{k \equiv j \pmod n} a_k \quad (10)$$

for  $j \in \mathbb{Z}/n\mathbb{Z}$ . Later on, the polynomial will be the even Poincaré polynomial of a smooth projective variety, and the  $\tilde{b}_j$  will be the index-periodic even Betti numbers.

**Lemma A.3.** *Assume that  $\tilde{b}_j \leq \tilde{b}_{jk}$  for all  $j, k \in \mathbb{Z}/n\mathbb{Z}$ . Then  $P(e^{2\pi i/n}) = \sum_{d|n} \mu(n/d) \tilde{b}_d$ , in particular, it is an integer.*

*Proof.* By assumption we have that  $\tilde{b}_j = \tilde{b}_{jk}$  if  $k \in (\mathbb{Z}/n\mathbb{Z})^\times$ . The orbits of  $(\mathbb{Z}/n\mathbb{Z})^\times$  on  $\mathbb{Z}/n\mathbb{Z}$  correspond to the divisors of  $n$ , thus we can write

$$P(e^{2\pi i/n}) = \sum_{j=1}^n \tilde{b}_j e^{2\pi i j/n} = \sum_{d|n} \tilde{b}_d \sum_{(j,n)=d} e^{2\pi i j/n} = \sum_{d|n} \tilde{b}_d \sum_{(j,n/d)=1} e^{2\pi i j/(n/d)} = \sum_{d|n} \mu(n/d) \tilde{b}_d \quad (11)$$

where the last step uses standard properties of Ramanujam sums [6, Theorem 271], and the latter expression is indeed an integer.  $\square$

We also recall the  $q$ -Lucas theorem on evaluations of Gaussian binomial coefficients  $\begin{bmatrix} a \\ b \end{bmatrix}_q$  at roots of unity [4, Proposition 2.2].

**Lemma A.4.** *For  $a, b \in \mathbb{N}$  and  $0 \leq r, s < n$ , we have*

$$\begin{bmatrix} an + r \\ bn + s \end{bmatrix}_{e^{2\pi i/n}} = \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} r \\ s \end{bmatrix}_{e^{2\pi i/n}}. \quad (12)$$

**Poincaré polynomials of Kronecker moduli.** Now we consider the Kronecker moduli spaces  $M_{(2,e)}^m$  for  $e$  odd and  $m, e \geq 3$ . There is a duality

$$M_{(2,e)}^m \cong M_{(2,2m-e)}^m, \quad (13)$$

(with both being empty when  $e > 2m$ ) see, e.g., [1, Corollary 4.1], thus we can assume  $e \leq m$ .

Their Poincaré polynomials admit a closed expression using a resolved Harder-Narasimhan type recursion [10, Section 7].

**Lemma A.5.** *Let  $e, m \geq 3$ , and assume that  $e \leq m$ , and  $e$  is odd. The Poincaré polynomial  $P_{(2,e)}^m(q)$  of  $M_{(2,e)}^m$  is given by*

$$P_{(2,e)}^m(q) = \frac{1}{q(q-1)} \left( \frac{1}{q+1} \begin{bmatrix} 2m \\ e \end{bmatrix}_q - \sum_{k=0}^{(e-1)/2} q^{(m-e+k)k} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} m \\ e-k \end{bmatrix}_q \right). \quad (14)$$

We can now derive the main observation.

**Lemma A.6.** *Let  $e, m \geq 3$ , and assume that  $e \leq m$ , and  $e$  is odd. Then  $P_{(2,e)}^m(e^{2\pi i/m})$  is strictly imaginary if and only if  $e = m \geq 5$ .*

*Proof.* If  $e < m$ , then all Gaussian binomial coefficients in Lemma A.5 evaluate to zero at  $q = e^{2\pi i/m}$  by Lemma A.4, because we always have  $r = 0$  and  $s = e \geq 1$ , and thus  $P_{(2,e)}^m(e^{2\pi i/m}) = 0$ .

If  $e = m$ , we find

$$\begin{aligned} P_{(2,m)}^m(e^{2\pi i/m}) &= \frac{1}{e^{2\pi i/m}(e^{2\pi i/m} - 1)} \left( \frac{1}{e^{2\pi i/m} + 1} \cdot 2 - 1 \right) \\ &= -\frac{1}{e^{2\pi i/m}(e^{2\pi i/m} + 1)}, \end{aligned} \tag{15}$$

as for  $k = 0$  in the summation in Lemma A.5 we have  $\begin{bmatrix} m \\ 0 \end{bmatrix}_{e^{2\pi i/m}} = \begin{bmatrix} m \\ m \end{bmatrix}_{e^{2\pi i/m}} = 1$  by definition, whilst all other Gaussian binomial coefficients in the sum evaluate to zero at  $q = e^{2\pi i/m}$  by Lemma A.4 because for  $k \geq 1$  we will have  $r = 0$  and  $s \geq 1$ , whereas  $\begin{bmatrix} 2m \\ m \end{bmatrix}_{e^{2\pi i/m}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{e^{2\pi i/m}} = 2$  by another application of Lemma A.4. This is strictly imaginary for  $m \geq 5$ .  $\square$

*Proof of Theorem A.2.* By Lemma A.6 we have that  $P_{(2,m)}^m(e^{2\pi i/m})$  is strictly imaginary (and thus nonzero). By Lemma A.3, this invalidates the criterion of Theorem 1.2, and thus the small quantum cohomology of  $M_{(2,m)}^m$  is not generically semisimple.  $\square$

Let us illustrate what happens in the smallest example, where  $m = 5$ .

**Example A.7.** The Kronecker moduli space  $M_{(2,5)}^5$  is a 22-dimensional smooth projective Fano variety, of index 5. Using the algorithm for Betti numbers of [10, Corollary 6.9], and implemented in [3], or by working out the special case from Lemma A.5, we obtain that the even Betti numbers are

$$1, 1, 3, 4, 8, 11, 17, 22, 30, 35, 41, 41, 41, 35, 30, 22, 17, 11, 8, 4, 3, 1, 1. \tag{16}$$

Hence, the index-periodic even Betti numbers  $\tilde{b}_0, \dots, \tilde{b}_4$  are

$$78, 77, 78, 77, 77. \tag{17}$$

We see that  $\tilde{b}_2 > \tilde{b}_4$ , hence by Theorem 1.2 we obtain that  $\text{QH}(M_{(2,5)}^5)$  cannot be generically semisimple.

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