

QUANTUM SCHUBERT CALCULUS FOR SMOOTH SCHUBERT DIVISORS OF $F\ell_n$

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ABSTRACT. We propose to study the quantum Schubert calculus for Schubert varieties, and investigate the smooth Schubert divisors X of the complete flag variety $F\ell_n$. We provide a Borel-type ring presentation of the quantum cohomology of X . We derive the quantum Monk-Chevalley formula for X by geometric arguments. We also show that the quantum Schubert polynomials for X are the same as that for $F\ell_n$ introduced by Fomin, Gelfand and Postnikov.

1. INTRODUCTION

In a forthcoming paper [LRY], the first and fourth authors, together with Rietsch, initiate a Peterson program for all Schubert subvarieties $X_{w,P}$ of flag varieties G/P , including mirror symmetry for Schubert varieties as one remarkable ingredient. They construct a uniform Landau-Ginzburg model mirror to $X_{w,P}$, generalizing the constructions in [Rie08, RW24], and conjecture that its Jacobi ring is isomorphic to the small quantum cohomology ring of $X_{w,P}$ whenever $X_{w,P}$ is smooth and Fano. This strongly motivates the study of quantum cohomology for smooth Schubert varieties. A natural starting point is provided by smooth Schubert varieties in the complete flag variety $F\ell_n := \{V_1 \leq \cdots \leq V_{n-1} \leq \mathbb{C}^n \mid \dim V_i = i, \forall 1 \leq i < n\}$, which is the quotient of $G = SL(n, \mathbb{C})$ by the Borel subgroup of upper triangular matrices in G . Let F_1 be a fixed one-dimensional vector subspace of $\mathbb{C}^n$. In this paper, we focus on the Schubert divisor

$$X = \{V_\bullet \in F\ell_n \mid F_1 \leq V_{n-1}\},$$

and study it systematically from the viewpoint of quantum Schubert calculus. Although X looks rather special and very close to the flag variety $F\ell_n$, we emphasize that our main results for X already suggest features of what a more general theory for smooth Fano Schubert varieties would look like; see the discussion at the end of the introduction.

We briefly recall the framework of Schubert calculus. Schubert problems, which count the number of geometric objects with given geometric constraints, are fundamental to enumerative geometry. Here the central objects are flag varieties G/P together with their Schubert subvarieties. The *classical Schubert calculus*, in modern language, is about the study of the integral cohomology ring $H^*(G/P, \mathbb{Z})$. The Schubert classes σ^u of the Schubert varieties form an additive basis of $H^*(G/P, \mathbb{Z})$. A satisfactory description of this ring typically consists of the following three parts:

- (1) A ring presentation of the form $H^*(G/P, \mathbb{Z}) = \mathbb{Z}[\mathbf{x}]/I$.

- (2) A (manifestly positive) formula of the Schubert constants $c_{u,v}^w$ in the cup product $\sigma^u \cup \sigma^v = \sum_w c_{u,v}^w \sigma^w$.
- (3) A Schubert polynomial $\mathfrak{S}_u(\mathbf{x}) \in \mathbb{Z}[\mathbf{x}]$ that represents the Schubert class σ^u in the aforementioned ring presentation $\mathbb{Z}[\mathbf{x}]/I$.

We refer to the very nice article [BGP25] and the references therein for the progress of classical Schubert calculus with an emphasis on the case $G = SL(n, \mathbb{C})$. With Gromov-Witten theory introduced in the 1990s, the classical cohomology $H^*(G/P, \mathbb{Z})$ can be deformed to the integral (small) quantum cohomology ring $QH^*(G/P, \mathbb{Z}) = (H^*(G/P, \mathbb{Z}) \otimes \mathbb{Z}[\mathbf{q}], \star)$, by incorporating 3-pointed, genus-0 Gromov-Witten invariants. In particular, we can write $\sigma^u \star \sigma^v = \sum_{w, \mathbf{d}} N_{u,v}^{w, \mathbf{d}} \sigma^w \mathbf{q}^{\mathbf{d}}$, where $N_{u,v}^{w, \mathbf{0}} = c_{u,v}^w$. There have been extensive studies of the *quantum Schubert calculus*, namely of the quantum versions of the above (1)-(3) for $QH^*(G/P, \mathbb{Z})$ (see e.g. the survey [LL17] and the references therein).

All the Schubert varieties, including flag varieties as special cases, have CW complex structures by Schubert cells. In return, the integral cohomology of a Schubert variety X_w inside G/P is torsion free, and has an additive basis of Schubert classes indexed by the Weyl group elements u satisfying $u \leq w$ with respect to the Bruhat order.

Question 1.1. *What is the Schubert calculus for Schubert varieties?*

The natural inclusion $\iota : X_w \hookrightarrow G/P$ induces a surjective ring homomorphism $\iota^* : H^*(G/P, \mathbb{Z}) \rightarrow H^*(X_w, \mathbb{Z})$ with kernel $I_w = \langle \sigma^u : u \not\leq w \rangle$. Hence, the classical Schubert calculus for Schubert varieties is trivial in the sense that all points (1)-(3) can be reduced to that for the flag varieties, at the price that the ring presentation $H^*(X_w, \mathbb{Z}) = H^*(G/P, \mathbb{Z})/I_w$ is not good enough. We refer to [ALP92, GR02, DMR07, RWY11, DY24] for the study of the ring presentation of the cohomology of Schubert varieties.

The quantum Schubert calculus for Schubert varieties is highly nontrivial. First of all, we have to restrict to the smooth ones, since there is no Gromov-Witten theory for singular (Schubert) varieties yet. To our knowledge, there have been very few pioneering studies [Pec13, MS19, HKLS25] in different contexts. The odd symplectic Grassmannian $IG(k, 2n+1)$ is a smooth Schubert variety of the symplectic Grassmannian $IG(k, 2n+2)$, a flag variety G/P with $G = Sp(2n, \mathbb{C})$. In [Pec13], Pech studied the case $k = 2$, which happens to be a general hyperplane section of the complex Grassmannian $Gr(2, 2n+1)$. She did a relatively complete quantum Schubert calculus for a non-homogeneous Schubert variety for the first time, by providing a ring presentation, the quantum Pieri formula (a partial formula for the quantum version of point (2), see also [GLLX25]), and the quantum Giambelli formula (i.e. the quantum version of (3)). In [MS19], Mihalcea and Shifler provided the (equivariant) quantum Chevalley formula for $IG(k, 2n+1)$ by using the curve neighborhood technique [BM15]. In [HKLS25], Hu, Ke, Li and Song provided a ring presentation for the quantum cohomology of the blowup of $Gr(2, n)$ along $Gr(2, n-1)$ for the purpose of studying mirror symmetry, which happens to be a Schubert divisor in a two-step flag variety. The special case when $n = 3$ is the blowup of $\mathbb{P}^2$ at a point, which has been well studied much earlier. Despite being a very natural extension from the viewpoint of Schubert calculus, the

quantum Schubert calculus for smooth non-homogeneous varieties is still largely uncharted territory, with many aspects awaiting exploration.

Let $F_\bullet$ denote the standard complete flag. Each permutation $w \in S_n$ labels a Schubert variety X_w of dimension $\ell(w)$ defined by rank conditions of the form $X_w = \{V_\bullet \mid \dim(V_i \cap F_j) \geq m(i, j, w), \forall i, j\}$. Note that the permutation $w_0 = n \cdots 21$ in one-line notation is the longest element in S_n , and $s_i := (i, i+1)$, $i < n$, denote the simple transpositions. Note $X = X_{w_0 s_{n-1}} \cong X_{w_0 s_1}$, while all the other Schubert divisors $X_{w_0 s_i}$, $2 \leq i \leq n-2$, are singular.

Denote the following $n \times n$ matrices

$$\begin{pmatrix} x_1 & q_1 & & & & \\ -1 & x_2 & q_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & x_{n-2} & q_{n-2} & \\ & & & -1 & x_{n-1} & q_{n-1} \\ & & & & -1 & x_n \end{pmatrix} \quad \begin{pmatrix} x_1 & q_1 & & & & \\ -1 & x_2 & q_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & x_{n-2} & q_{n-2} & -q_{n-1}q_{n-2} \\ & & & -1 & x_{n-1} & -q_{n-1}x_{n-1} \\ & & & & -1 & x_n \end{pmatrix}$$

by $M_{F\ell_n}$ and $M_{X_{w_0 s_{n-1}}}$ respectively. Write

$$(1.1) \quad \det(I_n + \lambda M_{F\ell_n}) = \sum_{i=0}^n E_i^n \lambda^i, \quad \det(I_n + \lambda M_{X_{w_0 s_{n-1}}}) = \sum_{i=0}^n \hat{E}_i^n \lambda^i.$$

The coefficients E_i^n , $\hat{E}_i^n$ may be viewed as quantizations of the i -th elementary symmetric polynomial $e_i^n(x_1, \dots, x_n)$. Moreover, we notice $\hat{E}_1^n = E_1^n$ and $\hat{E}_n^n = (x_n - q_{n-1})E_{n-1}^{n-1}$, while the difference between $\hat{E}_i^n$ and E_i^n is a bit involved for $1 < i < n$. As shown in [GK95, CF95, Ki96], there is a canonical ring isomorphism

$$\Phi_q : QH^*(F\ell_n, \mathbb{Z}) \longrightarrow \mathbb{Z}[x_1, \dots, x_n, q_1, \dots, q_{n-1}] / (E_1^n, \dots, E_n^n).$$

Its classical limit at $\mathbf{q} = \mathbf{0}$ gives Borel's ring isomorphism $\Phi : H^*(F\ell_n, \mathbb{Z}) \xrightarrow{\sim} \frac{\mathbb{Z}[x_1, \dots, x_n]}{(e_1^n, \dots, e_n^n)}$ [Bo53], where $e_i^n = e_i^n(x)$ denotes the i -th elementary symmetric polynomial in variables $x_1, \dots, x_n$. Our first main result is a similar quantum ring presentation for X .

Theorem 1.2 (Borel-type ring presentation). *There is a canonical ring isomorphism*

$$\Psi_q : QH^*(X, \mathbb{Z}) \longrightarrow \mathbb{Z}[x_1, \dots, x_n, q_1, \dots, q_{n-1}] / (\hat{E}_1^n, \dots, \hat{E}_{n-1}^n, E_{n-1}^{n-1}).$$

In particular, we obtain the canonical ring isomorphism $\Psi : H^*(X, \mathbb{Z}) \xrightarrow{\sim} \frac{\mathbb{Z}[x_1, \dots, x_n]}{(e_1^n, \dots, e_{n-1}^n, e_{n-1}^{n-1})}$ obtained in [GR02, RWY11] by taking the classical limit at $\mathbf{q} = \mathbf{0}$. By “canonical” above, we mean that x_i represents the first Chern class of a specific tautological line bundle (see Equation (2.8) for $F\ell_n$ and Equation (2.10) for X).

For any $u \in S_n$, the Schubert class σ^u in $H^{2\ell(u)}(F\ell_n, \mathbb{Z})$ is given by the Poincaré dual of the homology class $[X_{w_0 u}]$ in $H_{2\ell(w_0) - 2\ell(u)}(F\ell_n, \mathbb{Z})$. The pullback Schubert classes $\{\xi^u := \iota^*(\sigma^u)\}_{u \leq w_0 s_{n-1}}$ form an additive basis of $H^*(X, \mathbb{Z})$, and the divisor classes ξ^{s_i} 's generate $QH^*(X, \mathbb{Z})$ as a $\mathbb{Z}[\mathbf{q}]$ -algebra. Whenever referring to a transposition $t_{ij} = (i, j)$, we always assume $i < j$. We say $u \leq_k ut_{ij}$ (resp. $u \leq_k^q ut_{ij}$) in the (quantum) k -Bruhat order, if both

$i \leq k < j$ and $\ell(ut_{ij}) = \ell(u) + 1$ (resp. $\ell(ut_{ij}) = \ell(u) - \ell(t_{ij})$) hold. Our second main result is the following quantum Monk-Chevalley formula, in analogy with the quantum Monk's formula for $QH^*(F\ell_n, \mathbb{Z})$ [FGP97] or more generally the quantum Chevalley formula for $QH^*(G/P, \mathbb{Z})$ [FW04]. It is a special case of the quantum version of point (2) for X but it determines the entire quantum product because the divisor classes generate $QH^*(X, \mathbb{Z})$.

Theorem 1.3 (Quantum Monk-Chevalley formula). *Let $1 \leq k \leq n - 1$ and $u \in S_n$ with $u \leq w_0 s_{n-1}$. In $QH^*(X, \mathbb{Z})$, we have*

$$\xi^{s_k} \star \xi^u = \sum \xi^{ut_{ab}} + \sum \xi^{ut_{ab}} q_a \cdots q_{b-1} + \sum \xi^w q_a \cdots q_{n-1} - \delta_{k,n-1} \xi^u q_{n-1},$$

where the first sum is over $u \leq_k ut_{ab} \leq w_0 s_{n-1}$, the second sum is over $u \leq_k^q ut_{ab}$ with $b < n$, and the third sum is over (w, a) that satisfies $wt_{an} \leq_k^q w$ and $u \leq_{n-1} wt_{an} \in S_n$.

By Remark 4.12, the permutation w in the third sum satisfies $w \leq w_0 s_{n-1}$; the same is true of ut_{ab} in the second sum.

In the above expression of $\xi^{s_k} \star \xi^u$, the first two sums are obtained by truncating the quantum product $\sigma^{s_k} \star \sigma^u$, and the third sum is about the quantum terms involving q_{n-1} and appearing in $\sigma^{s_k} \star (\sigma^u \cup \sigma^{s_{n-1}})$. We emphasize that the minus sign in the fourth part is necessary and has a geometric explanation, which gives rise to a key ingredient in our geometric arguments. Intuitively, a smooth curve with two marked points of degree d with $d_{n-1} = 1$ lies on X if and only if the two marked points are both on X . For stable maps, an additional correction term must be taken into consideration, due to the presence of nodal curves. The fourth part may or may not be canceled by the third part.

Example 1.4. *For the Schubert divisor $X_{w_0 s_3}$ of $F\ell_4$, we have*

$$\begin{aligned} \xi^{s_3 s_2} \star \xi^{s_3} &= \xi^{s_3 s_2 s_3} + 0 + (q_3 \xi^{s_3 s_2} + q_2 q_3) - q_3 \xi^{s_3 s_2} = \xi^{s_3 s_2 s_3} + q_2 q_3; \\ \xi^{s_1 s_3} \star \xi^{s_3} &= \xi^{s_2 s_1 s_3} + 0 + (q_3 \xi^{s_1 s_2} + q_3 \xi^{s_2 s_1}) - q_3 \xi^{s_1 s_3}. \end{aligned}$$

To show the vanishing of Gromov-Witten invariants of higher degrees, we need a special treatment with involved geometric arguments for degrees of the form $(0, \dots, 0, 1, \dots, 1, 2)$, in addition to the use of the standard curve neighborhood technique developed by Buch and Mihalcea [BM15] for Gromov-Witten invariants of other higher degrees.

Remark 1.5. *Our proof of Theorem 1.3 is purely geometric, and all morphisms involved are torus-equivariant. Consequently, Theorem 4.2 extends equivariantly by simply interpreting $N_{v,u}^{w,d}$ as equivariant quantum Schubert structure constants for $F\ell_n$. Therefore, the equivariant quantum extension of Theorem 1.3 is simply obtained by adding into it the single term $(\omega_k - u(\omega_k))\xi^u$, where ω_i 's denote the fundamental weights.*

We work only in the non-equivariant setting in the main text, to keep the presentation concise and aligned with the predictions of the Peterson program [LRY].

The Schubert polynomials $\mathfrak{S}_w \in \mathbb{Z}[x_1, \dots, x_n]$, $w \in S_n$, were introduced by Lascoux and Schützenberger [LS82], and satisfy $\Phi(\sigma^w) = [\mathfrak{S}_w]$ in Borel's presentation of $H^*(F\ell_n, \mathbb{Z})$. Moreover, every $\mathfrak{S}_w$ admits a unique linear expansion $\mathfrak{S}_w = \sum \alpha_{i_1 \dots i_{n-1}} e_{i_1}^1 e_{i_2}^2 \cdots e_{i_{n-1}}^{n-1}$. In [FGP97], Fomin, Gelfand and Postnikov introduced the quantum Schubert polynomial

$$\mathfrak{S}_w^q = \sum \alpha_{i_1 \dots i_{n-1}} E_{i_1}^1 E_{i_2}^2 \cdots E_{i_{n-1}}^{n-1}.$$

They showed that $\Phi_q(\sigma^w) = [\mathfrak{S}_w^q]$ under the aforementioned canonical ring isomorphism Φ_q for $QH^*(F\ell_n, \mathbb{Z})$. Our third main result is the following. Recall the ring isomorphism Ψ_q in Theorem 1.2.

Theorem 1.6 (Quantum Schubert polynomials). *For any $w \leq w_0 s_{n-1}$, we have*

$$\Psi_q(\xi^w) = [\mathfrak{S}_w^q].$$

The key ingredient in our proof is a careful application of Theorem 1.3, combined with a transition equation (in Proposition 5.3), showing that quantum Schubert polynomials can be completely determined by a very small part of the quantum Monk-Chevalley formula. This principle was noticed and applied early in [LOTRZ25, Theorem 4 and Remark 3.15] in the study of a combinatorial model for quantum double Schubert polynomials for $F\ell_n$.

On the one hand, even though the quantum (resp. classical) Schubert polynomial of the class ξ^w for X is the same as the corresponding polynomial for the class σ^w for $F\ell_n$, the quantum cohomology ring structures differ substantially. As Theorem 1.3 shows, the expansion of $\mathfrak{S}_{s_i}^q \mathfrak{S}_u^q$ for X is not obtained by simply truncating the formula for $F\ell_n$. Thus the naive linear pullback $\sigma^u \mapsto \iota^*(\sigma^u) = \xi^u$ does not extend to a ring homomorphism $QH^*(F\ell_n, \mathbb{Z}) \rightarrow QH^*(X, \mathbb{Z})$.

On the other hand, the following modified pullback is a ring homomorphism.

Theorem 1.7 (Quantum Lefschetz hyperplane principle). *There is a ring homomorphism $\iota_q^* : QH^*(F\ell_n, \mathbb{Z}) \rightarrow QH^*(X, \mathbb{Z})$, defined by $\sigma^{s_i} \mapsto \xi^{s_i}$ and $q_i \mapsto q_i$ for $1 \leq i \leq n-2$, and*

$$\sigma^{s_{n-1}} \mapsto \xi^{s_{n-1}} + q_{n-1}, \quad q_{n-1} \mapsto q_{n-1} \xi^{s_{n-1}} + q_{n-1}^2.$$

Notable features of the main results. A fundamental insight of Dale Peterson [Pet97] assembles the spectra of the quantum cohomology rings $QH^*(G/P) = QH^*(G/P, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$, as P varies, into the Peterson variety in the Langlands dual complete flag variety; see [Rie03, Chow22]. In the forthcoming work [LRY], a Peterson program is proposed for Schubert varieties in arbitrary flag varieties. As mentioned earlier, our main results above indicate features of a more general theory of smooth Fano Schubert varieties predicted in the Peterson program as follows.

- (1) Theorem 1.2 suggests that the quantum cohomology of a smooth Fano Schubert variety in a partial flag variety $F\ell_{n_1, \dots, n_r; n}$ should admit a presentation governed by a matrix generalizing $M_{X_{w_0 s_{n-1}}}$. The known ring presentation of $QH^*(F\ell_{n_1, \dots, n_r; n})$ in [AS95, CF99, Ki99a] has the quotient ideal generated by non-constant coefficients of a matrix characteristic polynomial. Related matrices arise in an intermediate study of the Peterson program for Schubert varieties in $F\ell_{n_1, \dots, n_r; n}$, and specialize to $M_{X_{w_0 s_{n-1}}}$ in the present case.
- (2) Theorem 1.6 reflects a predicted functoriality for the quantum cohomology of smooth Fano Schubert varieties. By [Rie03, Proposition 11.1], a Schubert class σ_P^u in $QH^*(F\ell_{n_1, \dots, n_r; n})$, viewed as a regular function on the corresponding Peterson stratum, coincides with the *restriction* of the regular function σ_B^u on the Peterson stratum corresponding to $QH^*(F\ell_n)$. The Peterson program predicts an extension of this phenomenon to smooth Fano Schubert varieties, and Theorem 1.6 provides evidence for this prediction.

Our main results may also have an impact in the wider field of genus zero Gromov-Witten theory.

- (3) Theorem 1.3 is very likely to play an important role in the study of Galkin-Golyshev-Iritani's Gamma conjectures I and II [GGI16] in the case of $X_{w_0 s_{n-1}}$, following the strategy in [HKLS26].
- (4) Through private communication with Galkin and Iritani, we learn that our Theorem 1.7 may be viewed as a special case of a new formulation of the quantum Lefschetz hyperplane principle at the level of ring homomorphisms in their forthcoming paper [GI].

The results of this paper, together with the earlier work on odd symplectic Grassmannians [Pec13, MS19] and blowups of Grassmannians [HKLS25], provide further motivation for a systematic study of quantum Schubert calculus for smooth Schubert varieties.

The paper is organized as follows. In Section 2, we introduce the necessary background. In Section 3, we derive the ring presentation of $QH^*(X, \mathbb{Z})$ by investigating the quantum differential equations. In Section 4, we provide the quantum Monk-Chevalley formula, by investigating the moduli space of stable maps of certain degrees and using the curve neighborhood technique for higher degrees. Finally in Section 5, we show that the quantum Schubert polynomials for X coincide with those for $F\ell_n$, by using induction based on a transition equation for quantum Schubert polynomials.

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2. NOTATIONS

We review some background, and refer to [BB05, GR02, CK99] for more details.

2.1. Combinatorics of S_n . The symmetric group S_n is generated by simple reflections $s_i := (i, i+1)$, $1 \leq i < n$. Every element $w \in S_n$ can be written as $w(1) \cdots w(n)$ in one-line notation, and its length $\ell(w)$ is the cardinality of the inversion set $\{(i, j) \mid i < j, w(i) > w(j)\}$. The (unique) longest element in S_n is given by $w_0 := n \cdots 21$. For any permutation w , we have $\ell(w_0 w) = \ell(w_0) - \ell(w)$.

Whenever referring to a transposition $t_{ab} := (a, b)$, we always assume $a < b$ in this paper. Let $u, w \in S_n$. We say $u \triangleleft w$ (i.e. u is covered by w), if there exists t_{ab} , such that $w = ut_{ab}$ and $\ell(w) = \ell(u) + 1$. Then $u \leq w$ in the Bruhat order, if u can be transformed into w by a series of transpositions t_{ij} that each increase the inversion number by 1. Note that $u \leq w$ holds if and only if for any $1 \leq j \leq n$, the increasingly sorted list of $u(1), \dots, u(j)$ is less than or equal to that of $w(1), \dots, w(j)$ in the usual partial order. In particular, we have

$$(2.1) \quad u \leq w_0 s_{n-1} = n \cdots 4312 \iff u(n) \neq 1.$$

Note $\ell(t_{ab}) = 2b - 2a - 1$. Let $1 \leq k < n$. The quantum Bruhat cover $\triangleleft^q$ is defined by

$$u \triangleleft^q ut_{ab} \quad \text{if and only if} \quad \ell(ut_{ab}) = \ell(u) - \ell(t_{ab}).$$

We can further define the (quantum) k -Bruhat cover by

$$(2.2) \quad u \lessdot_k ut_{ab} \quad \text{if and only if} \quad \ell(ut_{ab}) = \ell(u) + 1 \text{ and } a \leq k < b;$$

$$(2.3) \quad u \lessdot_k^q ut_{ab} \quad \text{if and only if} \quad \ell(ut_{ab}) = \ell(u) - \ell(t_{ab}) \text{ and } a \leq k < b.$$

2.2. Schubert varieties of $F\ell_n$. Consider the complete flag variety

$$F\ell_n := \{V_1 \leq \cdots \leq V_{n-1} \leq \mathbb{C}^n \mid \dim V_i = i, \forall 1 \leq i < n\}.$$

Let $F_\bullet \in F\ell_n$ be the standard complete flag in $\mathbb{C}^n$. For any permutation $w \in S_n$, we define the Schubert cell $X_w^\circ \subseteq F\ell_n$ (associated to $F_\bullet$) by the following rank conditions.

$$X_w^\circ := \{V_\bullet \in F\ell_n \mid \dim(V_p \cap F_q) = \#\{k \in \mathbb{Z}_{>0} \mid k \leq p, w(k) \leq q\}, \forall 1 \leq p, q \leq n\}.$$

Then $X_w^\circ \cong \mathbb{C}^{\ell(w)}$, and $X_u^\circ \cap X_w^\circ = \emptyset$ for any $u \neq w$. The Schubert variety X_w is given by

$$X_w := \overline{X_w^\circ} = \{V_\bullet \in F\ell_n \mid \dim(V_p \cap F_q) \geq \#\{k \in \mathbb{Z}_{>0} \mid k \leq p, w(k) \leq q\}, \forall 1 \leq p, q \leq n\},$$

and admits a cell decomposition by Schubert cells (with respect to the Bruhat order):

$$(2.4) \quad X_w = \bigsqcup_{u \leq w} X_u^\circ.$$

In particular, $F\ell_n$ is the (biggest) Schubert variety X_{w_0} . The Schubert divisors are given by $X_{w_0 s_i}$, $1 \leq i \leq n-1$. Throughout this paper, we will focus on the Schubert divisor $X_{w_0 s_{n-1}}$, whose defining rank conditions can be reduced to the single one:

$$(2.5) \quad X := X_{w_0 s_{n-1}} = \{V_\bullet \in F\ell_n \mid F_1 \subset V_{n-1}\}.$$

Let $pr : F\ell_n \rightarrow Gr(n-1, n)$ be the natural projection that sends $V_\bullet$ to V_{n-1} . Then the divisor X can be viewed as the zero locus of a section of the line bundle over $F\ell_n$,

$$(2.6) \quad \mathcal{L}_{\varpi_{n-1}} := pr^* \mathcal{O}_{Gr(n-1, n)}(1).$$

We notice that $X_{w_0 s_1} \cong X_{w_0 s_{n-1}}$ is smooth and isomorphic to an $F\ell_{n-1}$ -bundle over $\mathbb{P}^{n-2}$, while the varieties $X_{w_0 s_i}$'s are all singular for $1 < i < n-1$.

2.2.1. Topology of $F\ell_n$. For $w \in S_n$, denote by $\sigma^w \in H^{2\ell(w)}(F\ell_n, \mathbb{Z})$ the Schubert class defined by the Poincaré dual of the homology class $[X_{w_0 w}] \in H_{2\ell(w_0 w)}(F\ell_n, \mathbb{Z})$. We have

$$(\sigma^{w_0 u}, \sigma^w) := \int_{F\ell_n} \sigma^{w_0 u} \cup \sigma^w = \langle [X_u], \sigma^w \rangle = \delta_{u, w}.$$

Here $(\cdot, \cdot)$ denotes the Poincaré pairing for $F\ell_n$, and $\langle \cdot, \cdot \rangle$ (resp. $\delta_{u, w}$) denote the Kronecker pairing (resp. symbol). It follows from Equation (2.4) that $H^*(F\ell_n, \mathbb{Z}) = \bigoplus_{w \in S_n} \mathbb{Z} \sigma^w$.

Denote the i -th elementary symmetric polynomial in variables $x_1, \dots, x_n$ as

$$(2.7) \quad e_i^n = e_i^n(x_1, \dots, x_n).$$

The following result is due to Borel [Bo53].

Proposition 2.1. *There is a canonical ring isomorphism*

$$\Phi : H^*(F\ell_n, \mathbb{Z}) \longrightarrow \mathbb{Z}[x_1, x_2, \dots, x_n] / (e_1^n, e_2^n, \dots, e_n^n).$$

Denote $V_0 = \{0\}$ and $V_n = \mathbb{C}^n$. Let $\mathcal{S}_i$ be the i -th tautological subbundle of $F\ell_n$, $0 \leq i \leq n$, namely the fiber of $\mathcal{S}_i$ at a point $V_\bullet \in F\ell_n$ is given by the vector space V_i . By “canonical” in the above proposition, we mean

$$(2.8) \quad \Phi^{-1}([x_i]) = c_1((\mathcal{S}_i/\mathcal{S}_{i-1})^\vee) \in H^2(F\ell_n, \mathbb{Z}).$$

2.2.2. Topology of X . It also follows from the cell decomposition Equation (2.4) that $H^*(X, \mathbb{Z})$ is torsion free and has an additive basis $\{PD([X_w])\}_{w \leq w_0 s_{n-1}}$. Let $\{\xi^w\}_{w \leq w_0 s_{n-1}}$ denote the dual basis with respect to Poincaré pairing, namely $(PD([X_u]), \xi^w) = \langle [X_u], \xi^w \rangle = \delta_{u,w}$ for any $u, w \leq w_0 s_{n-1}$. The natural inclusion map $\iota : X \hookrightarrow F\ell_n$ induces a surjective ring homomorphism

$$(2.9) \quad \iota^* : H^*(F\ell_n, \mathbb{Z}) \longrightarrow H^*(X, \mathbb{Z}); \quad \iota^*(\sigma^w) = \begin{cases} \xi^w, & \text{if } w \leq w_0 s_{n-1}, \\ 0, & \text{otherwise.} \end{cases}$$

We therefore call the classes ξ^w the pullback Schubert classes. Moreover, the following ring presentation was obtained in [GR02, RWY11].

Proposition 2.2. *There is a canonical ring isomorphism*

$$\Psi : H^*(X, \mathbb{Z}) \longrightarrow \mathbb{Z}[x_1, x_2, \dots, x_n] / (e_1^n, e_2^n, \dots, e_{n-1}^n, e_n^{n-1}).$$

Here by “canonical”, we mean

$$(2.10) \quad \Psi^{-1}([x_i]) = c_1((\iota^* \mathcal{S}_i / \iota^* \mathcal{S}_{i-1})^\vee) \in H^2(X, \mathbb{Z}).$$

2.3. Quantum cohomology. To simplify the descriptions, we only review the necessary notions for $Y \in \{F\ell_n, X\}$. Both $F\ell_n$ and X are Fano with the first Chern class

$$(2.11) \quad c_1(Y) = \begin{cases} 2\sigma^{s_1} + 2\sigma^{s_2} + \dots + 2\sigma^{s_{n-2}} + 2\sigma^{s_{n-1}}, & \text{if } Y = F\ell_n, \\ 2\xi^{s_1} + 2\xi^{s_2} + \dots + 2\xi^{s_{n-2}} + \xi^{s_{n-1}}, & \text{if } Y = X. \end{cases}$$

(See [LRY25] for a criterion for a factorial Schubert variety of general Lie type being Fano, as well as a formula of the first Chern class.) Uniformly denote by γ^w the cohomology class σ^w (resp. ξ^w) labeled by an appropriate permutation w for $Y = F\ell_n$ (resp. X).

Let $\overline{\mathcal{M}}_{0,m}(Y, d)$ denote the moduli space of m -pointed, genus-0 stable maps $(f : C \rightarrow Y; \text{pt}_1, \dots, \text{pt}_m)$ of degree $d \in H_2(Y; \mathbb{Z})$. For each i , let $ev_i : \overline{\mathcal{M}}_{0,m}(Y, d) \rightarrow Y$ denote the i -th evaluation map, which sends $(f : C \rightarrow Y; \text{pt}_1, \dots, \text{pt}_m)$ to $f(\text{pt}_i)$; let $\mathcal{L}_i$ denote the i -th universal cotangent line bundle on $\overline{\mathcal{M}}_{0,m}(Y, d)$. The Gromov-Witten invariant with gravitational descendants, associated with nonnegative integers a_i and cohomology classes $\gamma^{u_i} \in H^*(Y) = H^*(Y, \mathbb{C})$, is defined by

$$\langle \tau_{a_1} \gamma^{u_1}, \tau_{a_2} \gamma^{u_2}, \dots, \tau_{a_m} \gamma^{u_m} \rangle_{0,m,d}^Y := \int_{[\overline{\mathcal{M}}_{0,m}(Y,d)]^{\text{vir}}} \prod_{i=1}^m (c_1(\mathcal{L}_i)^{a_i} \cup ev_i^*(\gamma^{u_i})).$$

Here $[\overline{\mathcal{M}}_{0,m}(Y, d)]^{\text{vir}}$ denotes the virtual fundamental class, which is of expected dimension $\dim Y + m - 3 + \int_d c_1(Y)$. In particular, Gromov-Witten invariants of degree d vanish unless d belongs to the Mori cone $\overline{\text{NE}}(Y) = \bigoplus_{j=1}^{n-1} \mathbb{Z}_{\geq 0} [X_{s_j}] \subset H_2(Y, \mathbb{Z})$ of effective curve classes of Y , which we simply denote as $d \geq 0$.

Let $\{\gamma_u\}_u \subset H^*(Y)$ be the dual basis of $\{\gamma^u\}_u$ with respect to the Poincaré pairing. The (small) quantum cohomology $QH^*(Y) = (H^*(Y) \otimes \mathbb{C}[\mathbf{q}], \star)$ is a $\mathbb{C}[\mathbf{q}]$ -algebra with the (small) quantum product defined by

$$(2.12) \quad \gamma^u \star \gamma^v = \sum_{d \in \overline{\text{NE}}(Y)} \sum_w \langle \gamma^u, \gamma^v, \gamma_w \rangle_{0,3,d}^Y \gamma^w q^d,$$

where $q^d := q_1^{d_1} \cdots q_{n-1}^{d_{n-1}}$ for $d = \sum_{j=1}^{n-1} d_j [X_{s_j}] \in \overline{\text{NE}}(Y)$. The quantum variable q_j is of degree

$$(2.13) \quad \deg q_j = \int_{[X_{s_j}]} c_1(Y) = \begin{cases} 1, & \text{if } Y = X \text{ and } j = n-1 \text{ both hold,} \\ 2, & \text{otherwise.} \end{cases}$$

Whenever $H^*(Y, \mathbb{Z}) \otimes \mathbb{Z}[\mathbf{q}]$ is closed under the quantum multiplication $\star$, we call it the integral quantum cohomology, simply denoted as $QH^*(Y, \mathbb{Z})$.

3. A BOREL-TYPE RING PRESENTATION OF $QH^*(X)$

For $Y \in \{F\ell_n, X\}$, we denote

$$(3.1) \quad \mathcal{H}(Y) := H^*(Y) \otimes_{\mathbb{C}} \mathbb{C}[\hbar][[\hbar^{-1}]]\langle q_1, \dots, q_{n-1} \rangle.$$

In this section, we introduce Givental's J -function $J^Y(t, \hbar)$ of Y , viewed as an element in $\mathcal{H}(Y)$. We then prove the Borel-type ring presentation of $QH^*(X)$ in Theorem 1.2, by finding quantum differential equations that annihilate $J^X(t, \hbar)$.

At the beginning and end of this section, we identify both first Chern classes in Equation (2.8) and Equation (2.10) with x_i by abuse of notation. Then $\gamma^{s_j} = x_1 + x_2 + \cdots + x_j$ for $1 \leq j \leq n-1$. Denoting $t = \sum_{i=1}^n t_i x_i \in H^2(Y)$, we view the quantum variables q_j as functions on $H^2(Y)$, by letting $q_j = e^{\int_{[X_{s_j}]} t} = e^{t_j - t_{j+1}}$. Then it follows from $\delta_{i,j} = \int_{[X_{s_i}]} \gamma^{s_j}$ that $q^d = e^{\int_d t} = e^{\int_d \sum_i t_i (\gamma_i - \gamma_{i-1})} = e^{\int_d \sum_i (t_i - t_{i+1}) \gamma_i} = q_1^{d_1} \cdots q_{n-1}^{d_{n-1}}$.

From Proposition 3.3 through Theorem 3.10, we use $\iota^* x_i$ for X to distinguish it from x_i for $F\ell_n$, in order to avoid confusion. For any $f(x, \hbar) \in \mathcal{H}(F\ell_n)$, by $f(\iota^* x, \hbar)$ we mean the element in $\mathcal{H}(X)$ simply obtained from $f(x, \hbar)$ by replacing all x_i with $\iota^* x_i$. In particular, by $\iota^* t$ we mean $\sum_i t_i \iota^* x_i$.

3.1. Givental's J -function of X . The (small) quantum connection acts on the trivial $H^*(Y)$ -bundle over $H^2(Y) \times \mathbb{C}^*$. Its derivations along the $H^2(Y)$ -direction are given by

$$\nabla_{\frac{\partial}{\partial t_i}} := \frac{\partial}{\partial t_i} + \frac{1}{\hbar} x_i \star, \quad 1 \leq i \leq n,$$

where $\hbar$ is the coordinate of $\mathbb{C}^*$. This quantum connection is flat, with the fundamental solution $L(t, \hbar)$ to the quantum differential equation $\nabla_{\frac{\partial}{\partial t_i}} (L(t, \hbar) \alpha) = 0$ given by [Gi96]

$$L(t, \hbar) \alpha := e^{-\frac{t}{\hbar}} \alpha + \sum_{d \neq 0} \sum_w \left\langle \frac{e^{-\frac{t}{\hbar}} \alpha}{-\hbar - c_1(\mathcal{L}_1)}, \gamma_w \right\rangle_{0,2,d}^Y \gamma^w q^d.$$

Here $\alpha \in H^*(Y)$, and we take the expansion $\frac{1}{-\hbar - c_1(\mathcal{L}_1)} = \sum_{i \geq 0} (-1)^{i+1} \hbar^{-i-1} (c_1(\mathcal{L}_1))^i$.

Definition 3.1. *Givental's (small) J -function of Y is defined by*

$$J^Y(t, \hbar) := L(t, \hbar)^{-1}(\mathbf{1}) = e^{\frac{t}{\hbar}} \left(1 + \sum_{d \neq 0} \sum_w \left\langle \frac{\gamma_w}{\hbar - c_1(\mathcal{L}_1)} \right\rangle_{0,1,d}^Y \gamma^w q^d \right);$$

$$J^Y(t_0, t, \hbar) := e^{\frac{t_0}{\hbar}} J^Y(t, \hbar).$$

Here $\mathbf{1} = \gamma^{\text{id}}$ is the identity element in $H^*(Y)$.

Proposition 3.2 ([BCK08, Theorem 1]). *Givental's J -function of $F\ell_n$ is given by*

$$J^{F\ell_n}(t, \hbar) = e^{\frac{t}{\hbar}} \left(\sum_{d \geq 0} q^d J_d^{F\ell_n}(x, \hbar) \right), \quad \text{where}$$

$$J_d^{F\ell_n}(x, \hbar) = \sum_{\sum_j d_{i,j} = d_i} \left(\prod_{i=1}^{n-1} \prod_{1 \leq j < j' \leq i} (-1)^{d_{i,j} - d_{i,j'}} \frac{x_j - x_{j'} + (d_{i,j} - d_{i,j'})\hbar}{x_j - x_{j'}} \right) \\ \left(\prod_{i=1}^{n-2} \prod_{\substack{1 \leq j \leq i \\ 1 \leq j' \leq i+1}} \frac{\prod_{k=-\infty}^0 (x_j - x_{j'} + k\hbar)}{\prod_{k=-\infty}^{d_{i,j} - d_{i+1,j'}} (x_j - x_{j'} + k\hbar)} \right) \prod_{1 \leq j \leq n-1} \frac{1}{\prod_{k=1}^{d_{n-1,j}} (x_j + k\hbar)^n}.$$

Here the sum is over nonnegative integers $d_{i,j}$, with $1 \leq j \leq i \leq n-1$.

Proposition 3.3. *Givental's J -function of X is given by*

$$J^X(\iota^* t, \hbar) = e^{-\frac{q_{n-1}}{\hbar}} I_{F\ell_n, X}(\iota^* t, \hbar),$$

where

$$I_{F\ell_n, X}(t, \hbar) := e^{\frac{t}{\hbar}} \left(\sum_{d \geq 0} q^d J_d^{F\ell_n}(x, \hbar) \prod_{k=1}^{\int_d \sigma^{s_{n-1}}} (-x_n + k\hbar) \right)$$

with the empty product $\prod_{k=1}^0$ read off as multiplication by 1 by convention.

Proof. Expanding $I_{F\ell_n, X}(t, \hbar)$ in powers of $\hbar^{-1}$, we obtain

$$I_{F\ell_n, X}(t, \hbar) = 1 + \frac{1}{\hbar} (q_{n-1} \mathbf{1} + t) + O(\hbar^{-2}).$$

Recall from Equation (2.6) that the smooth Schubert divisor X is realized as the zero locus of a section of the line bundle $\mathcal{L}_{\varpi_{n-1}}$ over $F\ell_n$ with $c_1(\mathcal{L}_{\varpi_{n-1}}) = \sigma^{s_{n-1}} = -x_n$. By using the quantum Lefschetz theorem [Ki99a, Theorem 1] (see also [CG07, Corollary 7]) with respect to $F\ell_n$ and X^1 , we obtain

$$J^X(q_{n-1} \mathbf{1} + \iota^* t, \hbar) = I_{F\ell_n, X}(\iota^* t, \hbar) = e^{\frac{\iota^* t}{\hbar}} \left(\sum_{d \geq 0} q^d J_d^{F\ell_n}(\iota^* x, \hbar) \prod_{k=1}^{\int_d \sigma^{s_{n-1}}} (-\iota^* x_n + k\hbar) \right).$$

This implies $J^X(\iota^* t, \hbar) = e^{-\frac{q_{n-1}}{\hbar}} I_{F\ell_n, X}(\iota^* t, \hbar)$. □

¹The function $I_{F\ell_n, X}$ in [CG07, Section 9] is a multiple of ours by $\hbar$ with the identification $z = \hbar$.

3.2. Quantum differential equations of X . In this subsection, we will investigate quantum differential equations of X , namely differential operators that annihilate $J^X(t, \hbar)$.

For any polynomial $P(x, q) \in \mathbb{C}[x, q]$, we write

$$P(x, q) = \sum_d \sum_{i_I} A_{i_I}^d x^{i_I} q^d$$

with $x^{i_I} := x_1^{i_1} \cdots x_n^{i_n}$, $q^d = q_1^{d_1} \cdots q_{n-1}^{d_{n-1}}$ and the coefficients $A_{i_I}^d \in \mathbb{C}$. Take the conventions:

$$(3.2) \quad \hbar \frac{\partial}{\partial t_{[1,m]}} := (\hbar \frac{\partial}{\partial t_1}, \hbar \frac{\partial}{\partial t_2} \cdots, \hbar \frac{\partial}{\partial t_m}), \quad q_{[1,m]} := (q_1, q_2, \dots, q_m),$$

where $q_j = e^{t_j - t_{j+1}}$ whenever it is treated as an operator. We then denote

$$P(\hbar \frac{\partial}{\partial t}, q) = P(\hbar \frac{\partial}{\partial t_{[1,n]}}, q_{[1,n-1]}) := \sum_d \sum_{i_I} A_{i_I}^d (\hbar \frac{\partial}{\partial t_1})^{i_1} \cdots (\hbar \frac{\partial}{\partial t_n})^{i_n} q_1^{d_1} \cdots q_{n-1}^{d_{n-1}},$$

and call the differential operator $P(\hbar \frac{\partial}{\partial t}, q)$ the *quantization* of $P(x, q)$.

Recall from Equation (1.1) that $E_i^n = E_i^n(x, q)$ are the coefficients of the expansion of the polynomial $\det(I_n + \lambda M_{F\ell_n}) = \sum_i E_i^n \lambda^i$ in variable λ , with respect to the matrix

$$M_{F\ell_n} = \begin{pmatrix} x_1 & q_1 & & & \\ -1 & x_2 & q_2 & & \\ & -1 & x_3 & q_3 & \\ & & \ddots & \ddots & \ddots \\ & & & -1 & x_{n-1} & q_{n-1} \\ & & & & -1 & x_n \end{pmatrix}.$$

Proposition 3.4 ([Gi97, Theorem 4]; [Ki99b, Theorem II]). *Let $D_i^n(\hbar \frac{\partial}{\partial t}, q)$ be the quantization of $E_i^n(x, q)$. Then $D_i^n(\hbar \frac{\partial}{\partial t}, q)(J^{F\ell_n}(t, \hbar)) = 0$ for any $1 \leq i \leq n$.*

Definition 3.5. *Viewing $q_j = e^{t_j - t_{j+1}}$, we define the operators*

$$\begin{aligned} \mathbb{T}(e^{\frac{t}{\hbar}} \sum_{d \geq 0} q^d f_d(x, \hbar)) &:= e^{\frac{\iota^* t}{\hbar}} \sum_{d \geq 0} q^d f_d(\iota^* x, \hbar) \prod_{k=1}^{\int_d \sigma^{s_{n-1}}} (-\iota^* x_n + k\hbar), \\ \mathbb{S}(e^{\frac{t}{\hbar}} \sum_{d \geq 0} q^d f_d(x, \hbar)) &:= e^{\frac{\iota^* t}{\hbar}} \sum_{d \geq 0} q^d f_d(\iota^* x, \hbar) \prod_{k=1}^{(\int_d \sigma^{s_{n-1}}) - 1} (-\iota^* x_n + k\hbar), \\ \mathbb{P}_0(e^{\frac{t}{\hbar}} \sum_{d \geq 0} q^d f_d(x, \hbar)) &:= e^{\frac{\iota^* t}{\hbar}} \sum_{d \geq 0, d_{n-1}=0} q^d f_d(\iota^* x, \hbar), \end{aligned}$$

where the products $\prod_{k=1}^0$ and $\prod_{k=1}^{-1}$ are read off as multiplication by 1.

Lemma 3.6. *As operators from $\mathcal{H}(F\ell_n)$ to $\mathcal{H}(X)$, we have*

$$[\hbar \frac{\partial}{\partial t_k}, \mathbb{T}] = [\hbar \frac{\partial}{\partial t_k}, \mathbb{S}] = [\hbar \frac{\partial}{\partial t_k}, \mathbb{P}_0] = [q_l, \mathbb{T}] = [q_l, \mathbb{S}] = [q_l, \mathbb{P}_0] = 0$$

for any $1 \leq k \leq n$ and $1 \leq l \leq n-2$. Moreover, we have

$$(3.3) \quad -\hbar \frac{\partial}{\partial t_n} \circ q_{n-1} \circ \mathbb{T} = \mathbb{T} \circ q_{n-1}, \quad q_{n-1} \circ \mathbb{T} = \mathbb{S} \circ q_{n-1}, \quad \mathbb{T} = -\mathbb{S} \circ \hbar \frac{\partial}{\partial t_n} + (1 + \iota^* x_n) \mathbb{P}_0,$$

as operators acting on $e^{t/\hbar} \sum_{d \geq 0} q^d f_d(x, \hbar)$ where f_d are independent of t_i .

Proof. Note that $\int_d \sigma^{s_{n-1}} = d_{n-1}$ and that the operators $\mathbb{T}$ and $\mathbb{S}$ are affected only by the power d_{n-1} of $q_{n-1} = e^{t_{n-1}-t_n}$. Thus the identities in the first half of the statement hold.

$$\begin{aligned} q_{n-1} \circ \mathbb{T}(e^{t/\hbar} \sum_{d \geq 0} q^d f_d(x, \hbar)) &= e^{\iota^* t/\hbar} \sum_{d=(d_1, \dots, d_{n-1}), d_{n-1} \geq 1} q^d g_d(\iota^* x, \hbar) \prod_{k=1}^{d_{n-1}-1} (-\iota^* x_n + k\hbar) \\ &= \mathbb{S} \circ q_{n-1}(e^{t/\hbar} \sum_{d \geq 0} q^d f_d(x, \hbar)), \end{aligned}$$

where $g_d(\iota^* x, \hbar) = f_{(d_1, \dots, d_{n-2}, d_{n-1}-1)}(\iota^* x, \hbar)$. Thus the second identity in Equation (3.3) holds. Denoting $g_d(x, \hbar) = f_{(d_1, \dots, d_{n-2}, d_{n-1}-1)}(x, \hbar)$, we have

$$\begin{aligned} \mathbb{T} \circ q_{n-1}(e^{t/\hbar} \sum_{d \geq 0} q^d f_d(x, \hbar)) &= \mathbb{T}(e^{t/\hbar} \sum_{d=(d_1, \dots, d_{n-1}), d_{n-1} \geq 1} q^d g_d(x, \hbar)) \\ &= e^{\iota^* t/\hbar} \sum_{d=(d_1, \dots, d_{n-1}), d_{n-1} \geq 1} q^d g_d(\iota^* x, \hbar) \prod_{k=1}^{d_{n-1}} (-\iota^* x_n + k\hbar) \\ &= -\hbar \frac{\partial}{\partial t_n} \circ q_{n-1} \circ \mathbb{T}(e^{t/\hbar} \sum_{d \geq 0} q^d f_d(x, \hbar)). \end{aligned}$$

Therefore the first identity in Equation (3.3) holds as well.

The last identity in Equation (3.3) follows directly from the definition of the operators $\mathbb{S}, \mathbb{T}, \mathbb{P}_0$. Precisely,

$$\begin{aligned} & -\mathbb{S} \circ \hbar \frac{\partial}{\partial t_n}(e^{t/\hbar} \sum_{d \geq 0} q^d f_d(x, \hbar)) \\ &= (-\iota^* x_n) \cdot e^{\iota^* t/\hbar} \sum_{d \geq 0, d_{n-1}=0} q^d f_d(\iota^* x, \hbar) + (-\iota^* x_n + d_{n-1}\hbar) \cdot e^{\iota^* t/\hbar} \sum_{d \geq 0, d_{n-1} \geq 1} q^d f_d(\iota^* x, \hbar) \\ &= \mathbb{T}(e^{t/\hbar} \sum_{d \geq 0} q^d f_d(x, \hbar)) - (1 + \iota^* x_n) \mathbb{P}_0(e^{t/\hbar} \sum_{d \geq 0} q^d f_d(x, \hbar)). \end{aligned}$$

□

Proposition 3.7. *For the quantizations $D_i^n(\hbar \frac{\partial}{\partial t_{[1,n]}}, q_{[1,n-1]})$ of $E_i^n(x, q)$,*

$$D_i^n(\hbar \frac{\partial}{\partial t_{[1,n]}}, q_{[1,n-2]}, (-\hbar \frac{\partial}{\partial t_n}) \circ q_{n-1})(I_{F\ell_n, X}(\iota^* t, \hbar)) = 0$$

holds for $1 \leq i \leq n-1$. Moreover, we have

$$(3.4) \quad (-D_{n-1}^{n-1} + D_{n-2}^{n-2} q_{n-1})(I_{F\ell_n, X}(\iota^* t, \hbar)) = 0.$$

Proof. It follows directly from the definition that $\mathbb{T}(J^{F\ell_n}(t, \hbar)) = I_{F\ell_n, X}(\iota^* t, \hbar)$. By Proposition 3.4 and Lemma 3.6, for $1 \leq i \leq n-1$, we obtain

$$D_i^n(\hbar \frac{\partial}{\partial t_{[1,n]}}, q_{[1,n-2]}, (-\hbar \frac{\partial}{\partial t_n}) \circ q_{n-1})(\mathbb{T}(J^{F\ell_n}(t, \hbar))) = \mathbb{T} \circ D_i^n(\hbar \frac{\partial}{\partial t}, q)(J^{F\ell_n}(t, \hbar)) = 0.$$

By Lemma 3.6, all $q_l, \frac{\partial}{\partial t_k}$ commute with $\mathbb{T}, \mathbb{S}, \mathbb{P}_0$ for $1 \leq l \leq n-2, 1 \leq k \leq n-1$. It follows that $[D_a^n, \mathbb{S}] = 0, [D_a^n, \mathbb{T}] = 0$ and $[D_a^n, \mathbb{P}_0] = 0$ for $a \in \{n-1, n-2\}$. Using the identities in Equation (3.3), we have

$$\begin{aligned} & (-D_{n-1}^{n-1} + D_{n-2}^{n-2} q_{n-1})(I_{F\ell_n, X}(\iota^* t, \hbar)) \\ &= (-D_{n-1}^{n-1} + D_{n-2}^{n-2} q_{n-1})(\mathbb{T}(J^{F\ell_n}(t, \hbar))) \\ &= (D_{n-1}^{n-1} \circ (\mathbb{S} \circ \hbar \frac{\partial}{\partial t_n} - (1 + \iota^* x_n) \mathbb{P}_0) + D_{n-2}^{n-2} \circ \mathbb{S} \circ q_{n-1})(J^{F\ell_n}(t, \hbar)) \\ &= (\mathbb{S} \circ (\hbar \frac{\partial}{\partial t_n}) \circ D_{n-1}^{n-1} + \mathbb{S} \circ D_{n-2}^{n-2} \circ q_{n-1} - (1 + \iota^* x_n) \mathbb{P}_0 \circ D_{n-1}^{n-1})(J^{F\ell_n}(t, \hbar)) \\ &= \mathbb{S}(D_n^n(J^{F\ell_n}(t, \hbar))) - (1 + \iota^* x_n) \mathbb{P}_0 \circ D_{n-1}^{n-1}(J^{F\ell_n}(t, \hbar)) \\ &= -(1 + \iota^* x_n) \mathbb{P}_0 \circ D_{n-1}^{n-1}(J^{F\ell_n}(t, \hbar)). \end{aligned}$$

Here the fourth equality holds by noting $D_n^n = (\hbar \frac{\partial}{\partial t_n}) D_{n-1}^{n-1} + D_{n-2}^{n-2} q_{n-1}$ and $D_n^n(J^{F\ell_n}) = 0$. Write $D_{n-1}^{n-1}(J^{F\ell_n}(t, \hbar)) = \sum_{i \geq 0} q_{n-1}^i A_i$, where each A_i is independent of q_{n-1} . Then it follows from $0 = D_n^n(J^{F\ell_n}) = (\hbar \frac{\partial}{\partial t_n} \circ D_{n-1}^{n-1} + D_{n-2}^{n-2} \circ q_{n-1})(J^{F\ell_n})$ that $\hbar \frac{\partial}{\partial t_n}(A_0) = x_n A_0 = 0$. By the projection formula, we have $\int_X 1_X \cup \iota^*(a \cup b) = \int_{F\ell_n} \iota_*(1_X) \cup (a \cup b) = \int_{F\ell_n} (-x_n) \cup a \cup b$. Thus if $x_n \cup a = 0 \in H^*(F\ell_n)$, then $\iota^* a = 0 \in H^*(X)$, following from the surjectivity of ι^* . Consequently, we have $\mathbb{P}_0 \circ D_{n-1}^{n-1}(J^{F\ell_n}(t, \hbar)) = \iota^* A_0 = 0$. $\square$

Lemma 3.8. *Both of the following identities hold.*

$$(\hbar \frac{\partial}{\partial t_{n-1}} + q_{n-1}) J^X(\iota^* t, \hbar) = e^{-\frac{q_{n-1}}{\hbar}} (\hbar \frac{\partial}{\partial t_{n-1}}) I_{F\ell_n, X}(\iota^* t, \hbar),$$

$$(\hbar \frac{\partial}{\partial t_n} - q_{n-1}) J^X(\iota^* t, \hbar) = e^{-\frac{q_{n-1}}{\hbar}} (\hbar \frac{\partial}{\partial t_n}) I_{F\ell_n, X}(\iota^* t, \hbar).$$

Proof. By Proposition 3.3, $J^X(\iota^*t, \hbar) = e^{-\frac{q_{n-1}}{\hbar}} I_{F\ell_{n,X}}(\iota^*t, \hbar)$. Then the statement follows from direct calculations by the Leibniz rule:

$$\begin{aligned} (\hbar \frac{\partial}{\partial t_{n-1}})(J^X) &= (-q_{n-1})e^{-\frac{q_{n-1}}{\hbar}} I_{F\ell_{n,X}} + e^{-\frac{q_{n-1}}{\hbar}} (\hbar \frac{\partial}{\partial t_{n-1}})(I_{F\ell_{n,X}}), \\ (\hbar \frac{\partial}{\partial t_n})(J^X) &= q_{n-1}e^{-\frac{q_{n-1}}{\hbar}} I_{F\ell_{n,X}} + e^{-\frac{q_{n-1}}{\hbar}} (\hbar \frac{\partial}{\partial t_n})(I_{F\ell_{n,X}}). \end{aligned}$$

□

3.3. Proofs of Theorem 1.2 and Theorem 1.7. As shown in [ST97, Proposition 2.2] by Siebert and Tian, the quantum cohomology $QH^*(X)$ of the Fano manifold X is of the form $\mathbb{C}[x, q]/I_q$, provided that $H^*(X) = \mathbb{C}[x]/I$ and I_q is generated by the corresponding quantized relations in $QH^*(X)$ of the generators of I for $H^*(X)$. Such relations can be found, by the following standard method due to Givental.

Proposition 3.9 ([Gi96, Corollary 6.4]). *If a differential operator $P(\hbar \frac{\partial}{\partial t_i}, e^{t_i - t_{i+1}}, \hbar)$ satisfies the equation $P(\hbar \frac{\partial}{\partial t_i}, e^{t_i - t_{i+1}}, \hbar)(J^Y(t, \hbar)) = 0$, then $P(x_i, q_i, 0) = 0$ in $QH^*(Y)$.*

We will first describe $QH^*(X)$, and then prove Theorem 1.2 as a consequence, by using [FP97, Proposition 11], which is a variation of [ST97, Proposition 2.2].

Theorem 3.10. *Let $\chi_j := \chi_j(\iota^*x, q)$ be given by $\det(I_n + \lambda \tilde{M}_X) = \sum_{i=0}^n \chi_i \lambda^i$ with respect to the matrix*

$$\tilde{M}_X = \begin{pmatrix} \iota^*x_1 & q_1 & & & & \\ -1 & \iota^*x_2 & q_2 & & & \\ & -1 & \iota^*x_3 & q_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & \iota^*x_{n-1} + q_{n-1} & q_{n-1}(-\iota^*x_n + q_{n-1}) \\ & & & & -1 & \iota^*x_n - q_{n-1} \end{pmatrix}.$$

Then $\chi_n = (\iota^*x_n - q_{n-1})E_{n-1}^{n-1}(\iota^*x_1, \dots, \iota^*x_{n-1}, q_1, \dots, q_{n-2})$. Furthermore, the quantum cohomology ring of X is canonically given by

$$QH^*(X) = \mathbb{C}[\iota^*x_1, \dots, \iota^*x_n, q_1, \dots, q_{n-1}] \Big/ \left(\chi_1, \chi_2, \dots, \chi_{n-1}, \frac{\chi_n}{\iota^*x_n - q_{n-1}} \right).$$

Proof. By Proposition 3.7 and Lemma 3.8, for any $1 \leq i \leq n-1$, we have

$$\begin{aligned} 0 &= e^{-\frac{q_{n-1}}{\hbar}} D_i^n(\hbar \frac{\partial}{\partial t_{[1,n]}}, q_{[1,n-2]}, (-\hbar \frac{\partial}{\partial t_n}) \circ q_{n-1})(I_{F\ell_{n,X}}) \\ &= D_i^n(\hbar \frac{\partial}{\partial t_{[1,n-2]}}, \hbar \frac{\partial}{\partial t_{n-1}} + q_{n-1}, \hbar \frac{\partial}{\partial t_n} - q_{n-1}, q_{[1,n-2]}, (-\hbar \frac{\partial}{\partial t_n} - q_{n-1}) \circ q_{n-1} + \hbar G_1 + \hbar^2 G_2)(J^X) \end{aligned}$$

where $G_1 = G_1(\hbar \frac{\partial}{\partial t}, q)$ and $G_2 = G_2(\hbar \frac{\partial}{\partial t}, q)$ are differential operators. Therefore by using Proposition 3.9, we have

$D_i^n(\iota^*x_1, \dots, \iota^*x_{n-2}, \iota^*x_{n-1} + q_{n-1}, \iota^*x_n - q_{n-1}, q_1, q_2, \dots, q_{n-2}, q_{n-1}(-\iota^*x_n + q_{n-1})) = 0$ in $QH^*(X)$. That is, $\chi_i(\iota^*x, q) = 0$ holds in $QH^*(X)$.

The next two equalities follow directly from Lemma 3.8 and Proposition 3.7 respectively.

$$\begin{aligned} & \left(-D_{n-1}^{n-1} \left(\hbar \frac{\partial}{\partial t_{[1,n-2]}}, \hbar \frac{\partial}{\partial t_{n-1}} + q_{n-1}, q_{[1,n-2]} \right) + D_{n-2}^{n-2} \left(\hbar \frac{\partial}{\partial t_{[1,n-2]}}, q_{[1,n-2]} \right) q_{n-1} + \hbar H_1 \right) (J^X) \\ &= e^{-\frac{q_{n-1}}{\hbar}} \left(-D_{n-1}^{n-1} \left(\hbar \frac{\partial}{\partial t_{[1,n-2]}}, q_{[1,n-2]} \right) + D_{n-2}^{n-2} \left(\hbar \frac{\partial}{\partial t_{[1,n-2]}}, q_{[1,n-2]} \right) q_{n-1} \right) (I_{F\ell_n, X}) = 0, \end{aligned}$$

where $H_1 = H_1(\hbar \frac{\partial}{\partial t}, q)$ is a differential operator. Therefore, by Proposition 3.9, in $QH^*(X)$ we have

$$-E_{n-1}^{n-1}(\iota^* x_1, \dots, \iota^* x_{n-2}, \iota^* x_{n-1} + q_{n-1}, q_1, \dots, q_{n-2}) + E_{n-2}^{n-2}(\iota^* x_1, \dots, \iota^* x_{n-2}, q_1, \dots, q_{n-2}) q_{n-1} = 0.$$

The left hand side of the above equality equals $-E_{n-1}^{n-1}(\iota^* x_1, \dots, \iota^* x_{n-1}, q_1, \dots, q_{n-2})$ and hence equals $-\frac{\chi_n(\iota^* x, q)}{\iota^* x_n - q_{n-1}}$ by Laplace expansion of matrices. Then we are done by using Proposition 2.2 and [ST97, Proposition 2.2]. $\square$

Now we are ready to show the ring presentation of the integral quantum cohomology $QH^*(X, \mathbb{Z})$ in **Theorem 1.2**, as well as the ring homomorphism in **Theorem 1.7**.

Proof of Theorem 1.2. Abusing the notation for $\iota^* x_i$ and x_i , we notice $M_X = A \tilde{M}_X A^{-1}$ with $A = I_n + q_{n-1} B_{n-1, n}$, where $B_{n-1, n}$ is the matrix with 1 in the $(n-1, n)$ -entry and zeros elsewhere. Hence, the matrices M_X and $\tilde{M}_X$ have the same characteristic polynomial, implying $\hat{E}_i^n = \chi_i$ for all $1 \leq i \leq n$. Again we note $\frac{\chi_n}{x_n - q_{n-1}} = E_{n-1}^{n-1}$. Thus all $\hat{E}_1^n, \dots, \hat{E}_{n-1}^n, E_{n-1}^{n-1}$ vanish in $QH^*(X) = QH^*(X, \mathbb{Z}) \otimes \mathbb{C}$, so do in $QH^*(X, \mathbb{Z})$. Their evaluations at $q = 0$ give the ideal $I = (e_1^n, \dots, e_{n-1}^n, e_{n-1}^{n-1})$, providing the ring presentation $H^*(X, \mathbb{Z}) = \mathbb{Z}[x_1, \dots, x_n]/I$ by Proposition 2.2. Therefore the statement follows from [FP97, Proposition 11]. $\square$

Proof of Theorem 1.7. Define $\iota_q^*(x_i) = x_i$ and $\iota_q^*(q_i) = q_i$ for $1 \leq i \leq n-2$. Define

$$\iota_q^*(x_{n-1}) = x_{n-1} + q_{n-1}, \quad \iota_q^*(x_n) = x_n - q_{n-1}, \quad \iota_q^*(q_{n-1}) = -q_{n-1}x_n + q_{n-1}^2.$$

This induces a ring homomorphism $\iota_q^*: \mathbb{Z}[x, q] \rightarrow \mathbb{Z}[x, q]$ with

$$\iota_q^*(E_i^n(x, q)) = E_i^n(x_1, \dots, x_{n-2}, \iota_q^*(x_{n-1}), \iota_q^*(x_n), q_1, \dots, q_{n-2}, \iota_q^*(q_{n-1})) = \chi_i(x, q)$$

for all $1 \leq i \leq n$. That is, $\iota_q^*(E_i^n(x, q)) = \hat{E}_i^n(x, q)$ for $1 \leq i \leq n-1$, and $\iota_q^*(E_n^n(x, q)) = (x_n - q_{n-1})E_{n-1}^{n-1}(x, q)$ by the proof of Theorem 1.2. Hence it further induces a ring homomorphism

$$\iota_q^*: QH^*(F\ell_n, \mathbb{Z}) = \frac{\mathbb{Z}[x, q]}{(E_1^n, \dots, E_n^n)} \longrightarrow \frac{\mathbb{Z}[x, q]}{(\hat{E}_1^n, \dots, \hat{E}_{n-1}^n, E_{n-1}^{n-1})} = QH^*(X, \mathbb{Z}).$$

Hence, we are done, by noting $\sigma^{s_i} = [x_1 + \dots + x_i]$ on the left hand side, and $\xi^{s_i} = [x_1 + \dots + x_i]$ on the right hand side. (In particular, $\xi^{s_{n-1}} = [-x_n]$.) $\square$

4. QUANTUM MONK-CHEVALLEY FORMULA FOR X

Let $Y \in \{F\ell_n, X\}$. By d we always mean $d = (d_1, \dots, d_{n-1}) = \sum_{i=1}^{n-1} d_i [X_{s_i}] \in H_2(Y, \mathbb{Z})$. For $1 \leq a < b \leq n$, we denote

$$(4.1) \quad \alpha_{ab} := \sum_{i=a}^{b-1} [X_{s_i}], \quad q_{ab} := q_a q_{a+1} \cdots q_{b-1} = q^{\alpha_{ab}}.$$

The following quantum Monk's formula for $F\ell_n$ was proved in [FGP97, Theorem 1.3] (see [FW04] for the quantum Chevalley formula for general G/P), where $\prec_r$ (resp. $\prec_r^q$) denotes the (quantum) r -Bruhat order defined in Equation (2.2) (resp. Equation (2.3)).

Proposition 4.1 (Quantum Monk's formula). *For $u \in S_n$ and $1 \leq r < n$, in $QH^*(F\ell_n, \mathbb{Z})$ we have*

$$\sigma^{s_r} \star \sigma^u = \sum_{u \prec_r ut_{ab}} \sigma^{ut_{ab}} + \sum_{u \prec_r^q ut_{ck}} q_{ck} \sigma^{ut_{ck}}.$$

Since $\{\sigma^u\}_u$ form a $\mathbb{Z}[q]$ -basis of $QH^*(F\ell_n, \mathbb{Z})$, we can write

$$\sigma^v \star \sigma^u = \sum_{w,d} N_{v,u}^{w,d} q^d \sigma^w.$$

We further denote by $N_{v,u \cup u'}^{w,d}$ the coefficient of $q^d \sigma^w$ in the product $\sigma^v \star (\sigma^u \cup \sigma^{u'})$. In particular if $v = s_r$ and $d \neq 0$, then the classical Monk's formula expresses $\sigma^u \cup \sigma^{u'}$ as a sum of distinct $\sigma^{\tilde{u}}$, which contribute nonzero quantum Schubert structure constants (equal to 1) only if $d = \alpha_{ab}$ and $\tilde{u} = wt_{ab}$. Therefore, the quantum Monk-Chevalley formula in the form of Theorem 1.3 is equivalent to the following description.

Theorem 4.2. *Let $1 \leq r \leq n-1$ and $u \in S_n$ with $u \leq w_0 s_{n-1}$. In $QH^*(X)$, we have*

$$\xi^{s_r} \star \xi^u = \sum_{w \leq w_0 s_{n-1}} N_{s_r, u}^{w,0} \xi^w + \sum_{d_{n-1}=0} N_{s_r, u}^{w,d} \xi^w q^d + \sum_{d_{n-1}=1} N_{s_r, (u \cup s_{n-1})}^{w,d} \xi^w q^d - \delta_{r,n-1} q_{n-1} \xi^u.$$

The constraint $w \leq w_0 s_{n-1}$ is a priori required in the second and third sums of the formula $\xi^{s_r} \star \xi^u$, but turns out to be redundant (see Remark 4.12).

This section is devoted to a proof of the above theorem. We use the current form in Theorem 4.2, to indicate our approach: degree- d Gromov-Witten invariants of X with $d_{n-1} \leq 1$ can be reduced those of $F\ell_n$, while those with $d_{n-1} \geq 2$ vanish.

4.1. Degree- d Gromov-Witten invariants with $d_{n-1} \leq 1$. In this subsection, we compute Gromov-Witten invariants $\langle \beta, \gamma \rangle_{0,2,d}$ of X with $d_{n-1} \leq 1$.

4.1.1. Unobstructedness of moduli spaces. Recall the line bundle $\mathcal{L}_{\omega_{n-1}}$ over $F\ell_n$ defined in (2.6), the zero locus of a section s of which defines the smooth Schubert divisor X .

Lemma 4.3. *Let $f : \mathbb{P}^1 \rightarrow X$ be a morphism satisfying $f_*[\mathbb{P}^1] = d$ with $d_{n-1} \leq 1$. Then we have $H^1(\mathbb{P}^1, f^* T_X) = 0$.*

Proof. Set $E := \mathcal{L}_{\omega_{n-1}}|_X \cong N_{X/F\ell_n}$. The exact sequence

$$0 \rightarrow T_X \rightarrow T_{F\ell_n}|_X \rightarrow E \rightarrow 0,$$

pulling back to $\mathbb{P}^1$, and then tensoring with $\mathcal{O}_{\mathbb{P}^1}(-1)$, induces a long exact sequence:

$$\cdots \rightarrow H^0(\mathbb{P}^1, (f^*E)(-1)) \rightarrow H^1(\mathbb{P}^1, (f^*T_X)(-1)) \rightarrow H^1(\mathbb{P}^1, (f^*T_{F\ell_n}|_X)(-1)) \rightarrow \cdots.$$

Since $T_{F\ell_n}$ is globally generated and vector bundles over $\mathbb{P}^1$ split, we have

$$(f^*T_{F\ell_n}|_X)(-1) = \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(a_i - 1)$$

with $a_i \geq 0$ for all i . Thus $H^1(\mathbb{P}^1, (f^*T_{F\ell_n}|_X)(-1)) = 0$. On the other hand, $(f^*E)(-1) = \mathcal{O}_{\mathbb{P}^1}(d_{n-1} - 1)$ with $d_{n-1} \leq 1$, which implies

$$(4.2) \quad \dim H^1(\mathbb{P}^1, (f^*T_X)(-1)) \leq 1.$$

Suppose $(f^*T_X)(-1) = \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(b_i - 1)$, Equation (4.2) implies all $b_i \geq 0$ except at most one $b_{i_0} = -1$. In other words, $f^*T_X = \bigoplus_{i \neq i_0} \mathcal{O}_{\mathbb{P}^1}(b_i) \oplus \mathcal{O}(-1)$ with $b_i \geq 0$ or $f^*T_X = \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(b_i)$ with $b_i \geq 0$, which implies $H^1(\mathbb{P}^1, f^*T_X) = 0$. $\square$

Proposition 4.4. *Let $d \geq 0$ with $d_{n-1} \leq 1$. Then the virtual fundamental class $[\overline{\mathcal{M}}_{0,k}(X, d)]^{\text{vir}}$ coincides with the usual fundamental class $[\overline{\mathcal{M}}_{0,k}(X, d)]$.*

Proof. Let $ft : \overline{\mathcal{M}}_{0,k+1}(X, d) \rightarrow \overline{\mathcal{M}}_{0,k}(X, d)$ be the forgetful morphism obtained by forgetting the last marked point (which is the universal curve over $\overline{\mathcal{M}}_{0,k}(X, d)$ [BM96]). For any $(f : C \rightarrow X; pt_1, \dots, pt_k) \in \overline{\mathcal{M}}_{0,k}(X, d)$, by using Lemma 4.3 and following (the proof of) [FP97, Lemma 10], we have $H^1(C, f^*T_X) = 0$. Therefore we have $R^1ft_*ev_{k+1}^*T_X = 0$. Then by [BeFa97, Proposition 7.3], $[\overline{\mathcal{M}}_{0,k}(X, d)]^{\text{vir}}$ is the usual fundamental class $[\overline{\mathcal{M}}_{0,k}(X, d)]$. $\square$

4.1.2. Computation of Gromov-Witten invariants with $d_{n-1} = 1$. In this subsection, we always assume $d \geq 0$ with $d_{n-1} = 1$. For a decomposition $d = d' + d''$, we always require both $d' \geq 0$ and $d'' \geq 0$. Recall the natural projection map $pr : F\ell_n \rightarrow Gr(n-1, n) = (\mathbb{P}^{n-1})^*$. Note that $H := pr(X)$ is a hyperplane in $(\mathbb{P}^{n-1})^*$. The inclusion $\iota : X \hookrightarrow F\ell_n$ induces a natural inclusion $\overline{\mathcal{M}}_{0,2}(X, d) \hookrightarrow \overline{\mathcal{M}}_{0,2}(F\ell_n, \iota_*d)$ denoted as ι by abuse of notation. Denote

$$\begin{aligned} \mathcal{A}^\circ &:= \{(f : C \rightarrow F\ell_n; pt_1, pt_2) \in \overline{\mathcal{M}}_{0,2}(F\ell_n, \iota_*d) \mid f(pt_i) \in X, pr(f(pt_1)) \neq pr(f(pt_2))\}; \\ \mathcal{B} &:= \{(f : C \rightarrow F\ell_n; pt_1, pt_2) \in \overline{\mathcal{M}}_{0,2}(F\ell_n, \iota_*d) \mid pr(f(pt_1)) = pr(f(pt_2)) \in H\}. \end{aligned}$$

Since $d_{n-1} = 1$, $pr \circ f(C)$ is a line in $(\mathbb{P}^{n-1})^*$, containing $pr(f(pt_1)), pr(f(pt_2))$. It follows that $f(C) \subset X$ for any stable map f in $\mathcal{A}^\circ$, because the line through the two distinct points $pr(f(pt_1)), pr(f(pt_2))$ in H must lie in H . Note that $\mathcal{A}^\circ$ is a Zariski open dense subset of $\iota(\overline{\mathcal{M}}_{0,2}(X, d))$. Thus for the evaluation maps $ev_i : \overline{\mathcal{M}}_{0,2}(F\ell_n, \iota_*d) \rightarrow F\ell_n$, we have

$$(4.3) \quad ev_1^{-1}(X) \cap ev_2^{-1}(X) = \mathcal{A} \cup \mathcal{B}, \quad \text{with} \quad \mathcal{A} := \overline{\mathcal{A}^\circ} = \iota(\overline{\mathcal{M}}_{0,2}(X, d)).$$

Here we note that $\overline{\mathcal{M}}_{0,2}(X, d)$ is proper, so $\iota(\overline{\mathcal{M}}_{0,2}(X, d))$ is closed in $\overline{\mathcal{M}}_{0,2}(F\ell_n, \iota_*d)$.

Denote by $X_{k,d} = ev_k^{-1}(X) \subset \overline{\mathcal{M}}_{0,k}(F\ell_n, \iota_*d)$ the space of stable maps whose last marked point is mapped to X . A stable map $(f : C \rightarrow F\ell_n, pt_1, pt_2)$ in $\mathcal{B}$ is of the form

- (1) $C = C_1 \cup C_2$, where the marked points pt_1, pt_2 are contained in C_1 ;
- (2) $f_1 = f|_{C_1}$ is a stable map to $pr^{-1}(p)$ for some $p \in H$;
- (3) $f_2 = f|_{C_2}$ is a stable map to $F\ell_n$.

Actually, we can take C_1 to be the union of components of C which is maximal with respect to (1) and (2). The union of the remaining components C_2 is connected since the image $pr(f(C))$ is a line. As a result, $\mathcal{B}$ can be written as the following union, in analogy with the boundary divisors of the moduli space of stable maps (see e.g. [FP97, Section 6.2]).

$$(4.4) \quad \mathcal{B} = \bigcup_{\substack{d'+d''=d \\ d'_{n-1}=0}} \mathcal{B}_{d',d'', \text{ with } \mathcal{B}_{d',d''} := X_{3,d'} \times_X X_{1,d''}.$$

Here the fiber product over X stands for the constraint $f_1(pt_3) = f_2(pt) \in X$ for the nodal point of C . By direct calculations, we have

$$\dim \mathcal{B}_{d',d''} = \langle \iota_* d', c_1(F\ell_n) \rangle + \dim X + \langle \iota_* d'', c_1(F\ell_n) \rangle - 2 = \langle \iota_* d, c_1(F\ell_n) \rangle + \dim X - 2.$$

That is, $\mathcal{B}$ is of pure dimension. Note $\mathcal{A}$ and $\mathcal{B}$ are both of codimension 2 in $\overline{\mathcal{M}}_{0,2}(F\ell_n, \iota_* d)$.

Proposition 4.5. *Let $d \geq 0$ with $d_{n-1} = 1$. For $u, w \in S_n$ with $u, w \leq w_0 s_{n-1}$, we have*

$$\langle \xi^u, \xi^w \rangle_{0,2,d}^X = \langle \sigma^u \cup \sigma^{s_{n-1}}, \sigma^w \cup \sigma^{s_{n-1}} \rangle_{0,2,\iota_* d}^{F\ell_n} - \delta_{q^d, q_{n-1}} \langle \sigma^u, \sigma^w, \sigma^{s_{n-1}} \rangle_{0,3,0}^{F\ell_n}.$$

Proof. Note that the Schubert divisor X defines the Schubert class $\sigma^{s_{n-1}} = PD([X])$, and $ev_i^{-1}(X)$ are divisors of $\overline{\mathcal{M}}_{0,2}(F\ell_n, \iota_* d)$. As discussed above, we have the decomposition $ev_1^{-1}(X) \cap ev_2^{-1}(X) = \overline{\mathcal{M}}_{0,2}(X, d) \cup \bigcup_{(d',d'')} X_{3,d'} \times_X X_{1,d''}$ with each irreducible component of codimension 2 in $\overline{\mathcal{M}}_{0,2}(F\ell_n, \iota_* d)$. Hence, we have

$$\begin{aligned} & \langle \sigma^u \cup \sigma^{s_{n-1}}, \sigma^w \cup \sigma^{s_{n-1}} \rangle_{0,2,\iota_* d}^{F\ell_n} \\ &= \int_{[\overline{\mathcal{M}}_{0,2}(F\ell_n, \iota_* d)]} ev_1^* \sigma^u \cup ev_2^* \sigma^w \cup ev_1^* \sigma^{s_{n-1}} \cup ev_2^* \sigma^{s_{n-1}} \\ &= \int_{[ev_1^{-1}(X) \cap ev_2^{-1}(X)]} ev_1^* \sigma^u \cup ev_2^* \sigma^w \\ &= \int_{[\overline{\mathcal{M}}_{0,2}(X, d)]} ev_1^* \sigma^u \cup ev_2^* \sigma^w + \sum_{(d',d'')} \int_{[X_{3,d'} \times_X X_{1,d''}]} ev_1^* \sigma^u \cup ev_2^* \sigma^w. \end{aligned}$$

The first term in the last equality is equal to

$$\int_{[\overline{\mathcal{M}}_{0,2}(X, d)]} \iota^*(ev_1^* \sigma^u \cup ev_2^* \sigma^w) = \int_{[\overline{\mathcal{M}}_{0,2}(X, d)]} ev_1^* \iota^* \sigma^u \cup ev_2^* \iota^* \sigma^w = \langle \xi^u, \xi^w \rangle_{0,2,d}^X.$$

Note $ev_1 \times ev_2 : X_{3,d'} \times_X X_{1,d''} \rightarrow X \times X$ factors through

$$X_{3,d'} \times_X X_{1,d''} \xrightarrow{\phi} X_{3,d'} \xrightarrow{ev_1 \times ev_2} X \times X,$$

where ϕ is a fibration with generic fibers of dimension the same as that of the generic fiber of the evaluation map $\overline{\mathcal{M}}_{0,1}(F\ell_n, \iota_* d'') \rightarrow F\ell_n$, namely of dimension $\langle \iota_* d'', c_1(T_{F\ell_n}) \rangle + 1 - 3$. Hence, $\phi_*[X_{3,d'} \times_X X_{1,d'']} = 0$ unless $\langle \iota_* d'', c_1(T_{F\ell_n}) \rangle = 2$, i.e. $d'' = [X_{s_{n-1}}] = \alpha_{n-1,n}$. In this case, $\overline{\mathcal{M}}_{0,1}(F\ell_n, d'') \cong F\ell_n$ ([Str02]), thus $X_{3,d'} \times_X X_{1,d''} \rightarrow X_{3,d'}$ is of degree 1. Hence, we have

$$\begin{aligned}
\sum_{(d', d'')} \int_{[X_{3, d'} \times_X X_{1, d''}]} ev_1^* \sigma^u \cup ev_2^* \sigma^w &= \sum_{(d', d'')} \int_{[X_{3, d'} \times_X X_{1, d''}]} \phi^* \circ (ev_1 \times ev_2)^* (\sigma^u \boxtimes \sigma^w) \\
&= \sum_{(d', \alpha_{n-1, n})} \int_{[X_{3, d'}]} (ev_1 \times ev_2)^* (\sigma^u \boxtimes \sigma^w) \\
&= \sum_{(d', \alpha_{n-1, n})} \int_{[\overline{\mathcal{M}}_{0,3}(F\ell_n, \iota_* d')] } ev_1^* \sigma^u \cup ev_2^* \sigma^w \cup ev_3^* \sigma^{s_{n-1}}.
\end{aligned}$$

Recall that $d'_{n-1} = 0$. If $d' \neq 0$, then by the divisor axiom, we have $\int_{[\overline{\mathcal{M}}_{0,3}(F\ell_n, \iota_* d')] } ev_1^* \sigma^u \cup ev_2^* \sigma^w \cup ev_3^* \sigma^{s_{n-1}} = \int_{\iota_* d'} \sigma^{s_{n-1}} \int_{\overline{\mathcal{M}}_{0,2}(F\ell_n, \iota_* d')} ev_1^* \sigma^u \cup ev_2^* \sigma^w = 0$. Hence, the above sum is nonzero only if $d = \alpha_{n-1, n} = d''$, in which case we have

$$\sum_{(d', d'')} \int_{[X_{3, d'} \times_X X_{1, d''}]} ev_1^* \sigma^u \cup ev_2^* \sigma^w = \int_{[\overline{\mathcal{M}}_{0,3}(F\ell_n, 0)]} ev_1^* \sigma^u \cup ev_2^* \sigma^w \cup ev_3^* \sigma^{s_{n-1}} = \langle \sigma^u, \sigma^w, \sigma^{s_{n-1}} \rangle_{0,3,0}^{F\ell_n}.$$

Hence, the statement follows. $\square$

4.1.3. Computation of Gromov-Witten invariants with $d_{n-1} = 0$. The analysis for $d_{n-1} = 0$ is similar to but much simpler than that for $d_{n-1} = 1$.

Proposition 4.6. *Let $d \geq 0$ with $d_{n-1} = 0$ and $d \neq 0$. For $u, w \in S_n$ with $u, w \leq w_0 s_{n-1}$, we have*

$$\langle \xi^u, \xi^w \rangle_{0,2,d}^X = \langle \sigma^u, \sigma^w \cup \sigma^{s_{n-1}} \rangle_{0,2,\iota_* d}^{F\ell_n}.$$

Proof. For $i \in \{1, 2\}$ and $(f : C \rightarrow F\ell_n; pt_1, pt_2) \in ev_i^{-1}(X) \subset \overline{\mathcal{M}}_{0,2}(F\ell_n, \iota_* d)$, we have $pr_*([f(C)]) = 0$, so that $pr(f(C))$ consists of a point in $\mathbb{P}^{n-1}$. Moreover, $pr(f(pt_1)) = pr(f(pt_2)) = pr(ev_i(f)) \in H$. It follows that $f(C) \subset X$. Hence, $ev_i^{-1}(X) = \iota(\overline{\mathcal{M}}_{0,2}(X, d))$ for any $i \in \{1, 2\}$. Recall $\xi^u = \iota^* \sigma^u$ and note $PD([ev_i^{-1}(X)]) = ev_i^* \sigma^{s_{n-1}}$. Hence, we have

$$\begin{aligned}
\langle \xi^u, \xi^w \rangle_{0,2,d}^X &= \int_{[\overline{\mathcal{M}}_{0,2}(X, d)]} ev_1^* \iota^* \sigma^u \cup ev_2^* \iota^* \sigma^w \\
&= \int_{[\overline{\mathcal{M}}_{0,2}(X, d)]} \iota^* (ev_1^* \sigma^u \cup ev_2^* \sigma^w) \\
&= \int_{[\iota(\overline{\mathcal{M}}_{0,2}(X, d))]} ev_1^* \sigma^u \cup ev_2^* \sigma^w \\
&= \int_{[\overline{\mathcal{M}}_{0,2}(F\ell_n, \iota_* d)]} ev_1^* \sigma^u \cup ev_2^* \sigma^w \cup ev_3^* \sigma^{s_{n-1}}.
\end{aligned}$$

$\square$

4.2. Vanishing of Gromov-Witten invariants with $d_{n-1} \geq 2$.

4.2.1. Vanishing by curve neighborhood technique. We use the curve neighborhood technique developed by Buch and Mihalcea [BM15], to show the vanishing of Gromov-Witten invariants of degree d with $d_{n-1} \geq 2$ and $d \neq \alpha_{in} + \alpha_{n-1,n}$.

Definition 4.7. Let $d \geq 0$ and $u \in S_n$, which further satisfies $u(n) \neq 1$ for $Y = X$. The curve neighborhood $\Gamma_d^Y(X_u)$ of X_u of degree d is a reduced subscheme of Y defined by

$$\Gamma_d^Y(X_u) = \text{ev}_2(\text{ev}_1^{-1}(X_u)).$$

The permutation $z_d \in S_n$ associated with $d \geq 0$ is defined by using the Hecke product $\bullet$ on S_n as follows. Note

$$w \bullet s_i = \begin{cases} ws_i, & \text{if } \ell(ws_i) > \ell(w), \\ w, & \text{otherwise.} \end{cases}$$

Take a sequence $(\alpha_{i_1 j_1}, \alpha_{i_2 j_2}, \dots, \alpha_{i_k j_k})$ of maximal elements $\alpha_{i_r j_r} = \sum_{m=i_r}^{j_r-1} [X_{s_m}]$ with respect to d ; that is, each $\alpha_{i_r j_r}$ is maximal in the sense $\alpha_{i_r j_r} \in A_r := \{\alpha_{ab} \mid d - \alpha_{ab} - \sum_{m=1}^{r-1} \alpha_{i_m j_m} \geq 0\}$ with $j_r - i_r = \max\{b - a \mid \alpha_{ab} \in A_r\}$. Then k depends only on d , and

$$z_d := t_{i_1 j_1} \bullet t_{i_2 j_2} \bullet \dots \bullet t_{i_k j_k} \in S_n$$

is also independent of the choice of a sequence of maximal elements with respect to d .

Proposition 4.8 ([BM15, Theorem 5.1]). $\Gamma_d^{F\ell_n}(X_u) = X_{u \bullet z_d}$, for $u \in S_n$ and $d \geq 0$.

Lemma 4.9. Let $d \geq 0$ with $d_{n-1} \geq 2$. Then we have $\ell(z_d) \leq \langle d, c_1(T_X) \rangle - 1$, with equality holding only if $d = \alpha_{in} + \alpha_{n-1,n}$ for some i .

Proof. Take a sequence $(\alpha_{i_1 j_1}, \alpha_{i_2 j_2}, \dots, \alpha_{i_k j_k})$ of maximal elements with respect to d . We may arrange that $j_r = n$ for $1 \leq r \leq d_{n-1}$ (by noting that the corresponding transpositions of the form t_{cn} are disjoint with that of the form t_{ab} with $b < n$; otherwise, $b = c$, and α_{an} would be a bigger element than α_{cn}). Note $\ell(t_{ab}) = 2b - 2a - 1 = 2|\alpha_{ab}| - 1$, $\ell(t_{an} \bullet t_{bn}) \leq \ell(t_{an}) + \ell(t_{bn}) - 1$ (since t_{an} (resp. t_{bn}) has a reduced expression ending (resp. starting) with s_{n-1}). Hence, $k \geq d_{n-1} \geq 2$, and we have

$$\begin{aligned} \ell(z_d) &\leq \ell(t_{i_1 n} \bullet \dots \bullet t_{i_{d_{n-1}} n}) + \ell(t_{i_{d_{n-1}+1} j_{d_{n-1}+1}} \bullet \dots \bullet t_{i_k j_k}) \\ &\leq \sum_{r=1}^{d_{n-1}} \ell(t_{i_r n}) - (d_{n-1} - 1) + \sum_{r=d_{n-1}+1}^k \ell(t_{i_r j_r}) \\ &= 2|d| - k - (d_{n-1} - 1) \\ &\leq 2|d| - d_{n-1} - (d_{n-1} - 1) = \langle d, c_1(T_X) \rangle - d_{n-1} + 1. \end{aligned}$$

Hence, $\ell(z_d) \leq \langle d, c_1(T_X) \rangle - 1$, with equality holding only if $k = d_{n-1} = 2$. When equality holds, all the above inequalities are equalities. In particular, $d = \alpha_{in} + \alpha_{jn}$ for some $i \leq j$, and $\ell(t_{in} \bullet (s_{n-1} t_{jn})) = \ell(t_{in} \bullet t_{jn}) = \ell(t_{in}) + \ell(t_{jn}) - 1$. Consequently $t_{in} \bullet (s_{n-1} t_{jn}) = t_{in}(s_{n-1} t_{jn})$, and it has a reduced expression ending with s_{n-1} . It follows that $t_{in}(s_{n-1} t_{jn})(n-1) > t_{in}(s_{n-1} t_{jn})(n)$. This results in a contradiction $i > t_{in}(j)$ in the case $j < n-1$. $\square$

Proposition 4.10. Let $d \geq 0$ satisfy $d_{n-1} \geq 2$ and $d \neq \alpha_{in} + \alpha_{n-1,n}$ for $1 \leq i \leq n-1$. Then for any $\beta, \gamma \in H^*(X)$, we have $\langle \beta, \gamma \rangle_{0,2,d} = 0$.

Proof. Take any $u, v \in S_n$ with $u, v \leq w_0 s_{n-1}$. Note $\Gamma_d^X(X_u) \subseteq \Gamma_d^{F\ell_n}(X_u) = X_{u \bullet z_d}$ by Proposition 4.8. Since $d_{n-1} \geq 2$ and $d \neq \alpha_{in} + \alpha_{n-1, n}$, by Lemma 4.9, we have

$$\dim \Gamma_d^X(X_u) \leq \dim(X_{u \bullet z_d}) = \ell(u \bullet z_d) \leq \ell(u) + \ell(z_d) < \ell(u) + \langle d, c_1(T_X) \rangle - 1.$$

Denote $PD([X_u])$ as $[X_u]$ by abuse of notation. Using projection formula, we have

$$\int_{[\overline{\mathcal{M}}_{0,2}(X, d)]^{\text{vir}}} ev_1^*([X_u]) \cup ev_2^*([X_v]) = \int_X (ev_2)_*(ev_1^*([X_u]) \cap [\overline{\mathcal{M}}_{0,2}(X, d)]^{\text{vir}}) \cup [X_v].$$

The cycle $(ev_2)(ev_1^{-1}([X_u]))$ is supported on the curve neighborhood $\Gamma_d^X(X_u)$, and the pushforward $(ev_2)_*(ev_1^*([X_u]) \cap [\overline{\mathcal{M}}_{0,2}(X, d)]^{\text{vir}})$ is non-zero only if the curve neighborhood $\Gamma_d^X(X_u)$ has components of dimension

$$\expdim \overline{\mathcal{M}}_{0,2}(X, d) - (\dim X - \ell(u)) = \langle d, c_1(T_X) \rangle - 1 + \ell(u).$$

However, such components do not exist by the above estimation of $\dim \Gamma_d^X(X_u)$.

Hence, $\langle [X_u], [X_v] \rangle_{0,2,d} = \int_{[\overline{\mathcal{M}}_{0,2}(X, d)]^{\text{vir}}} ev_1^*([X_u]) \cup ev_2^*([X_v]) = 0$. Since $\{[X_u]\}_{u \leq w_0 s_{n-1}}$ is a basis of $H^*(X, \mathbb{Z})$, the statement follows. $\square$

4.2.2. Vanishing for specific degrees. It remains to show the vanishing of Gromov-Witten invariants for $d = \alpha_{in} + \alpha_{n-1, n}$.²

Let us consider

$$\mathcal{P} = F\ell_{1, \dots, n-2; n}, \quad Y = \{V_\bullet \in \mathcal{P} \mid F_1 \subseteq V_{n-2}\}.$$

We have a natural projection $\pi : X \rightarrow \mathcal{P}$ by forgetting V_{n-1} . The fiber over a point of Y is $\mathbb{P}^1$, while the fiber over a point of $\mathcal{P} \setminus Y$ is a point. Note

$$\dim \mathcal{P} = \dim X, \quad \dim Y = \dim \mathcal{P} - 2.$$

Proposition 4.11. *Let $d = \alpha_{in} + \alpha_{n-1, n}$ where $1 \leq i \leq n-1$. Then for any $\beta_1, \beta_2 \in H^*(X)$, we have $\langle \beta_1, \beta_2 \rangle_{0,2,d} = 0$.*

Proof. We discuss the degrees in two cases.

Case $i = n-1$. In this case, $\pi_* d = 0$. Therefore any stable map of degree d is contained in a fiber of π at some point of Y . This defines a morphism from the moduli space to $\mathcal{P}$ whose image is contained in Y . Denoting $ev = ev_1 \times ev_2$ and by Δ the diagonal map, we have

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,2}(X, d) & \xrightarrow{ev} & X \times X \\ \downarrow & & \downarrow \\ \mathcal{P} & \xrightarrow{\Delta} & \mathcal{P} \times \mathcal{P} \end{array}$$

The space $\overline{\mathcal{M}}_{0,2}(X, d)$ has expected dimension

$$\dim X + \deg_X q_{n-1}^2 + 2 - 3 = \dim X + 1.$$

²Special treatments for Gromov-Witten invariants of specific degrees were also required in certain blowups of Grassmannians in [HKLS24, Lemma 3.7] and [HKLS25, Theorem 4.1 (b)].

The image of ev lies in the preimage of Y of $X \times_{\mathcal{P}} X$, which is a $\mathbb{P}^1 \times \mathbb{P}^1$ -bundle over Y . Thus its dimension is

$$\dim Y + 2 = \dim X - 2 + 2 = \dim X.$$

Thus $\text{ev}_*[\overline{\mathcal{M}}_{0,2}(X, d)]^{\text{vir}} = 0$.

Case $i < n - 1$. We have a similar commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,2}(X, d) & \xrightarrow{\text{ev}} & X \times X \\ \hat{\pi} \downarrow & & \downarrow \pi \times \pi \\ \overline{\mathcal{M}}_{0,2}(\mathcal{P}, \pi_* d) & \xrightarrow{\text{ev}_{\mathcal{P}}} & \mathcal{P} \times \mathcal{P}. \end{array}$$

Note that $\deg_X \alpha_{j,j+1} = 2 = \deg_{\mathcal{P}} \pi_* \alpha_{j,j+1}$ for $j < n - 2$, $\deg_X \alpha_{n-2,n-1} = 2 = \deg_{\mathcal{P}} \pi_* \alpha_{n-2,n-1} - 1$ and $\deg_X \alpha_{n-1,n} = 1$. Hence, we have

$$\begin{aligned} \exp \dim \overline{\mathcal{M}}_{0,2}(X, d) &= \dim X + \deg_X q^d + 2 - 3, \\ \exp \dim \overline{\mathcal{M}}_{0,2}(\mathcal{P}, \pi_* d) &= \dim \overline{\mathcal{M}}_{0,2}(\mathcal{P}, \pi_* d) = \exp \dim \overline{\mathcal{M}}_{0,2}(X, d) - 1. \end{aligned}$$

Moreover the map $\hat{\pi}$ cannot be surjective, since any stable map in the image has an extra constrain that C meets Y . Combining with the known fact that $\overline{\mathcal{M}}_{0,2}(\mathcal{P}, \pi_* d)$ is irreducible, this reduces one more dimension:

$$\dim(\text{im}(\hat{\pi})) \leq \dim \overline{\mathcal{M}}_{0,2}(\mathcal{P}, \pi_* d) - 1.$$

As a result, $\text{ev}_*[\overline{\mathcal{M}}_{0,2}(X, d)]^{\text{vir}}$ is supported over

$$(\pi \times \pi)^{-1}(\text{im}(\text{ev}_{\mathcal{P}} \circ \hat{\pi})).$$

Since the fibers of $\pi \times \pi$ have dimension at most 2, it has dimension at most

$$\begin{aligned} \dim(\text{im}(\text{ev}_{\mathcal{P}} \circ \hat{\pi})) + 2 &\leq \dim(\text{im} \hat{\pi}) + 2 \\ &\leq \dim \overline{\mathcal{M}}_{0,2}(\mathcal{P}, \pi_* d) - 1 + 2 = \exp \dim \overline{\mathcal{M}}_{0,2}(X, d). \end{aligned}$$

If $\text{ev}_*[\overline{\mathcal{M}}_{0,2}(X, d)]^{\text{vir}} \neq 0$, then equality must hold throughout. Since $\pi \times \pi$ has two-dimensional fibers only over points of $Y \times Y$, equality can hold only if $\text{ev}_{\mathcal{P}}$ restricts to a morphism $Z_1 \rightarrow Z_2$ that is generically finite, where Z_1 is a component of $\text{im} \hat{\pi}$ of codimension 1 in $\overline{\mathcal{M}}_{0,2}(\mathcal{P}, \pi_* d)$ and Z_2 is a component of $\text{im}(\text{ev}_{\mathcal{P}} \circ \hat{\pi})$ contained in $Y \times Y$. By (the proof of) [BCMP13, Lemma 3.8], Z_2 is contained in a locally trivial fibration over $Y_1 \subset Y$ with fiber $\Gamma_{\pi_* d}^{\mathcal{P}}(y)$, where Y_1 denotes the natural projection of $Z_2 \subset Y \times Y$ to the first factor. By [BM15, Theorem 5.1], the fiber $\Gamma_{\pi_* d}^{\mathcal{P}}(y)$ is a Schubert variety of $\mathcal{P}$, indexed by the permutation $z_{\pi_* d}^{\mathcal{P}} = z_{\alpha_{in}} s_{n-1} = t_{in} s_{n-1}$. Thus it is of dimension $\ell(z_{\pi_* d}^{\mathcal{P}}) = \ell(t_{in}) - 1$.

$$\begin{aligned} \dim Z_2 &\leq \dim Y_1 + \dim \Gamma_{\pi_* d}^{\mathcal{P}}(y) \leq \dim Y + \ell(t_{in}) - 1 = \dim X - 2 + 2(n - i) - 1 - 1, \\ \dim Z_1 &= \dim \overline{\mathcal{M}}_{0,2}(\mathcal{P}, \pi_* d) - 1 = \exp \dim \overline{\mathcal{M}}_{0,2}(X, d) - 2 = \dim X + 2(n - i) - 1 - 2, \end{aligned}$$

resulting in a contradiction $0 = \dim Z_2 - \dim Z_1 \leq -1$. $\square$

4.3. Proof of Theorem 4.2. Note that $\{PD([X_u])\}_u$ is the dual basis of $\{\xi^u\}_u$ with respect to the Poincaré pairing. Write $PD([X_u]) = \sum_{\gamma} a_{\gamma}^u \xi^{\gamma}$. Using the projection formula,

$$\delta_{u,w} = \int_{[X]} PD([X_u]) \cup \xi^w = \int_{[X]} \iota^* \left(\sum_{\gamma} a_{\gamma}^u \sigma^{\gamma} \right) \cup \iota^* \sigma^w = \int_{[F\ell_n]} \sum_{\gamma} a_{\gamma}^u \sigma^{\gamma} \cup \iota_* (\iota^* \sigma^w).$$

Note $\iota_* (\iota^* \sigma^w) = \iota_* (\iota^* \sigma^w \cup \xi^{\text{id}}) = \sigma^w \cup PD([X]) = \sigma^w \cup \sigma^{s_{n-1}}$. The permutation u varies in S_n with $u(n) \neq 1$. Hence,

$$\sum_{\gamma} a_{\gamma}^w \sigma^{\gamma} \cup \sigma^{s_{n-1}} = (\sigma^w)^{\vee} + \sum_{\eta(n)=1} b_{\eta} (\sigma^{\eta})^{\vee}.$$

We have

$$\begin{aligned} \xi^{s_r} \star \xi^u &= \sum_{w,d} \langle \xi^{s_r}, \xi^u, PD([X_w]) \rangle_{0,3,d}^X \xi^w q^d \\ &= \sum_{w \leq w_0 s_{n-1}} \langle \xi^{s_r}, \xi^u, PD([X_w]) \rangle_{0,3,0}^X \xi^w + \sum_{w \leq w_0 s_{n-1}, d \neq 0} \langle \xi^{s_r}, \xi^u, PD([X_w]) \rangle_{0,3,d}^X \xi^w q^d. \end{aligned}$$

By Proposition 4.10 and Proposition 4.11, there are no q^d -terms in the second sum whenever $d_{n-1} \geq 2$. By the divisor axiom in Gromov-Witten theory, for $d \neq 0$ we have $\langle \xi^{s_r}, \xi^u, PD([X_v]) \rangle_{0,3,d}^X = \int_d \xi^{s_r} \langle \xi^u, \sum_{\gamma} a_{\gamma}^v \xi^{\gamma} \rangle_{0,2,d}$.

For $d_{n-1} = 1$, by Proposition 4.5, we have

$$\begin{aligned} &\int_d \xi^{s_r} \langle \xi^u, \sum_{\gamma} a_{\gamma}^w \xi^{\gamma} \rangle_{0,2,d}^X q^d \\ &= \int_d \sigma^{s_r} \langle \sigma^u \cup \sigma^{s_{n-1}}, \sum_{\gamma} a_{\gamma}^w \sigma^{\gamma} \cup \sigma^{s_{n-1}} \rangle_{0,2,d}^{F\ell_n} q^d - \int_d \xi^{s_r} \delta_{q^d, q_{n-1}} \langle \sigma^u, \sum_{\gamma} a_{\gamma}^w \sigma^{\gamma}, \sigma^{s_{n-1}} \rangle_{0,3,0}^{F\ell_n} q_{n-1} \\ &= \int_d \sigma^{s_r} \langle \sigma^u \cup \sigma^{s_{n-1}}, \sum_{\gamma} a_{\gamma}^w \sigma^{\gamma} \cup \sigma^{s_{n-1}} \rangle_{0,2,d}^{F\ell_n} q^d - \int_d \xi^{s_r} \delta_{q^d, q_{n-1}} q_{n-1} \int_{[F\ell_n]} \sigma^u \cup \left(\sum_{\gamma} a_{\gamma}^w \sigma^{\gamma} \right) \cup \sigma^{s_{n-1}} \\ &= \langle \sigma^{s_r}, \sigma^u \cup \sigma^{s_{n-1}}, (\sigma^w)^{\vee} + \sum_{\eta(n)=1} b_{\eta} (\sigma^{\eta})^{\vee} \rangle_{0,3,d}^{F\ell_n} q^d - \int_d \xi^{s_r} \delta_{q^d, q_{n-1}} q_{n-1} \int_{[F\ell_n]} \sigma^u \cup ((\sigma^w)^{\vee} + \sum_{\eta(n)=1} b_{\eta} (\sigma^{\eta})^{\vee}) \\ &= \langle \sigma^{s_r}, \sigma^u \cup \sigma^{s_{n-1}}, (\sigma^w)^{\vee} \rangle_{0,3,d}^{F\ell_n} q^d - \int_d \xi^{s_r} \delta_{q^d, q_{n-1}} q_{n-1} \delta_{u,w} \end{aligned}$$

The last equality holds by noting $\int_{[F\ell_n]} \sigma^u \cup (\sigma^{\eta})^{\vee} = 0$ (since $u(n) \neq 1$) and

$$\langle \sigma^{s_r}, \sigma^u \cup \sigma^{s_{n-1}}, (\sigma^{\eta})^{\vee} \rangle_{0,3,d}^{F\ell_n} = \sum_{\tilde{u}} \langle \sigma^{s_r}, \sigma^{\tilde{u}}, (\sigma^{\eta})^{\vee} \rangle_{0,3,d}^{F\ell_n} = \sum_{\tilde{u}} N_{s_r, \tilde{u}}^{\eta, d} = 0$$

for any permutation η with $\eta(n) = 1$. Indeed, by the quantum Monk's formula for $F\ell_n$, $u \leq_{n-1} \tilde{u}$ and for $\hat{w} \in S_n$, $N_{s_r, \tilde{u}}^{\hat{w}, d} \neq 0$ only if $\tilde{u} \leq^q \hat{w} = \tilde{u} t_{an}$ for some a (since $d_{n-1} = 1$).

- i) If $\tilde{u} \leq w_0 s_{n-1}$, then $\hat{w} \leq \tilde{u} \leq w_0 s_{n-1}$, i.e. $\hat{w}(n) \neq 1$;
- ii) If $\tilde{u} \not\leq w_0 s_{n-1}$, i.e. $\tilde{u}(n) = 1$, then $\hat{w}(n) = \tilde{u} t_{an}(n) = \tilde{u}(a) \neq \tilde{u}(n) = 1$.

Thus the sum vanishes for any η with $\eta(n) = 1$.

For $d \neq 0$ with $d_{n-1} = 0$, by Proposition 4.6, we have

$$\begin{aligned} \int_d \xi^{s_r} \langle \xi^u, \sum_{\gamma} a_{\gamma}^w \xi^{\gamma} \rangle_{0,2,d}^X q^d &= \int_d \sigma^{s_r} \langle \sigma^u, \sum_{\gamma} a_{\gamma}^w \sigma^{\gamma} \cup \sigma^{s_{n-1}} \rangle_{0,2,d}^{F\ell_n} q^d \\ &= \langle \sigma^{s_r}, \sigma^u, (\sigma^w)^{\vee} + \sum_{\eta(n)=1} b_{\eta} (\sigma^{\eta})^{\vee} \rangle_{0,3,d}^{F\ell_n} q^d \\ &= \langle \sigma^{s_r}, \sigma^u, (\sigma^w)^{\vee} \rangle_{0,3,d}^{F\ell_n} q^d. \end{aligned}$$

The last equality holds again by noting $\langle \sigma^{s_r}, \sigma^u, (\sigma^w)^{\vee} \rangle_{0,3,d}^{F\ell_n} q^d = 0$ unless $u \triangleleft_r^q \hat{w}$, implying $\hat{w}(n) \neq 1$. Hence, we are done. $\square$

Remark 4.12. The arguments i) and ii) in the above proof say that for any $u, w \in S_n$ and any a, k , the hypothesis $u \triangleleft_{n-1} w t_{an} \triangleleft_k^q w$ implies $w(n) \neq 1$, i.e. $w \leq w_0 s_{n-1}$.

5. QUANTUM SCHUBERT POLYNOMIALS FOR X

This section is devoted to a proof of Theorem 1.6, namely for any w , the quantum Schubert polynomial $\mathfrak{S}_w^q$ of Fomin, Gelfand and Postnikov represents the pullback Schubert class ξ^w , under the canonical ring isomorphism in Theorem 1.2.

Recall

$$q_{ab} := q_a q_{a+1} \cdots q_{b-1} \quad \text{for } 1 \leq a < b \leq n.$$

5.1. Quantum Schubert polynomials. The classical Schubert polynomials were introduced by Lascoux and Schützenberger [LS82] by using the divided difference operators ∂_i of Bernstein, Gelfand and Gelfand [BGG73]. Precisely, for $f = f(x_1, \dots, x_n) \in \mathbb{Z}[x]$ and $w \in S_n$, denote $wf = f(x_{w^{-1}(1)}, \dots, x_{w^{-1}(n)})$. Then $\partial_i f := \frac{f - s_i f}{x_i - x_{i+1}} \in \mathbb{Z}[x]$, and the classical Schubert polynomials $\mathfrak{S}_w(x)$ are recursively defined by

$$(5.1) \quad \mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1} \quad \text{and} \quad \mathfrak{S}_{ws_i} = \partial_i \mathfrak{S}_w \quad \text{whenever} \quad \ell(ws_i) = \ell(w) - 1.$$

The following were shown in [LS82].

- (1) $\Phi(\sigma^w) = [\mathfrak{S}_w(x)]$ under the canonical ring isomorphism Φ in Proposition 2.1.
- (2) $\{e_{i_1}^1 e_{i_2}^2 \cdots e_{i_{n-1}}^{n-1}\}_{0 \leq i_j \leq j}$ form a $\mathbb{Z}$ -basis of $\mathbb{Z}[x]$.

Therefore, we have the linear expansion $\mathfrak{S}_w = \sum \alpha_{i_1 \dots i_{n-1}} e_{i_1}^1 e_{i_2}^2 \cdots e_{i_{n-1}}^{n-1}$. In [FGP97], Fomin, Gelfand and Postnikov introduced the quantum Schubert polynomial

$$(5.2) \quad \mathfrak{S}_w^q := \sum \alpha_{i_1 \dots i_{n-1}} E_{i_1}^1 E_{i_2}^2 \cdots E_{i_{n-1}}^{n-1}.$$

They also showed $\Phi_q(\sigma^w) = [\mathfrak{S}_w^q]$, under the canonical ring isomorphism [GK95, CF95, Ki96]

$$(5.3) \quad \Phi_q : QH^*(F\ell_n, \mathbb{Z}) \longrightarrow \mathbb{Z}[x_1, \dots, x_n, q_1, \dots, q_{n-1}] / (E_1^n, \dots, E_n^n).$$

Recall the (quantum) r -Bruhat order $\triangleleft_r$ (resp. $\triangleleft_r^q$) defined in Equation (2.2) (resp. Equation (2.3)). The following is the quantum Monk's formula on the level of polynomials, proved in the first half of [FGP97, Theorem 7.1].

Proposition 5.1 (Quantum Monk's formula). *For $u \in S_n$ and $1 \leq r < n$, in $\mathbb{Z}[x]$ we have*

$$\mathfrak{S}_{s_r}^q \mathfrak{S}_u^q = (x_1 + \cdots + x_r) \mathfrak{S}_u^q = \sum_{u \leq_r ut_{ab}} \mathfrak{S}_{ut_{ab}}^q + \sum_{u \leq_r^q ut_{ck}} q_{ck} \mathfrak{S}_{ut_{ck}}^q.$$

Lemma 5.2. *Let $u, w \in S_n$ and $1 \leq a < b \leq n$.*

- (1) *$u \leq ut_{ab}$ if and only if $u(a) < u(b)$ and for any $a < c < b$, we have $u(c) \notin [u(a), u(b)]$.*
- (2) *$u \leq^q ut_{ab}$ if and only if $u(a) > u(b)$ and for any $a < c < b$, we have $u(c) \in [u(b), u(a)]$.*

Proof. Note $\ell(t_{ab}) = 2b - 2a - 1$. The statement follows from a direct counting of the number of inversions, which defines the length of a permutation. $\square$

The next proposition is a special case of the second half of [FGP97, Theorem 7.1], with a slightly more precise description than that in loc. cit.; see also [LOTRZ25, Theorem 4]. This special case will play a crucial role in our proof of Theorem 1.6. A permutation $w \in S_n$ is said to have a descent at the k -th position if $w(k+1) < w(k)$.

Proposition 5.3 (Transition equation). *Let $w \in S_n \setminus \{\text{id}\}$. Denote by i the last descent position of w . Take the maximal j with $w(j) < w(i)$. Then $u := wt_{ij}$ satisfies $u \leq w$, and we have*

$$\mathfrak{S}_w^q = x_i \mathfrak{S}_u^q + \sum_{u \leq ut_{hi}} \mathfrak{S}_{ut_{hi}}^q + \sum_{u \leq^q ut_{hi}} q_{hi} \mathfrak{S}_{ut_{hi}}^q - \sum_{u \leq^q ut_{ik}} q_{ik} \mathfrak{S}_{ut_{ik}}^q.$$

Proof. It follows directly from the definition that $i < j$ and that for any $i < c < j$, $u(c) = w(c) < w(j) = u(i)$ (where the inequality holds since i is the last descent position). Thus $u \leq w$ by Lemma 5.2 (1).

By Proposition 5.1, we compare the two quantum Monk's formulas

$$\begin{aligned} (x_1 + x_2 + \cdots + x_{i-1}) \mathfrak{S}_u^q &= \sum_{u \leq_{i-1} ut_{ab}} \mathfrak{S}_{ut_{ab}}^q + \sum_{u \leq_{i-1}^q ut_{ck}} q_{ck} \mathfrak{S}_{ut_{ck}}^q; \\ (x_1 + x_2 + \cdots + x_i) \mathfrak{S}_u^q &= \sum_{u \leq_i ut_{ab}} \mathfrak{S}_{ut_{ab}}^q + \sum_{u \leq_i^q ut_{ck}} q_{ck} \mathfrak{S}_{ut_{ck}}^q. \end{aligned}$$

Notice that if $c \leq i-1 < i < k$, then $u \leq_{i-1}^q ut_{ck}$ implies $u \leq_i^q ut_{ck}$. Therefore the difference between the quantum parts of the two products occurs exactly when either $k = i$ in the first product or $c = i$ in the second product. Hence, the quantum part of the statement follows.

For the classical part, the same argument applies once we show that $u \leq ut_{ib}$ implies $b = j$. Indeed, assume $u \leq ut_{ib}$ for some $b \neq j$. Then $w(j) = u(i) < u(b) = w(b)$ by Lemma 5.2. Since i is the last descent position of w , it follows that $j < b$. Furthermore, we have $u(j) = w(i) < w(b) = u(b)$, since j is maximal with respect to $w(j) < w(i)$. But then we would have $i < j < b$ and $u(i) < u(j) < u(b)$, contradicting $u \leq ut_{ib}$ by Lemma 5.2. $\square$

Lemma 5.4. *Let $w \in S_n \setminus \{\text{id}\}$ with $w(n) \neq 1$. Then all the permutations v occurring on the right hand side of the formula of $\mathfrak{S}_w^q$ in Proposition 5.3 satisfy $v(n) \neq 1$.*

Proof. With the same notation as in Proposition 5.3, if $v = u = wt_{ij}$, then $u(n) \geq u(j) > u(i) \geq 1$. If $v = ut_{hi}$, then $v(n) = u(n) \neq 1$ by noting $i < n$. It remains to discuss the case $v = ut_{ik}$. Since $u \prec^q ut_{ik}$, by Lemma 5.2, $u(k) < u(i) = w(j)$, so $k < j$ since i is the last descent position. In particular, $k \neq n$. Thus, $v(n) = u(n) \neq 1$. $\square$

Definition 5.5. For $u, w \in S_n$, we say $u \prec w$ if and only if

$$(\ell(u), -u(n), -u(n-1), \dots, -u(1)) < (\ell(w), -w(n), -w(n-1), \dots, -w(1))$$

with respect to the lexicographic order. This defines a total order $\prec$ on S_n .

Lemma 5.6. Let $w \in S_n \setminus \{\text{id}\}$. Then all the permutations occurring on the right hand side of the formula of $\mathfrak{S}_w^q$ in Proposition 5.3 are strictly smaller than w with the order $\prec$.

Proof. Let u be as in Proposition 5.3. Then $u \prec w$ since $\ell(u) = \ell(w) - 1$. For $u \prec ut_{hi}$, we have $\ell(w) = \ell(ut_{hi}) = \ell(u) + 1$. By Lemma 5.2, we have $w(j) = u(i) < u(j) = ut_{hi}(j)$. Combining this with the property $w(a) = u(a) = ut_{hi}(a)$ for $a > j$, we obtain $ut_{hi} \prec w$. Permutations in the quantum part are all of length smaller than $\ell(w)$, and hence are strictly smaller than w with respect to the total order $\prec$. $\square$

5.2. Proof of Theorem 1.6. To achieve our aim, we first show that the pullback Schubert classes ξ^w admit exactly the same transition equations in the quantum cohomology $QH^*(X)$ as that for $\mathfrak{S}_w^q$ on the level of polynomials.

Proposition 5.7. With the same notation as in Proposition 5.3, in $QH^*(X)$ we have

$$\xi^w = x_i \xi^u + \sum_{u \prec ut_{hi}} \xi^{ut_{hi}} + \sum_{u \prec^q ut_{hi}} q_{hi} \xi^{ut_{hi}} - \sum_{u \prec^q ut_{ik}} q_{ik} \xi^{ut_{ik}}.$$

Proof. Recall $w = ut_{ij}$ and that i is the last descent of w , so $i \leq n-1$ and $x_i = \xi^{s_i} - \xi^{s_{i-1}}$, where $\xi^{s_0} := 0$. We compare the two quantum Monk-Chevalley formulas

$$\begin{aligned} \xi^{s_{i-1}} \star \xi^u &= \sum_{u \prec_{i-1} ut_{ab} \leq w_0 s_{n-1}} \xi^{ut_{ab}} + \sum_{u \prec_{i-1}^q ut_{ck} \text{ with } k < n} \xi^{ut_{ck}} q_{ck} + \sum_{u \prec ut_{an} \prec_{i-1}^q ut_{an} t_{cn} \leq w_0 s_{n-1}} \xi^{ut_{an} t_{cn}} q_{cn} - 0, \\ \xi^{s_i} \star \xi^u &= \sum_{u \prec_i ut_{ab} \leq w_0 s_{n-1}} \xi^{ut_{ab}} + \sum_{u \prec_i^q ut_{ck} \text{ with } k < n} \xi^{ut_{ck}} q_{ck} + \sum_{u \prec ut_{an} \prec_i^q ut_{an} t_{cn} \leq w_0 s_{n-1}} \xi^{ut_{an} t_{cn}} q_{cn} - \delta_{i,n-1} q_{n-1} \xi^u. \end{aligned}$$

The classical part follows from the same argument as in Proposition 5.3, where the constraint $ut_{hi} \leq w_0 s_{n-1}$ is redundant by Lemma 5.4. Again note that if $c \leq i-1 < i < k$, then $u \prec_{i-1}^q ut_{ck}$ implies $u \prec_i^q ut_{ck}$. Therefore the difference between the quantum parts of the two quantum products occurs exactly when either $k = i$ in the first product or $c = i$ in the second product. The $k = i$ part in the first product is exactly the second sum in the equation for ξ^w in the statement. It remains to show that the rest is given by the $c = i$ part of $\xi^{s_i} \star \xi^u$ together with $\delta_{i,n-1} q_{n-1} \xi^u$, namely to show

$$(5.4) \quad \sum_{u \prec^q ut_{ik}} \xi^{ut_{ik}} q_{ik} = \sum_{u \prec^q ut_{ik} \text{ with } k < n} \xi^{ut_{ik}} q_{ik} + \sum_{u \prec ut_{an} \prec_i^q ut_{an} t_{in} \leq w_0 s_{n-1}} \xi^{ut_{an} t_{in}} q_{in} - \delta_{i,n-1} q_{n-1} \xi^u.$$

Denote by RHS (resp. LHS) the right (resp. left) hand side of the above equation to prove.

- (1) Case $i = n - 1$. Then $j = n$, the sum on the LHS is empty (otherwise we would have $u \leq^q ut_{ik} = ut_{n-1,n} = w$, contradicting $u \leq w$), and the first sum on the RHS is empty as well. The constraints

$$u \leq ut_{an} \leq^q ut_{ant_{in}}$$

imply $a = n - 1$ by Lemma 5.2. (Otherwise, $a < n - 1 = i$, then $ut_{an}(i) = u(i) < u(a) = ut_{an}(n)$, a contradiction.) Then we have $ut_{ant_{in}} = u$, which automatically satisfies $u \leq w_0 s_{n-1}$ by Lemma 5.4. Hence, $\text{RHS} = 0 + \xi^u q_{n-1} - \xi^u q_{n-1} = 0 = \text{LHS}$.

- (2) Case $i < n - 1$. Then $\delta_{i,n-1} = 0$.

For $u \leq^q ut_{ik}$ on the LHS, we have $k \neq j$ since $u \leq ut_{ij} = w$. By Lemma 5.2, $w(k) = u(k) < u(i) = w(j)$. Since i is the last descent, we have $k < j$. In particular, $k < n$, thus the LHS is equal to the first sum of the RHS. It remains to show that the second sum on the RHS is zero. Suppose we have $u \leq ut_{an} \leq^q ut_{ant_{in}}$. By the choice of i, j , u also has no descent after i . We also note that j is the minimal integer greater than i that satisfies $u(i) < u(j)$. These two properties will be used over and over again in the following argument. If $i < a < n - 1$, then $u(n - 1) \in [u(a), u(n)]$, contradicting Lemma 5.2. Therefore, either $a = n - 1$ or $a \leq i$. If $a = n - 1$, by Lemma 5.2, we have $u(n - 1) = ut_{an}(n) < ut_{an}(i) = u(i)$, so $j = n$. Then $ut_{an}(n - 1) = u(n) = u(j) > u(i) = ut_{an}(i)$. Then $ut_{an}(n - 1) \notin [ut_{an}(n), ut_{an}(i)]$, contradicting Lemma 5.2. If $a = i$, then from $u \leq ut_{an}$ and Lemma 5.2, we have $j = n$. Then $ut_{an}(n - 1) = u(n - 1) < u(i) = ut_{an}(n)$, so $ut_{an}(n - 1) \notin [ut_{an}(n), ut_{an}(i)]$, contradicting Lemma 5.2. If $a < i$, then by Lemma 5.2, we have $ut_{an}(n - 1) = u(n - 1) < u(a) = ut_{an}(n)$, so $ut_{an}(n - 1) \notin [ut_{an}(n), ut_{an}(i)]$, contradicting Lemma 5.2. $\square$

Proof of Theorem 1.6. By (the proof of) Theorem 1.2, the canonical ring isomorphism

$$\Psi_q : QH^*(X, \mathbb{Z}) \longrightarrow \mathbb{Z}[x_1, \dots, x_n, q_1, \dots, q_{n-1}] / (\hat{E}_1^n, \dots, \hat{E}_{n-1}^n, E_{n-1}^{n-1}).$$

satisfies $\Psi_q(\xi^{s_i}) = [x_1 + x_2 + \dots + x_i]$ for $1 \leq i \leq n - 1$. Thus $\Psi_q(\xi^w) = [\mathfrak{S}_w^q]$ holds for $w = \text{id}$ and for $w \in S_n$ with $\ell(w) = 1$ (all of which satisfy $w \leq w_0 s_{n-1}$).

By (2.1), $w \leq w_0 s_{n-1}$ holds if and only if $w(n) \neq 1$. By Proposition 5.7, Lemma 5.4 and Lemma 5.6, every ξ^w with $w(n) \neq 1$ can be written as a $\mathbb{Z}[q]$ -linear combination of classes ξ^v with $v \prec w$ and $v(n) \neq 1$. By Proposition 5.3, $\mathfrak{S}_w^q$ can also be written as exactly the same $\mathbb{Z}[q]$ -linear combination of $\mathfrak{S}_v^q$ on the level of polynomials. Hence, the statement follows immediately from induction on the totally ordered subset $(\{w\}_{w \leq w_0 s_{n-1}}, \prec)$. $\square$

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