GAMMA CONJECTURE II VIA GLOBAL GAMMA-I

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ABSTRACT. For a Fano manifold X, Gamma conjecture II aims to use $\mathcal{D}^b_{\mathrm{coh}}(X)$ to describe the asymptotic behavior of its Dubrovin connection via $\widehat{\Gamma}$-integral structure. It was proposed by Galkin, Golyshev and Iritani, and can be regarded as a quantitative refinement of Dubrovin's conjecture on Fano manifolds with semisimple big quantum cohomology. As a step toward Gamma conjecture II, we define the Gamma-I property at points satisfying the (SR) condition, arising from the original Gamma conjecture I. We prove that this property holds globally in the following sense: if it holds at one such point, then it holds throughout the connected component of the (SR)-region containing that point. Based on this global Gamma-I property, we establish a strategy-type theorem relating Gamma conjecture II to the Gamma-I property at a possibly non-semisimple point, together with an analysis of small quantum cohomology. We further apply this theorem to prove Gamma conjecture II for del Pezzo surfaces; the proof combines Iritani's Galois action with additional elementary operations on exceptional collections, and its most technically involved step consists in verifying the required global Gamma-I property.

CONTENTS

1. INTRODUCTION

The structural correspondence between Gromov–Witten theory and classical algebraic geometry is a central theme in symplectic geometry and complex geometry. This relationship has motivated a series of profound conjectures and the development of new categorical frameworks. In his 1998 ICM talk [Dub98], Dubrovin proposed a conjecture for Fano manifolds X, describing an intriguing relationship between the big quantum cohomology $QH_{\rm big}(X)$ and the bounded derived category $\mathcal{D}^b_{\rm coh}(X)$ of coherent sheaves on X. Its first part asserts, qualitatively, that $QH_{\rm big}(X)$ is generically semisimple if and only if $\mathcal{D}^b_{\rm coh}(X)$ admits a full exceptional collection. Gamma conjecture II, proposed by Galkin, Golyshev, and Iritani [GGI16], can be regarded as a refinement of the quantitative part of Dubrovin's conjecture (see also Iritani's 2022 ICM talk [Iri23]). It aims to describe the Dubrovin connection associated with $QH_{\rm big}(X)$ in terms of $\mathcal{D}^b_{\rm coh}(X)$ via Iritani's Gamma-integral structure, when the latter admits a full exceptional collection. An independent refinement of Dubrovin's conjecture was also proposed by Cotti, Dubrovin, and Guzzetti [CDG24].

To describe Gamma conjecture II, we consider the Dubrovin connection ∇. Modulo convergence issues, it is a meromorphic flat connection on the trivial $H^*(X)$-bundle over $H^*(X)\times\mathbb{P}^1$ given by

$$\nabla = \mathrm{d} + \sum_{i=0}^{s-1}\frac{1}{z}(T_i\star_{\mathbf{t}})\mathrm{d}t_i - \left(\frac{1}{z}\left(E(\mathbf{t})\star_{\mathbf{t}}\right)-\mu\right)\frac{\mathrm{d}z}{z}.$$

Here z is the inhomogeneous coordinate on $\mathbb{P}^1$, $\{T_i\}_i$ is a homogeneous basis of $H^*(X)$, $\star_{\mathbf{t}}$ denotes the quantum product of $QH_{\rm big}(X)$, $E(\mathbf{t})$ is the Euler vector field, and $\mu\in\mathrm{End}(H^*(X))$ is a grading operator. Denote by $K(X)$ the Grothendieck group of topological complex vector bundles on X, and by $\mathcal{S}$ the space of ∇-flat sections. The $\widehat{\Gamma}$-integral structure, introduced by Iritani [Iri09], is the image of a specified group homomorphism

$$\mathcal{Z}^K : K(X)\to\mathcal{S},$$

defined using the Gamma class $\widehat{\Gamma}_X$ [Lib99, Lu08, KKP08, Iri09] of X, a modification of Chern character, and a canonical $\mathbb{C}$-linear isomorphism $\mathcal{Z} : H^*(X)\to\mathcal{S}$ given by Givental's operator. The Dubrovin connection ∇ has two singularities:

(i) the regular singularity at $z=\infty$, where the $\widehat{\Gamma}$-integral structure in $\mathcal{S}$ is involved;
(ii) the irregular singularity at $z=0$, where the Stokes structure in $\mathcal{S}$ is involved and characterized by the exponential growth of flat sections.

Gamma conjecture II asserts that these two structures are compatible in the following sense. Assume that $QH_{\rm big}(X)$ is semisimple at $\mathbf{t}\in H^*(X)$, so that $E\star_{\mathbf{t}}$ has eigenvalues $\{u_i\}_{i=1}^s$ with corresponding normalized idempotent eigenvectors $\{\Psi_i\}_{i=1}^s\subset H^*(X)$, where $s=\dim H^*(X)$. We say that a flat section $y\in\mathcal{S}$ *respects* (u_i,Ψ_i) *with the phase* $\phi\in\mathbb{R}$ *at* $\mathbf{t}$, if

$$e^{\frac{u_i(\mathbf{t})}{z}}y(\mathbf{t},z)\longrightarrow\Psi_i,\ \text{as } z\to 0 \text{ along the sector } |\arg z-\phi|<\frac{\pi}{2}+\epsilon \text{ for some } \epsilon>0.$$

Such y uniquely exists when ϕ is admissible at $\mathbf{t}$, i.e. $\mathrm{e}^{\mathbf{i}\phi}$ is not parallel to any nonzero difference $u_{i_1}(\mathbf{t})-u_{i_2}(\mathbf{t})$.

Gamma conjecture II ([GGI16, Conjecture 4.6.1])**.** *Assume that $QH_{\rm big}(X)$ is convergent and semisimple at some $\mathbf{t}\in H^*(X)$ and that $\mathcal{D}^b_{\rm coh}(X)$ admits a full exceptional collection. Then for any phase $\phi\in\mathbb{R}$ admissible at $\mathbf{t}$, there exists a full exceptional collection $(E_1,\dots,E_s)$ of $\mathcal{D}^b_{\rm coh}(X)$, such that the flat section $\mathcal{Z}^K(E_i)$ respects (u_i,Ψ_i) at $\mathbf{t}$ with the phase ϕ for all i.*

We refer to Section 2.3 for a more precise review of Gamma conjecture II.

Currently, the only Fano manifolds for which Gamma conjecture II has been proved are the following: complex Grassmannians via quantum Satake by Galkin, Golyshev and Iritani [GGI16] (see also [CDG24]), toric Fano manifolds via homological mirror symmetry by Fang and Zhou [FZ19], and quadrics via direct computations by Hu and Ke [HK23].

As noted in [GGI16, Section 4.6], Gamma conjecture II refines part (3) of Dubrovin's conjecture, with the latter consisting of three parts. As recalled above, part (1) asserts that $QH_{\text{big}}(X)$ is generically semisimple if and only if $\mathcal{D}^b_{\text{coh}}(X)$ admits a full exceptional collection; this statement was made more precise in [Bay04, HMT09]. Parts (2) and (3) concern, respectively, the Stokes matrix and the central connection matrix of the Dubrovin connection.

Besides Gamma conjecture II, Gamma conjecture I and the underlying Conjecture $\mathcal{O}$ were also proposed in [GGI16]. It is a fundamental fact that for a Fano manifold X, $\mathcal{D}^b_{\text{coh}}(X)$ contains the structure sheaf $\mathcal{O}_X$ as an exceptional object. Gamma conjecture I concerns the asymptotic behavior near $z = 0$ of the flat section corresponding to $\mathcal{O}_X$ at $\mathbf{0} \in H^*(X)$. To be more precise, we consider the restriction of $QH_{\text{big}}(X)$ to the small quantum cohomology $QH_{\text{sm}}(X) = (H^*(X) \otimes \mathbb{C}[e^{\mathbf{t}^{(2)}}], \star_{\mathbf{t}^{(2)}})$, where $\mathbf{t}^{(2)} \in H^2(X)$. Accordingly, we have the small Dubrovin connection ∇^{sm} on the trivial $H^*(X)$-bundle over $H^2(X) \times \mathbb{P}^1$, the space $\mathcal{S}^{\text{sm}}$ of ∇^{sm}-flat sections, as well as the $\widehat{\Gamma}$-integral structure $\mathcal{Z}^{K,\text{sm}} : K(X) \to \mathcal{S}^{\text{sm}}$. Consider the linear operator $\hat{c}_1(\mathbf{t}^{(2)}) = c_1(X)\star_{\mathbf{t}^{(2)}}$ on $H^*(X)$. In that work, it was shown that whenever the spectral radius of $\hat{c}_1(\mathbf{0})$ is a simple eigenvalue, the vector subspace $\mathcal{A}(\mathbf{0})$ of $\mathcal{S}^{\text{sm}}$ with the smallest asymptotics as $z \to 0$ is one-dimensional. Gamma conjecture I asserts that $\mathcal{A}(\mathbf{0}) = \mathbb{C}\widehat{\Gamma}_X$, or equivalently, $\mathcal{Z}^{K,\text{sm}}(\mathcal{O}_X)(\mathbf{t}^{(2)}, z)|_{\mathbf{t}^{(2)}=\mathbf{0}} \in \mathcal{A}(\mathbf{0})$.

Gamma conjecture I has been verified in a number of cases, including complex Grassmannians [GGI16], Fano threefolds of Picard rank one [GZ16], Fano complete intersections in $\mathbb{P}^n$ [GI19, SS20, Ke24], del Pezzo surfaces [HKLY21], the blow-up of $\mathbb{P}^n$ along $\mathbb{P}^r$ [Yan22], and flag varieties [Cho25]. In a recent collaboration [GHIKLS24] between S. Galkin, H. Iritani and the present authors, counterexamples to the original Gamma conjecture I were discovered among toric Fano manifolds. Moreover, modified formulations were provided and proved to hold for all toric Fano manifolds, and Gamma conjecture I over the Kähler moduli was analyzed for these counterexamples.

The statement of Gamma conjecture I appears to be less directly related to Gamma conjecture II, and the existence of counterexamples might even seem to weaken its importance. Nevertheless, one of the main insights of the present paper, inspired by the detailed investigation of the counterexamples in [GHIKLS24], is that Gamma conjecture I is much more deeply connected with Gamma conjecture II (see the Strategy-Theorem below) than previously understood; moreover, the aforementioned counterexamples do not affect this new perspective.

We say that X satisfies condition (SR) at $\mathbf{t}^{(2)} \in H^2(X)$, if the linear operator $\hat{c}_1(\mathbf{t}^{(2)})$ on $H^*(X)$ has a simple rightmost eigenvalue, namely, a simple eigenvalue with strictly largest real part. If such an eigenvalue exists, then it is unique, and is denoted by $u^{\text{SR}}(\mathbf{t}^{(2)})$. By [GGI16, Remark 3.1.9], the subspace $\mathcal{A}(\mathbf{t}^{(2)})$ of $\mathcal{S}^{\text{sm}}$ that consists of $y(\mathbf{t}^{(2)}, z)$ with $\mathrm{e}^{\frac{u^{\text{SR}}(\mathbf{t}^{(2)})}{z}} y(\mathbf{t}^{(2)}, z)$ having moderate growth as $z \to 0$ along the positive real axis is one-dimensional. We then say that X *satisfies property **Gamma-I** at* $\mathbf{t}^{(2)} \in H^2(X)$, if X satisfies condition (SR) at $\mathbf{t}^{(2)}$ and $\mathcal{Z}^{K,\text{sm}}(\mathcal{O}) \in \mathcal{A}(\mathbf{t}^{(2)})$. Note that the validity of the original Gamma conjecture I is a special case when X satisfies Gamma-I at $\mathbf{0}$, and that the aforementioned counterexample does satisfy Gamma-I at some nonzero $\mathbf{t}^{(2)} \in H^2(X)$. By the (SR)-region of X, we mean the subset of $H^2(X)$ consisting of points at which X satisfies condition (SR). As one of the main results, we prove the following theorem in **Theorem 3.7**, which partially answers the third question in [GHIKLS24, Question 6.12].

Theorem 1.1 (Global Gamma-I)**.** *Let U be a domain inside the (SR)-region of X. If X satisfies property Gamma-I at some point of U, then it satisfies property Gamma-I at every point of U.*

This leads to the following strategy-type theorem (i.e. **Theorem 3.10**). Let $K_{ts} \subset H^2(X)$ be the set of tame semisimple points in the Kähler moduli, i.e. for $\mathbf{t}^{(2)} \in K_{ts}$, $\hat{c}_1(\mathbf{t}^{(2)})$ has only simple eigenvalues.

Strategy-Theorem. *Let X be a Fano manifold. Assume the following conditions.*

(1) The big quantum cohomology of X is convergent near the large radius limit.
(2) There exists $\mathbf{t}_0^{(2)} \in H^2(X)$ at which X satisfies property Gamma-I.
(3) There exists a domain U inside the (SR)-region of X such that $\mathbf{t}_0^{(2)} \in U$ and $U \cap K_{ts} \neq \varnothing$.
(4) Denote $V_1 := \mathcal{O}$. There exists $\mathbf{t}_1^{(2)} \in U \cap K_{ts}$ such that:
  (a) there exist $V_2, \ldots, V_s \in \mathcal{D}^b_{\mathrm{coh}}(X)$ such that each V_i is obtained from $\mathcal{O}$ via the Galois action along some path $\gamma_i \subset H^2(X)$ starting at the same point $\mathbf{t}_1^{(2)}$;
  (b) there exists a full exceptional collection $(\tilde{V}_1, \ldots, \tilde{V}_s)$ in $\mathcal{D}^b_{\mathrm{coh}}(X)$, obtained from the set $\{V_1, \ldots, V_s\}$ by "elementary operations";
  (c) there exists an admissible phase ϕ at $\mathbf{t}_1^{(2)}$ such that $\mathcal{Z}^{K,\mathrm{sm}}(\tilde{V}_i)$ respects the pair (u_i, Ψ_i) at $\mathbf{t}_1^{(2)}$ with phase ϕ for all i, where the u_i are ordered so that $\mathrm{Im}(\mathrm{e}^{-\mathbf{i}\phi} u_1(\mathbf{t}_1^{(2)})) > \cdots > \mathrm{Im}(\mathrm{e}^{-\mathbf{i}\phi} u_s(\mathbf{t}_1^{(2)}))$.

Then Gamma conjecture II holds for X.

There are several notable features of the above theorem, whenever it is applicable.

i) The starting point $\mathbf{t}_0^{(2)}$ may be a non-semisimple point. Indeed, for the del Pezzo surfaces X_r investigated below, the quantum cohomology is non-semisimple at the starting point $\mathbf{t}_0^{(2)} = \mathbf{0}$ when $5 \leq r \leq 8$ [BM04].
ii) The validity of property Gamma-I relies only on the small quantum cohomology and does not require convergence of the big quantum cohomology. In particular, both $\mathbf{t}_0^{(2)}$ and $\mathbf{t}_1^{(2)}$ may lie outside the domain of convergence of the big quantum cohomology.
iii) The assumption (4.a) provides a strategy for constructing as many exceptional objects as possible, using the Galois action established by Iritani [Iri09]. The obtained objects are "initial data" for (4.b), where the "elementary operations" involved require individual treatment, including mutations, taking differences, and related procedures, and therefore cannot be defined uniformly in a precise way.
iv) Under the strategy-style assumptions (2), (3), (4.a) and (4.b), the Strategy-Theorem becomes a reconstruction theorem, rather than merely a strategy, once assumption (4.c) is imposed; this is what we prove in Theorem 3.10.

Remark 1.2. The method of combining Gamma conjecture I with the Galois action has been used to study Gamma conjecture II in [GGI16, Iri20]. The main distinctions between these earlier applications and our Strategy-Theorem are the features described above, beyond the use of Galois action. The heart of the first half of the above theorem is the Global Gamma-I theorem. It guarantees that one can move from a starting point $\mathbf{t}_0^{(2)}$ to a possibly distant endpoint $\mathbf{t}_1^{(2)}$, although a major difficulty in applying the theorem lies in finding such a path in the (SR)-region. Motivated by [GHIKLS24, Sections 6 and 7], the endpoint may be chosen near a boundary region of the Kähler moduli space where the spectrum of $c_1(X)\star_{\mathbf{t}^{(2)}}$ organizes itself in a way that reflects the geometry of X and makes condition (4) in the Strategy-Theorem accessible.

Remark 1.3. Assumption (3) implicitly requires the small quantum cohomology to be generically semisimple, and the current strategy works primarily when $\mathcal{D}^b_{\rm coh}(X)$ is generated by line bundles. The above theorem provides a transparent approach to studying Gamma conjecture II in this setting. We anticipate that the Strategy-Theorem may admit generalizations to Fano manifolds with non-semisimple small quantum cohomology and to cases in which $\mathcal{D}^b_{\rm coh}(X)$ is not generated by line bundles, by investigating an analogue of the Global Gamma-I property in a more general setting.

It is known that a 2-dimensional Fano manifold is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$, or a del Pezzo surface X_r (i.e. the blow-up of $\mathbb{P}^2$ at r general points) where $1 \leq r \leq 8$. The del Pezzo surface X_r is toric if and only if $1 \leq r \leq 3$. Since its proposal in [GGI16], Gamma conjecture II has remained open for (non-toric) del Pezzo surfaces for a decade, even though the geometry of these low-dimensional cases is relatively well understood. As another highlight of this paper, we prove the following theorem, by verifying all assumptions of the Strategy-Theorem in a precise manner. The proof involves a study of Puiseux expansions of eigenvalues of $c_1(X)\star_{\mathbf{t}^{(2)}}$.

Theorem 1.4. *Gamma conjecture II holds for all del Pezzo surfaces.*

Part (1) of Dubrovin's conjecture for del Pezzo surfaces X_r was established in [BM04]. As shown in [GGI16, Proposition 4.6.6], Gamma conjecture II implies parts (2) and (3) of Dubrovin's conjecture. Therefore, we obtain the following result, as an immediate corollary of the above theorem.

Corollary 1.5. *Dubrovin's conjecture holds for all del Pezzo surfaces.*

We remark that part (2) of Dubrovin's conjecture was shown for X_6 [Ued05].

Gamma conjecture II can be formulated straightforwardly for smooth projective manifolds with semisimple big quantum cohomology. Let S_r be an r-fold blow-up of $\mathbb{P}^2$, i.e., S_r is obtained by $S_r \xrightarrow{b_r} S_{r-1} \xrightarrow{b_{r-1}} \cdots \xrightarrow{b_2} S_1 \xrightarrow{b_1} \mathbb{P}^2$, where each b_i is a blow-up at one point. The rational surfaces are not Fano in general, but they can be deformed to X_r when $r \leq 8$. As an application of Theorem 1.4, we obtain

Corollary 1.6. *Gamma conjecture II holds for S_r with $1 \leq r \leq 8$.*

We anticipate that our Strategy-Theorem has broader applications, thanks to the extensive existing literature on Gamma conjecture I. Indeed, combined with mirror symmetry, it can be applied to verify Gamma conjecture II for Milnor hypersurfaces (i.e. smooth degree-$(1,1)$ hypersurfaces in $\mathbb{P}^n \times \mathbb{P}^m$) [HKLS26].

Let us briefly describe how to apply the Strategy-Theorem to prove Theorem 1.4. We need to show that conditions (1)-(4) are satisfied. Condition (1) follows from the convergence criterion of Iritani [Iri07, Corollary 5.9]. Condition (2) is satisfied by choosing $\mathbf{t}_0^{(2)} = \mathbf{0}$ and using the original Gamma conjecture I proved in [HKLY21]. For condition (4), we choose the endpoint $\mathbf{t}_1^{(2)} \in K_{ts}$ near the boundary of the Kähler moduli reflecting the blow-up structure of X_r from $\mathbb{P}^2$; heuristically, this corresponds to letting the volumes of the exceptional divisors tend to infinity. After applying Galois actions to find initial objects $V_1, \dots, V_s$ as in (4.1), we succeed in finding "elementary operations" on these initial objects to produce a full exceptional collection $(\tilde{V}_1, \dots, \tilde{V}_s)$ as in (4.2), which satisfies condition (4.3).

The most technically involved step in the proof of Theorem 1.4 is to verify condition (3), i.e., to find a suitable domain U inside the (SR)-region of X containing both $\mathbf{t}_0^{(2)}$

and $\mathbf{t}_1^{(2)}$. This depends on the distribution of the eigenvalues of $c_1(X)\star_{\mathbf{t}^{(2)}}$. The Perron–Frobenius theorem for nonnegative matrices plays a key role, in analogy with the proof of Conjecture $\mathcal{O}$ for flag varieties [ChLi17]. We achieve this by reducing to a positivity problem for a matrix representing $c_1(X_r)\star_{\mathbf{t}^{(2)}}$. As $\mathbf{t}^{(2)}$ moves from the origin toward the boundary corresponding to the blow-up structure of X_r from $\mathbb{P}^2$, we choose a basis $\mathbf{1}, aH - \sum_i E_i, E_1, \cdots, E_r, [pt]$, with a suitable parameter $a > 0$. To apply the Perron–Frobenius theorem, we need careful entrywise estimates in this basis along a specific path approaching the boundary; these are based on the two key observations (o1) and (o2) made in Section 5.2. They produce the desired domain inside the (SR)-region, where the Perron–Frobenius theorem gives the required simple rightmost eigenvalue.

We point out that our proof of Gamma conjecture II requires a rather involved manipulation of eigenvalues. In particular, the finer estimates for del Pezzo surfaces goes beyond what is provided by the general results on the quantum spectrum of blow-ups of algebraic surfaces in [GS25]. Moreover, the proof relies on remarkably little explicit information regarding the Gromov-Witten invariants of X_r, as provided in (4.2) and Lemma 5.4.

Remark 1.7. More generally, it is important to study the quantum spectrum of the Euler vector field $E(\mathbf{t})$. A conjecture concerning the quantum spectrum for blow-ups was proposed by Kontsevich in several talks. The general case was proved by Iritani [Iri25], and the surface case was also shown by Gyenge and Szabó [GS25]. The quantum spectrum plays an important role in the breakthrough work [KKPY25] by Katzarkov, Kontsevich, Pantev and Yu, which resolves the long-standing irrationality problem for very general cubic fourfolds.

Remark 1.8. In the context of noncommutative Hodge theory [KKP08], Katzarkov, Kontsevich, and Pantev proposed a $\mathbb{Q}$-structure on $QH_{\mathrm{big}}(X)$, which can be viewed as the rational counterpart of Iritani's $\widehat{\Gamma}$-integral structure. Gamma conjecture II can be understood as a compatibility between the Betti structure and the Stokes structure, as discussed by Hertling and Sevenheck [HS07] in the context of TERP structures. Gamma conjecture II was formulated under the assumption that $QH_{\mathrm{big}}(X)$ is generically semisimple. For the setting in which $QH_{\mathrm{big}}(X)$ is non-semisimple, Sanda and Shamoto [SS20] proposed a corresponding generalization, referred to as a Dubrovin-type conjecture. In this case, there was also a topological formulation proposed by Galkin and described in [Iri23], which is coined as "Gamma conjecture III" and makes sense for any compact symplectic manifolds.

The paper is organized as follows. In Section 2, we review the precise statement of Gamma conjecture II. In Section 3, we prove the Global Gamma-I theorem and the Strategy-Theorem. In Section 4, we show that Gamma conjecture II holds for del Pezzo surfaces. Finally, in Section 5, we prove a technical proposition needed in the proof of Theorem 1.4.

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2. Brief review of Gamma conjecture II

In this section, we review Gamma conjecture II proposed by Galkin, Golyshev and Iritani [GGI16]. It was originally stated for Fano manifolds with semisimple big quantum cohomology, and can be straightforwardly generalized to projective manifolds with the same property. By [HMT09], any such projective manifold has vanishing odd cohomology. We therefore begin with a projective manifold X satisfying $H^*(X) = H^{\text{even}}(X, \mathbb{C})$.

2.1. **Quantum cohomology and Dubrovin connection.** Let $\text{Eff}(X) \subset H_2(X, \mathbb{Z})$ be the semigroup of effective curve classes. Denote by $\overline{M}_{0,k}(X, \mathbf{d})$ the moduli space of k-pointed, genus-zero stable maps of degree $\mathbf{d} \in \text{Eff}(X)$. The genus-zero Gromov-Witten invariants of X are defined as

$$\langle \prod_{i=1}^{k} \tau_{a_i}(\gamma_i) \rangle_{\mathbf{d}}^X := \int_{[\overline{M}_{0,k}(X,\mathbf{d})]^{\text{vir}}} \prod_{i=1}^{k} c_1(L_i)^{a_i} \text{ev}_i^* \gamma_i,$$

where $\gamma_i \in H^*(X)$, $a_i \in \mathbb{Z}_{\geq 0}$, L_i is the universal cotangent line bundle at the i-th marked point, $\overline{M}_{0,k}(X, \mathbf{d}) \xrightarrow{\text{ev}_i} X$ is the evaluation map at the i-th marked point, and $[\overline{M}_{0,k}(X, \mathbf{d})]^{\text{vir}}$ is the *virtual fundamental class*. They satisfy a number of axioms [CK99, Chapter 7.3], such as Degree Axiom (i.e., the dimension constraint) and Divisor Axiom. Roughly speaking, $\langle \gamma_1, \ldots, \gamma_k \rangle_{\mathbf{d}}^X := \langle \prod_{i=1}^{k} \tau_0(\gamma_i) \rangle_{\mathbf{d}}^X$ is the (virtual) number of degree-$\mathbf{d}$ rational curves in X intersecting the Poincaré dual cycles of $\gamma_1, ..., \gamma_k$. We refer the reader to [CK99] for general introductions of Gromov-Witten invariants.

Let $\{T_i\}_{i=0}^{s-1}$ be a homogeneous basis of $H^*(X)$, such that $T_0 = \mathbf{1}$ and $\{T_i\}_{i=1}^{s'}$ is a nef basis in $H^2(X, \mathbb{Z})$. Write a general element of $H^*(X)$ as $\mathbf{t} = \sum_{i=0}^{s-1} t_i T_i$. We decompose $\mathbf{t} = \mathbf{t}^{(2)} + \mathbf{t}'$, where $\mathbf{t}^{(2)} \in H^2(X)$ and $\mathbf{t}' \in \bigoplus_{p \neq 1} H^{2p}(X)$. The potential function of primary invariants is defined by $\mathcal{F}_X(\mathbf{t}) := \sum_{\mathbf{d} \in \text{Eff}(X)} \langle \exp(\tau_0(\mathbf{t}')) \rangle_{\mathbf{d}}^X \cdot \mathrm{e}^{\mathbf{t}^{(2)}(\mathbf{d})}$. By the nef assumption of $\{T_i\}_{i=1}^{s'}$, we see that $\mathcal{F}_X(\mathbf{t}) \in \mathbb{C}[[t_0, q_1, ..., q_{s'}, t_{s'+1}, \ldots, t_{s-1}]]$, with $q_i := \mathrm{e}^{t_i}$. The quantum product $\star_{\mathbf{t}}$ on $H^*(X)$ is defined by

$$(T_i \star_{\mathbf{t}} T_j, T_k) = \partial_{t_i} \partial_{t_j} \partial_{t_k} \mathcal{F}_X(\mathbf{t}), \tag{2.1}$$

where $(\cdot, \cdot)$ is the Poincaré pairing on $H^*(X)$. The family $(H^*(X), \star_{\mathbf{t}})$ is called the big quantum cohomology of X.

We impose the following convergence assumption:

(2.2) the potential function $\mathcal{F}_X(\mathbf{t})$ **converges near the large radius limit**.

That is, there exists $\delta > 0$ such that if $|t_0|, |\mathrm{e}^{t_1}|, ..., |\mathrm{e}^{t_{s'}}|, |t_{s'+1}|, ..., |t_{s-1}| < \delta$, then $\mathcal{F}_X(\mathbf{t})$ converges absolutely as a series in $t_0, \mathrm{e}^{t_1}, \ldots, \mathrm{e}^{t_{s'}}, t_{s'+1}, \cdots, t_{s-1}$.

Remark 2.1. If X is Fano and $H^*(X)$ is generated by $H^2(X)$ (e.g. a del Pezzo surface), then $\mathcal{F}_X(\mathbf{t})$ converges near the large radius limit by [Iri07, Corollary 5.9].

Let $B \subset H^*(X)$ be the domain of convergence of the potential function $\mathcal{F}_X(\mathbf{t})$. For any $\mathbf{t} \in B$, $(H^*(X), \star_{\mathbf{t}})$ is a commutative and associative $\mathbb{C}$-algebra with identity $\mathbf{1}$; we say that $\mathbf{t}$ is a **semisimple** point if $(H^*(X), \star_{\mathbf{t}})$ is semisimple, namely if $(H^*(X), \star_{\mathbf{t}})$ is isomorphic to $\mathbb{C} \oplus \cdots \oplus \mathbb{C}$ as a $\mathbb{C}$-algebra. We further call $\mathbf{t}$ a **tame semisimple** point if $(E(\mathbf{t})\star_{\mathbf{t}})$ has only simple eigenvalues, where

$$E(\mathbf{t}) := c_1(X) + \sum_{i=0}^{s-1} \left(1 - \frac{1}{2} \deg T_i\right) t_i T_i$$

is called the **Euler vector field**. Let $B_{ss} \subset B$ be the subset of all semisimple points. We say that the big quantum cohomology of X is semisimple if $B_{ss} \neq \varnothing$. In this case, $B \setminus B_{ss}$ is a divisor in B.

The (big) **Dubrovin connection** ∇ [Dub96, Dub98, Dub99] is a meromorphic flat connection on the trivial $H^*(X)$-bundle over $B \times \mathbb{P}^1$. With respect to a frame of constant sections, it reads

$$\nabla = \mathrm{d} + \sum_{i=0}^{s-1} \frac{1}{z}(T_i \star_{\mathbf{t}})\mathrm{d}t_i - \left(\frac{1}{z}\left(E(\mathbf{t})\star_{\mathbf{t}}\right) - \mu\right)\frac{\mathrm{d}z}{z},$$

where z is the inhomogeneous coordinate on $\mathbb{P}^1$, and $\mu \in \mathrm{End}(H^*(X))$ is the grading operator defined by $\mu(T_i) = \frac{1}{2}\left(\deg T_i - \dim_{\mathbb{C}} X\right) T_i$. Let $\widetilde{\mathbb{C}^\times}$ be the universal cover of $\mathbb{C}^\times = \mathbb{P}^1 \setminus \{0, \infty\}$, and denote by $\widetilde{\nabla}$ the pullback of ∇ via $B \times \widetilde{\mathbb{C}^\times} \to B \times \mathbb{C}^\times$. Denote

$$\mathcal{S} := \{\widetilde{\nabla}\text{-flat sections of the trivial } H^*(X)\text{-bundle over } B \times \widetilde{\mathbb{C}^\times}\}.$$

For $\mathbf{t} \in B$, following Givental [Giv96, Corollary 6.2], we define a formal endomorphism $L(\mathbf{t}, z) \in \mathrm{End}(H^*(X))[[\frac{1}{z}]]$ such that the Poincaré pairing $(L(\mathbf{t}, z)T_i, T_j)$ equals

$$\left(\mathrm{e}^{-\frac{\mathbf{t}^{(2)}}{z}} T_i, T_j\right) + \sum_{m \geq 0} \left(\frac{-1}{z}\right)^{m+1} \sum_{\mathbf{d} \in \mathrm{Eff}(X)} \langle \tau_m \left(\mathrm{e}^{-\frac{\mathbf{t}^{(2)}}{z}} T_i\right) \exp\left(\tau_0(\mathbf{t}')\right) \tau_0(T_j)\rangle_{\mathbf{d}}^X \cdot \mathrm{e}^{\mathbf{t}^{(2)}(\mathbf{d})}.$$

The **cohomology framing** of $\mathcal{S}$ is the $\mathbb{C}$-linear isomorphism

$$\mathcal{Z} : \ H^*(X) \xrightarrow{\cong} \mathcal{S}; \ \alpha \mapsto \left((\mathbf{t}, z) \mapsto L(\mathbf{t}, z) z^{-\mu} z^{c_1(X)\cup} \alpha\right),$$

where $z^{-\mu} z^{c_1(X)\cup} := \mathrm{e}^{\mu \log z} \mathrm{e}^{(c_1(X)\cup)\log z}$. Let $K(X)$ be the Grothendieck group of topological complex vector bundles on X. The K**-group framing** of $\mathcal{S}$ is the group homomorphism

$$\mathcal{Z}^K : K(X) \longrightarrow \mathcal{S}; \ V \mapsto (2\pi)^{-\frac{1}{2}} \mathcal{Z}\left(\widehat{\Gamma}_X \cup \widetilde{\mathrm{ch}}(V)\right).$$

Here $\widetilde{\mathrm{ch}}$ denotes the modified Chern character defined as $\widetilde{\mathrm{ch}}(V) := \sum_{p \geq 0} (2\pi\mathbf{i})^p \,\mathrm{ch}_p(V)$. The Gamma class $\widehat{\Gamma}_X$ of X is defined by

$$\widehat{\Gamma}_X := \prod_{i=1}^{n} \Gamma(1 + \delta_i) \in H^*(X), \tag{2.3}$$

where δ_i are the Chern roots of X and $\Gamma(x)$ denotes Euler's Gamma function. Iritani's $\widehat{\Gamma}$**-integral structure** [Iri09] is the image $\mathcal{S}_{\mathbb{Z}} \subset \mathcal{S}$ of $\mathcal{Z}^K$, which is a lattice in $\mathcal{S}$ of rank $\dim_{\mathbb{C}} \mathcal{S}$.

Remark 2.2. Inspired by "remarkable identities" of Hosono, Klemm, Theisen, and Yau [HKTY95] in the context of mirror symmetry, Libgober [Lib99] introduced the (inverse) Gamma class. In [Iri09], Iritani introduced the $\widehat{\Gamma}$-integral structure, and showed that, when X is toric Fano, the structure matches with the natural integral structure in the Landau–Ginzburg B-model of X. A similar rational structure for quantum cohomology was also proposed by Katzarkov–Kontsevich–Pantec [KKP08] in their study of non-commutative Hodge structures. We refer the reader to [GGI16, Section 1.5] for a detailed account of references.

The free abelian group $\mathcal{S}_{\mathbb{Z}}$ is endowed with a $\mathbb{Z}$-valued bilinear pairing $[\cdot,\cdot)$ defined as follows. For any $s_1, s_2 \in \mathcal{S}$, it is known that the pairing $\big(s_1(\mathbf{t}, \mathrm{e}^{-\pi\mathbf{i}}z), s_2(\mathbf{t}, z)\big)$ is a constant function on $B \times \widetilde{\mathbb{C}^\times}$. This induces a $\mathbb{C}$-bilinear function $\mathcal{S} \times \mathcal{S} \xrightarrow{[\cdot,\cdot)} \mathbb{C}$ defined by $[s_1, s_2) := \big(s_1(\mathbf{t}, \mathrm{e}^{-\pi\mathbf{i}}z), s_2(\mathbf{t}, z)\big)$, which is nondegenerate, but is not necessarily symmetric or anti-symmetric. When restricted to $\mathcal{S}_{\mathbb{Z}}$, the pairing takes values in $\mathbb{Z}$. Moreover, for any $V_1, V_2 \in K(X)$, we have

$$[\mathcal{Z}^K(V_1), \mathcal{Z}^K(V_2)) = \int_X \mathrm{ch}(V_1^\vee)\,\mathrm{ch}(V_2)\mathrm{td}(X) =: \chi(V_1, V_2) \in \mathbb{Z}. \tag{2.4}$$

We refer the reader to [Iri09] for more discussions on the pseudolattice $(\mathcal{S}_{\mathbb{Z}}, [\cdot,\cdot))$.

2.2. **Exceptional collections.** We refer the reader to [BP94] for further details of this subsection. An object F in the bounded derived category $\mathcal{D}^b_{\mathrm{coh}}(X)$ of coherent sheaves on X is called **exceptional** if $\mathrm{Hom}(F, F) = \mathbb{C}$ and $\mathrm{Hom}(F, F[j]) = 0$, where $[j]$ refers to applying the shift functor j times, $j \neq 0$. An ordered tuple $(F_1, \ldots, F_s)$ of such exceptional objects forms an **exceptional collection** if $\mathrm{Hom}(F_{i_1}, F_{i_2}[j]) = 0$ for all integers j whenever $i_1 > i_2$. Two such collections are considered equivalent up to shift if they differ only by a sequence of integer shifts applied to their respective components. An exceptional collection is called **full** if it generates the category $\mathcal{D}^b_{\mathrm{coh}}(X)$.

Every exceptional object F induces **left** and **right mutation functors**, L_F and R_F, which are defined for any object $F' \in \mathcal{D}^b_{\mathrm{coh}}(X)$ via the distinguished triangles

$$L_F F'[-1] \to \mathrm{Hom}^\bullet(F, F') \otimes F \to F' \to L_F F', \text{ and}$$
$$R_F F' \to F' \to \mathrm{Hom}^\bullet(F', F)^* \otimes F \to R_F F'[1].$$

The set of exceptional collections of length N in $\mathcal{D}^b_{\mathrm{coh}}(X)$ admits a natural action by the braid group B_N. The generator σ_i acts by replacing the adjacent pair (F_i, F_{i+1}) with $(F_{i+1}, R_{F_{i+1}}F_i)$, while σ_i^{-1} replaces it with $(L_{F_i}F_{i+1}, F_i)$. The full triangulated subcategory generated by the collection is invariant under these mutations.

Throughout the present paper, we abuse notation by identifying an object of $\mathcal{D}^b_{\mathrm{coh}}(X)$ with its corresponding class in the Grothendieck group $K(X)$ of topological complex vector bundles on X. For derived objects E, F, the Euler pairing χ in (2.4) satisfies $\chi(E, F) = \sum_i (-1)^i \dim \mathrm{Ext}^i(E, F)$ by Hirzebruch-Riemann-Roch. Whenever $\mathcal{D}^b_{\mathrm{coh}}(X)$ possesses a full exceptional collection $(F_1, \ldots, F_s)$, the abelian group $K(X)$ is torsion-free and the classes $F_1, \ldots, F_s$ in $K(X)$ form a $\mathbb{Z}$-basis for $K(X)$; moreover, we have $\chi(F_i, F_i) = 1$ and $\chi(F_j, F_i) = 0$ for $j > i$. Thus, any full exceptional collection in $\mathcal{D}^b_{\mathrm{coh}}(X)$ naturally determines an exceptional basis for the pseudolattice $(K(X), \chi)$.

2.3. **Statement of Gamma conjecture II.** Assume $B_{ss} \neq \varnothing$. For a sufficiently small domain $U \subset B_{ss}$, Dubrovin [Dub99, Theorem 3.1, Lemma 3.2] proved that there exist holomorphic coordinates $\{u_i\}$ on U (called **canonical coordinates**), such that for each $\mathbf{t} \in U$, the vectors $\{\partial_{u_i}\mathbf{t}\}$ form an **idempotent basis** of $(H^*(X), \star_{\mathbf{t}})$ (i.e. $\partial_{u_i}\mathbf{t} \star_{\mathbf{t}} \partial_{u_j}\mathbf{t} = \delta_{ij}\partial_{u_i}\mathbf{t}$), and $E(\mathbf{t}) \star_{\mathbf{t}} \partial_{u_i}\mathbf{t} = u_i(\mathbf{t}) \cdot \partial_{u_i}\mathbf{t}$. Assume moreover that the **normalized idempotents** $\Psi_i(\mathbf{t}) := (\partial_{u_i}\mathbf{t}, \partial_{u_i}\mathbf{t})^{-\frac{1}{2}}\, \partial_{u_i}\mathbf{t}$ are well-defined holomorphic maps on U (which holds, for instance, if U is simply connected). Then we say that

$$U \text{ is } \textbf{properly-chosen} \text{ with respect to } \{(u_i, \Psi_i)\}. \tag{2.5}$$

The functions u_i are unique up to ordering, and Ψ_i are defined up to sign.

Definition 2.3. Let $U \subset B_{ss}$ be properly-chosen with respect to $\{(u_i, \Psi_i)\}$ and $\phi \in \mathbb{R}$. For $y \in \mathcal{S}$, we say that y **respects** (u_i, Ψ_i) **over** U **with phase** ϕ, if there exists $\varepsilon > 0$ such that $\mathrm{e}^{\frac{u_i(\mathbf{t})}{z}} y(\mathbf{t}, z) \to \Psi_i(\mathbf{t})$, as $z \to 0$ with $|\arg z - \phi| < \frac{\pi}{2} + \varepsilon$. We say that y respects (u_i, Ψ_i) **at** $\mathbf{t}_0 \in U$ with phase ϕ, if there exists $\varepsilon > 0$ such that $\mathrm{e}^{\frac{u_i(\mathbf{t}_0)}{z}} y(\mathbf{t}_0, z) \to \Psi_i(\mathbf{t}_0)$, as $z \to 0$ with $|\arg z - \phi| < \frac{\pi}{2} + \varepsilon$.

For $\mathbf{t} \in B$, we say that a phase $\phi \in \mathbb{R}$ is **admissible** at $\mathbf{t}$ if $\mathrm{e}^{\mathbf{i}\phi}$ is not parallel to any nonzero difference of eigenvalues of $E(\mathbf{t})\star_{\mathbf{t}}$. This is an open condition on $\mathbb{R}$.

Proposition 2.4. [GGI16, Proposition 2.5.1] *Suppose that $U \subset B_{ss}$ is properly-chosen with respect to $\{(u_i, \Psi_i)\}$, $\mathbf{t}_0 \in U$ and $\phi \in \mathbb{R}$ is an admissible phase at $\mathbf{t}_0$. Then there exists a unique basis $\{y_i\}$ of $\mathcal{S}$ satisfying the following property: there exists an open neighborhood $U' \subset U$ of $\mathbf{t}_0$ such that y_i respects (u_i, Ψ_i) over U' with phase ϕ for each i.*

The basis $\{y_i\}$ of $\mathcal{S}$ in the above proposition is called the **asymptotically exponential fundamental solution** (AEFS for short) to the equation $\widetilde{\nabla} s = 0$ near $\mathbf{t}_0$ associated to the phase ϕ with respect to $\{\Psi_i\}$. It follows from [HK23, Lemma 2.7] that the above AEFS $\{y_i\}$ can be uniquely characterized as follows: for each i, y_i respects (u_i, Ψ_i) **at** $\mathbf{t}_0$ with phase ϕ.

Roughly speaking, Gamma conjecture II expects that an AEFS lies in $\mathcal{S}_{\mathbb{Z}}$. The precise statement is as follows.

Gamma conjecture II. [GGI16, Conjecture 4.6.1] *Assume that: (i) the big quantum cohomology of X is semisimple; (ii) $\mathcal{D}^b_{\mathrm{coh}}(X)$ admits a full exceptional collection. Suppose that $U \subset B_{ss}$ is properly-chosen with respect to $\{(u_i, \Psi_i)\}$, $\mathbf{t}_0 \in U$ and $\phi \in \mathbb{R}$ is an admissible phase at $\mathbf{t}_0$, where the u_i are ordered so that $\mathrm{Im}\left(\mathrm{e}^{-\mathbf{i}\phi} u_1(\mathbf{t}_0)\right) \geq \ldots \geq \mathrm{Im}\left(\mathrm{e}^{-\mathbf{i}\phi} u_s(\mathbf{t}_0)\right)$. Then there exists a full exceptional collection $(E_1, \ldots, E_s)$ of $\mathcal{D}^b_{\mathrm{coh}}(X)$ such that for each i, $\mathcal{Z}^K(E_i)$ respects (u_i, Ψ_i) at $\mathbf{t}_0$ with phase ϕ.*

By [GGI16, Remark 4.6.3] and [GI19, Remark 4.13], the validity of Gamma conjecture II does not depend on the choices of semisimple points and admissible phases.

3. Gamma conjecture II via Gamma-I over Kähler moduli

Gamma conjecture I was proposed for Fano manifolds in terms of the small quantum cohomology at the specialization of $\mathbf{q} = \mathbf{1}$ [GGI16]. It appears to be largely independent of Gamma conjecture II. Nevertheless, in this section we establish the Strategy-Theorem (i.e. Theorem 3.10) that reveals a deep connection between Gamma conjecture II and property *Gamma-I* over the Kähler moduli space. We now further assume that X is a Fano manifold (or more generally, that X can be deformed to a Fano manifold, by deformation invariance of Gromov-Witten theory).

3.1. **Small quantum cohomology.** We use notations in Section 2.1. Recall $q_k = e^{t_k}$ for $k = 1, \ldots, s'$. For any $\mathbf{t}^{(2)} \in H^2(X)$, it follows from the dimension axiom of Gromov-Witten theory and the Fano property of X that

$$T_i \star_{\mathbf{t}^{(2)}} T_j \in \mathbb{C}[q_1, \ldots, q_{s'}] \otimes H^*(X).$$

Therefore, we obtain a $\mathbb{C}[\mathbf{q}]$-family of $\mathbb{C}$-algebras $(H^*(X), \star_{\mathbf{t}^{(2)}})$, called the small quantum cohomology of X.

Analogously to the big quantum cohomology case, the (small) Dubrovin connection ∇^{sm} is a meromorphic flat connection on the trivial $H^*(X)$-bundle over $H^2(X) \times \mathbb{P}^1$, and with respect to a frame of constant sections, it reads

$$\nabla^{\mathrm{sm}} = \mathrm{d} + \sum_{i=1}^{s'} \frac{1}{z}(T_i \star_{\mathbf{t}^{(2)}})\mathrm{d}t_i - \left(\frac{1}{z}\hat{c}_1(\mathbf{t}^{(2)}) - \mu\right)\frac{\mathrm{d}z}{z}, \text{ where } \hat{c}_1(\mathbf{t}^{(2)}) := c_1(X)\star_{\mathbf{t}^{(2)}} .$$

Moreover, we obtain the space $\mathcal{S}^{\mathrm{sm}}$ of $\widetilde{\nabla}^{\mathrm{sm}}$-flat sections, together with cohomology framing $\mathcal{Z}^{\mathrm{sm}}$, K-group framing $\mathcal{Z}^{K,\mathrm{sm}}$ and $\widehat{\Gamma}$-integral structure $\mathcal{S}^{\mathrm{sm}}_{\mathbb{Z}}$, defined as in the big case with the operator $L(\mathbf{t}, z)$ replaced by $L(\mathbf{t}^{(2)}, z)$.

In particular, for any $\mathbf{t}^{(2)} \in B \cap H^2(X)$, we have $\mathcal{Z}^K(V)(\mathbf{t}^{(2)}, z) = \mathcal{Z}^{K,\mathrm{sm}}(V)(\mathbf{t}^{(2)}, z)$ for any $V \in K(X)$.

We will use Iritani's **Galois action** of $H^2(X,\mathbb{Z})$ on $\mathcal{S}^{\mathrm{sm}}_{\mathbb{Z}}$, described as follows. For any $\xi \in H^2(X,\mathbb{Z})$, consider the isomorphism of trivial $H^*(X)$-bundles over $H^2(X) \times \mathbb{P}^1$ defined by

$$H^*(X) \times (H^2(X) \times \mathbb{P}^1) \to H^*(X) \times (H^2(X) \times \mathbb{P}^1); (\alpha, \mathbf{t}^{(2)}, z) \mapsto (\alpha, \mathbf{t}^{(2)} - 2\pi\mathbf{i}\xi, z).$$

Then it is an automorphism of ∇^{sm}, and it induces an action on $\mathcal{S}^{\mathrm{sm}}_{\mathbb{Z}}$

$$G(\xi) : \mathcal{S}^{\mathrm{sm}}_{\mathbb{Z}} \to \mathcal{S}^{\mathrm{sm}}_{\mathbb{Z}}; s \mapsto \Big((\mathbf{t}^{(2)}, z) \mapsto s(\mathbf{t}^{(2)} - 2\pi\mathbf{i}\xi, z)\Big). \tag{3.1}$$

To describe the action $G(\xi)$ in terms of K-group framing, let L_ξ be the topological complex line bundle with $c_1(L_\xi) = \xi$, and we have

$$\mathcal{Z}^{K,\mathrm{sm}}(V \otimes L_\xi) = G(\xi)\left(\mathcal{Z}^{K,\mathrm{sm}}(V)\right), \quad \forall V \in K(X). \tag{3.2}$$

We refer the reader to [Iri09] for more details. For any path $\phi : [0,1] \to H^2(X)$ with $\phi(0) = \mathbf{t}^{(2)}$ and $\phi(1) = \mathbf{t}^{(2)} - 2\pi\mathbf{i}\xi$, we will say that $V \otimes L_\xi$ **is obtained from** V **via Galois action given by the path** ϕ.

3.2. **Condition** (SR) **and Gamma-I.** In this subsection, we recall condition (SR), introduce property Gamma-I, and prove the global Gamma-I Theorem 3.7. We emphasize that the semisimplicity of the big/small quantum cohomology is *NOT* assumed in this subsection.

By condition (SR) at $\mathbf{t}^{(2)}$, we mean

(SR) $\qquad \hat{c}_1(\mathbf{t}^{(2)})$ has a simple rightmost eigenvalue.

That is, the linear operator $\hat{c}_1(\mathbf{t}^{(2)}) \in \mathrm{End}(H^*(X))$ has a (unique) simple eigenvalue u, such that $\mathrm{Re}\, u' < \mathrm{Re}\, u$ for every other eigenvalue u' of $\hat{c}_1(\mathbf{t}^{(2)})$. By the (SR)-region of X, we mean the subset $\{\mathbf{t}^{(2)} \in H^2(X) \mid X \text{ satisfies (SR) at } \mathbf{t}^{(2)}\}$ of $H^2(X)$. This is an open subset of $H^2(X)$, which need not be connected when nonempty.

Suppose that X satisfies condition (SR) at $\mathbf{t}^{(2)} \in H^2(X)$, and let $u \in \mathbb{C}$ be the simple rightmost eigenvalue of $\hat{c}_1(\mathbf{t}^{(2)})$. Denote by $\mathcal{A}(\mathbf{t}^{(2)})$ the $\mathbb{C}$-linear space consisting of $\widetilde{\nabla}^{\mathrm{sm}}$-flat sections $y(\mathbf{t}^{(2)}, z)$ such that $\mathrm{e}^{\frac{u}{z}} y(\mathbf{t}^{(2)}, z)$ has moderate growth as $z \to 0$ along the positive real axis (i.e. $|\arg z| = 0$). Then $\dim_{\mathbb{C}} \mathcal{A}(\mathbf{t}^{(2)}) = 1$ by [GGI16, Remark 3.1.9].

Definition 3.1 (Property Gamma-I)**.** We say that X **satisfies Gamma-I at** $\mathbf{t}^{(2)} \in H^2(X)$, if X satisfies condition (SR) at $\mathbf{t}^{(2)}$ and $\mathcal{Z}^{K,\mathrm{sm}}(\mathcal{O}) \in \mathcal{A}(\mathbf{t}^{(2)})$.

We point out that the property Gamma-I does *NOT* assume convergence or semisimplicity of big quantum cohomology.

Remark 3.2. The original Gamma conjecture I [GGI16, Conjecture 3.4.3] is slightly stronger than the statement that, if a Fano manifold satisfies condition (SR) at $\mathbf{0} \in H^2(X)$, then it satisfies Gamma-I at $\mathbf{0}$. Counter-examples were found in [GHIKLS24], for which the aforementioned statement fails but Gamma-I at some nonzero $\mathbf{t}^{(2)}$ is satisfied.

Assume that X satisfies (SR) at $\mathbf{t}_0^{(2)}$ with $u_0 \in \mathbb{C}$ the simple rightmost eigenvalue. The equation $\det\big(u - (\hat{c}_1(\mathbf{t}^{(2)}))\big) = 0$ determines, by the Implicit Function Theorem, a unique holomorphic function $u(\mathbf{t}^{(2)})$ near $\mathbf{t}_0^{(2)}$ with $u(\mathbf{t}_0^{(2)}) = u_0$. As a consequence, for any domain U inside the (SR)-region of X, there are unique holomorphic maps $U \xrightarrow{u} \mathbb{C}$ and $U \xrightarrow{\psi} H^*(X)$, such that for each $\mathbf{t}^{(2)} \in U$, $u(\mathbf{t}^{(2)})$ is the simple rightmost eigenvalue of $\hat{c}_1(\mathbf{t}^{(2)})$, and $\psi(\mathbf{t}^{(2)})$ is the idempotent generator of the associated one-dimensional eigenspace. Whenever the normalized idempotent $\Psi := (\psi,\psi)^{-\frac{1}{2}}\psi$ is a well-defined holomorphic map (which holds, for instance, if U is simply connected), we say that

$$(u, \Psi) \text{ is } \textbf{a simple rightmost pair} \text{ over } U. \tag{3.3}$$

Before we investigate Gamma-I, we need the following result from linear algebra. Note that with respect to the Poincaré pairing, $\hat{c}_1(\mathbf{t}^{(2)})$ is symmetric, and the grading operator μ is anti-symmetric by degree-counting.

Lemma 3.3. *Suppose that V is a finite-dimensional $\mathbb{C}$-linear space, and $(\cdot,\cdot)$ is a bilinear pairing on V which is symmetric and non-degenerate. Let $S, A \in \mathrm{End}_{\mathbb{C}}(V)$ be such that S is symmetric and A is anti-symmetric with respect to $(\cdot,\cdot)$. Suppose that λ is a simple eigenvalue of S with the corresponding eigenspace V_λ, and let V' be the direct sum of generalized eigenspaces of S with eigenvalues $\neq \lambda$. Then: (i) $V' = \mathrm{Im}(S - \lambda \cdot id)$; (ii) V_λ and V' are mutually orthogonal complements; (iii) $A(V_\lambda) \subset V'$.*

Proof. Statement (i) follows from the assumption $\dim_{\mathbb{C}} V_\lambda = 1$. The statement (ii) holds since generalized eigenspaces corresponding to distinct eigenvalues of S are mutually orthogonal. For (iii), let $v \in V_\lambda \setminus \{0\}$; noting that A is anti-symmetric, we have

$$(A(v), v) = -(v, A(v)) \Rightarrow (A(v), v) = 0 \Rightarrow A(v) \perp v.$$

Then $A(v) \perp V_\lambda$ since $\dim_{\mathbb{C}} V_\lambda = 1$, and hence $A(v) \in V'$ by (ii). This proves (iii). ☐

The following Lemma 3.4 and 3.5 concern a **simple pair** (u, Ψ) over a domain $U \subset H^2(X)$, that is, holomorphic maps $U \xrightarrow{u} \mathbb{C}$ and $U \xrightarrow{\Psi} H^*(X)$, such that for each $\mathbf{t}^{(2)} \in U$, $u(\mathbf{t}^{(2)})$ is a simple eigenvalue of $\hat{c}_1(\mathbf{t}^{(2)})$, and $\Psi(\mathbf{t}^{(2)})$ is a normalized idempotent in the associated one-dimensional eigenspace. A simple rightmost pair is a special case of a simple pair.

Lemma 3.4. *Suppose that (u, Ψ) is a simple pair over a domain $U \subset H^2(X)$. Then, for every $1 \le j \le s'$, we have $T_j \star_{\mathbf{t}^{(2)}} \Psi(\mathbf{t}^{(2)}) = \partial_{t_j} u(\mathbf{t}^{(2)}) \cdot \Psi(\mathbf{t}^{(2)})$, $\forall \mathbf{t}^{(2)} \in U$.*

Proof. From the flatness of ∇^{sm} and $\hat{c}_1(\mathbf{t}^{(2)})\Psi(\mathbf{t}^{(2)}) = u(\mathbf{t}^{(2)}) \cdot \Psi(\mathbf{t}^{(2)})$, we get

$$\begin{aligned} &\partial_{t_j} u(\mathbf{t}^{(2)}) \cdot \Psi(\mathbf{t}^{(2)}) + \Big(u(\mathbf{t}^{(2)}) - \hat{c}_1(\mathbf{t}^{(2)})\Big)\, \partial_{t_j}\Psi(\mathbf{t}^{(2)}) \\ =\ & T_j \star_{\mathbf{t}^{(2)}} \Psi(\mathbf{t}^{(2)}) + [(T_j \star_{\mathbf{t}^{(2)}}), \mu]\Psi(\mathbf{t}^{(2)}). \end{aligned}$$

Let $H'_{\mathbf{t}^{(2)}} \subset H^*(X)$ be the direct sum of generalized eigenspaces of $\hat{c}_1(\mathbf{t}^{(2)})$ with eigenvalues $\neq u(\mathbf{t}^{(2)})$. Then $H'_{\mathbf{t}^{(2)}} = \mathrm{Im}\,\big(u(\mathbf{t}^{(2)}) - \hat{c}_1(\mathbf{t}^{(2)})\big)$ by Lemma 3.3 (i). Observe that $T_j \star_{\mathbf{t}^{(2)}} \Psi(\mathbf{t}^{(2)}) \in \mathbb{C}\Psi(\mathbf{t}^{(2)})$, and so it suffices to show that $[(T_j \star_{\mathbf{t}^{(2)}}), \mu]\Psi(\mathbf{t}^{(2)}) \in H'_{\mathbf{t}^{(2)}}$.

Note that Lemma 3.3 (iii) implies $\mu(\mathbb{C}\Psi(\mathbf{t}^{(2)})) \in H'_{\mathbf{t}^{(2)}}$. So $\mu\left(T_j \star_{\mathbf{t}^{(2)}} \Psi(\mathbf{t}^{(2)})\right) \in H'_{\mathbf{t}^{(2)}}$. As a consequence, we have

$$\begin{aligned}0 &= \left(\mu\left(T_j \star_{\mathbf{t}^{(2)}} \Psi(\mathbf{t}^{(2)})\right), \Psi(\mathbf{t}^{(2)})\right) = -\left(T_j \star_{\mathbf{t}^{(2)}} \Psi(\mathbf{t}^{(2)}), \mu\left(\Psi(\mathbf{t}^{(2)})\right)\right)\\ &= -\left(\Psi(\mathbf{t}^{(2)}), T_j \star_{\mathbf{t}^{(2)}} \mu\left(\Psi(\mathbf{t}^{(2)})\right)\right).\end{aligned}$$

Therefore $T_j \star_{\mathbf{t}^{(2)}} \mu\left(\Psi(\mathbf{t}^{(2)})\right) \in H'_{\mathbf{t}^{(2)}}$. This finishes the proof of the lemma. □

We will use the Laplace-dual connection $\widehat{\nabla}^{\mathrm{sm}}$ [GGI16, Section 2.5] of ∇^{sm} to study ∇^{sm}-flat sections. Let $\mathbb{C}_\lambda$ be a copy of $\mathbb{C}$ with coordinate λ. The connection $\widehat{\nabla}^{\mathrm{sm}}$ is a meromorphic flat connection on the trivial $H^*(X)$-bundle over $H^2(X) \times \mathbb{C}_\lambda$. With respect to a frame of constant sections, it reads

$$\widehat{\nabla}^{\mathrm{sm}} = \mathrm{d} - \sum_{i=1}^{s'} (T_i \star_{\mathbf{t}^{(2)}}) \left(\lambda - \hat{c}_1(\mathbf{t}^{(2)})\right)^{-1} \mu \mathrm{d}t_i + \left(\lambda - \hat{c}_1(\mathbf{t}^{(2)})\right)^{-1} \mu \mathrm{d}\lambda.$$

When $|\lambda| \gg 0$, we have $\left(\lambda - \hat{c}_1(\mathbf{t}^{(2)})\right)^{-1} = \lambda^{-1} \sum_{k\geq 0} \left(\lambda^{-1}\hat{c}_1(\mathbf{t}^{(2)})\right)^k$. It follows that the connection $\widehat{\nabla}^{\mathrm{sm}}$ has logarithmic singularities along the smooth divisor $\{\lambda = \infty\}$.

Lemma 3.5. *Suppose that (u, Ψ) is a simple pair over a domain $U \subset H^2(X)$, and let $P_0 = (\mathbf{t}_0^{(2)}, u(\mathbf{t}_0^{(2)})) \in U \times \mathbb{C}_\lambda$. Then near P_0 there is a unique holomorphic $\widehat{\nabla}^{\mathrm{sm}}$-flat section $\hat{y}(\mathbf{t}^{(2)}, \lambda)$ satisfying $\hat{y}(\mathbf{t}^{(2)}, u(\mathbf{t}^{(2)})) = \Psi(\mathbf{t}^{(2)})$.*

Proof. Consider the function $\bar{\lambda}(\mathbf{t}^{(2)}, \lambda) := \lambda - u(\mathbf{t}^{(2)})$ on $U \times \mathbb{C}_\lambda$. Then $\bar{\lambda}, t_1, ..., t_{s'}$ are holomorphic coordinates on $U \times \mathbb{C}_\lambda$ and the divisor $\{\lambda = u\}$ is $\{\bar{\lambda} = 0\}$. In terms of these new coordinates, the connection $\widehat{\nabla}^{\mathrm{sm}}$ with respect to a frame of constant sections reads

$$\widehat{\nabla}^{\mathrm{sm}} = \mathrm{d} + \sum_{i=1}^{s'} \left(\partial_{t_i} u(\mathbf{t}^{(2)}) - (T_i \star_{\mathbf{t}^{(2)}})\right) M(\mathbf{t}^{(2)}, \bar{\lambda}) \mu \mathrm{d}t_i + M(\mathbf{t}^{(2)}, \bar{\lambda}) \mu \mathrm{d}\bar{\lambda}.$$

where $M(\mathbf{t}^{(2)}, \bar{\lambda}) := \left(\bar{\lambda} - (\hat{c}_1(\mathbf{t}^{(2)}) - u(\mathbf{t}^{(2)}))\right)^{-1}$.

We show that the endomorphisms $M_i(\mathbf{t}^{(2)}, \bar{\lambda}) := \left(\partial_{t_i} u(\mathbf{t}^{(2)}) - (T_i \star_{\mathbf{t}^{(2)}})\right) M(\mathbf{t}^{(2)}, \bar{\lambda})$ and $\bar{\lambda} M(\mathbf{t}^{(2)}, \bar{\lambda})$ are holomorphic near P_0, i.e., when $\mathbf{t}^{(2)}$ is near $\mathbf{t}_0^{(2)}$ and $\bar{\lambda}$ is near 0. Around P_0, we let $\Psi(\mathbf{t}^{(2)}), f_1(\mathbf{t}^{(2)}), ..., f_{s-1}(\mathbf{t}^{(2)})$ be a local frame, and set $e_j := f_j - (f_j, \Psi)\Psi$. Then $\underline{e} := (\Psi, e_1, ..., e_{s-1})$ is a local frame near P_0 satisfying $(\Psi, e_j) = 0$, since $(\Psi, \Psi) = 1$. It suffices to show that applying $M_i(\mathbf{t}^{(2)}, \bar{\lambda})$ and $\bar{\lambda} M(\mathbf{t}^{(2)}, \bar{\lambda})$ on $\underline{e}$, we get holomorphic sections. On the one hand, $\Psi(\mathbf{t}^{(2)})$ is an eigenvector of $\hat{c}_1(\mathbf{t}^{(2)})$ with eigenvalue $u(\mathbf{t}^{(2)})$, implying $M(\mathbf{t}^{(2)}, \bar{\lambda}) \left(\Psi(\mathbf{t}^{(2)})\right) = \bar{\lambda}^{-1} \Psi(\mathbf{t}^{(2)})$. Consequently, from Lemma 3.4, we have

$$M_i(\mathbf{t}^{(2)}, \bar{\lambda})(\Psi(\mathbf{t}^{(2)})) = 0, \quad \bar{\lambda} M(\mathbf{t}^{(2)}, \bar{\lambda})(\Psi(\mathbf{t}^{(2)})) = \Psi(\mathbf{t}^{(2)}).$$

On the other hand, let $H'_{\mathbf{t}^{(2)}} \subset H^*(X)$ be the direct sum of generalized eigenspaces of $\hat{c}_1(\mathbf{t}^{(2)})$ with eigenvalues $\neq u(\mathbf{t}^{(2)})$, and set $M'(\mathbf{t}^{(2)}) := \left(\hat{c}_1(\mathbf{t}^{(2)}) - u(\mathbf{t}^{(2)})\right)|_{H'_{\mathbf{t}^{(2)}}}$. Then $M'(\mathbf{t}^{(2)})$ is invertible, and

$$M(\mathbf{t}^{(2)}, \bar{\lambda})|_{H'_{\mathbf{t}^{(2)}}} = (-1)^{s-1} M'(\mathbf{t}^{(2)})^{-1} \sum_{k\geq 0} \left(\bar{\lambda} M'(\mathbf{t}^{(2)})^{-1}\right)^k,$$

when $|\bar{\lambda}|$ is sufficiently small. Note that $(\Psi, e_j) = 0$ gives $e_j(\mathbf{t}^{(2)}) \in H'_{\mathbf{t}^{(2)}}$ by Lemma 3.3 (ii). As a consequence, the section $M(\mathbf{t}^{(2)}, \bar{\lambda})e_j(\mathbf{t}^{(2)})$ is holomorphic near P_0. This proves that $M_i(\mathbf{t}^{(2)}, \bar{\lambda})$ and $\bar{\lambda}M(\mathbf{t}^{(2)}, \bar{\lambda})$ are holomorphic near P_0.

So near P_0, the connection $\widehat{\nabla}^{\mathrm{sm}}$ is either holomorphic or has logarithmic singularities along $\{\bar{\lambda} = 0\}$, and we only need to consider the singular case. The residue of $\widehat{\nabla}^{\mathrm{sm}}$ along $\{\bar{\lambda} = 0\}$ is the following endomorphism of the trivial $H^*(X)$-bundle over $\{\bar{\lambda} = 0\}$:

$$R(\mathbf{t}^{(2)}) := \left(\bar{\lambda}M(\mathbf{t}^{(2)}, \bar{\lambda})\mu\right)_{\bar{\lambda}=0}.$$

From the discussions in the preceding paragraph, we see that

$$\left(\bar{\lambda}M(\mathbf{t}^{(2)}, \bar{\lambda})\right)_{\bar{\lambda}=0}(\Psi(\mathbf{t}^{(2)})) = \Psi(\mathbf{t}^{(2)}), \quad \left(\bar{\lambda}M(\mathbf{t}^{(2)}, \bar{\lambda})\right)_{\bar{\lambda}=0}|_{H'_{\mathbf{t}^{(2)}}} = 0.$$

Moreover, degree-counting gives $(\mu(\Psi(\mathbf{t}^{(2)})), \Psi(\mathbf{t}^{(2)})) = 0$, implying $\mu(\Psi(\mathbf{t}^{(2)})) \in H'_{\mathbf{t}^{(2)}}$ since $H'_{\mathbf{t}^{(2)}}$ is the orthogonal complement of $\mathbb{C}\Psi(\mathbf{t}^{(2)})$ with respect to the Poincaré pairing. Now we can check that $R(\mathbf{t}^{(2)})\Psi(\mathbf{t}^{(2)}) = 0$ and $R(\mathbf{t}^{(2)})^2 e_j(\mathbf{t}^{(2)}) = 0$. So R is nilpotent, and as a consequence, a section near P_0 is $\widehat{\nabla}^{\mathrm{sm}}$-flat if and only if it is of the form (see, e.g., [YT75, Theorem 2 and Remark 2]):

$$(\mathbf{t}^{(2)}, \bar{\lambda}) \mapsto \hat{U}(\mathbf{t}^{(2)}, \bar{\lambda}) \exp\left(-R(\mathbf{t}_0^{(2)}) \log \bar{\lambda}\right)(v), \quad v \in H^*(X). \tag{3.4}$$

Here $\hat{U}(\mathbf{t}^{(2)}, \bar{\lambda})$ is a holomorphic endomorphism near P_0 with $\hat{U}(\mathbf{t}_0^{(2)}, 0) = \mathrm{id}$, and $R(\mathbf{t}_0^{(2)})$ is viewed as a constant endomorphism near P_0. Let $\hat{y}(\mathbf{t}^{(2)}, \bar{\lambda})$ be the $\widehat{\nabla}^{\mathrm{sm}}$-flat near P_0 with $v = \Psi(\mathbf{t}_0^{(2)})$. Then it is holomorphic near P_0 since $R(\mathbf{t}_0^{(2)})\Psi(\mathbf{t}_0^{(2)}) = 0$.

It remains to show that $\hat{y}(\mathbf{t}^{(2)}, 0) = \Psi(\mathbf{t}^{(2)})$. Let $\widehat{\nabla}^{\mathrm{sm,res}}$ be the residual connection of $\widehat{\nabla}^{\mathrm{sm}}$ along the divisor $\{\bar{\lambda} = 0\}$, and with respect to a frame of constant sections it reads $\widehat{\nabla}^{\mathrm{sm,res}} = \mathrm{d} + \sum_{i=1}^{s'} M_i(\mathbf{t}^{(2)}, 0)\mu \mathrm{d}t_i$. Note that $\hat{y}(\mathbf{t}^{(2)}, 0)$ is a $\widehat{\nabla}^{\mathrm{sm,res}}$-flat section over $\{\bar{\lambda} = 0\}$ and $\hat{y}(\mathbf{t}_0^{(2)}, 0) = \Psi(\mathbf{t}_0^{(2)})$. So it suffices to show that $\partial_{t_i}\Psi(\mathbf{t}^{(2)}) + M_i(\mathbf{t}^{(2)}, 0)\mu\Psi(\mathbf{t}^{(2)}) = 0$. Note that both terms are in $H'_{\mathbf{t}^{(2)}}$, and we only need to prove that $M'(\mathbf{t}^{(2)})\left(\partial_{t_i}\Psi(\mathbf{t}^{(2)}) + M_i(\mathbf{t}^{(2)}, 0)\mu\Psi(\mathbf{t}^{(2)})\right) = 0$, or equivalently,

$$\left(u(\mathbf{t}^{(2)}) - \hat{c}_1(\mathbf{t}^{(2)})\right)\partial_{t_i}\Psi(\mathbf{t}^{(2)}) + \left(\partial_{t_i}u(\mathbf{t}^{(2)}) - (T_i\star_{\mathbf{t}^{(2)}})\right)\mu\Psi(\mathbf{t}^{(2)}) = 0.$$

This follows from the flatness of $\widehat{\nabla}^{\mathrm{sm}}$ and Lemma 3.4.

This proves the existence. Uniqueness follows from (3.4). □

We use Definition 2.3 for elements in $\mathcal{S}^{\mathrm{sm}}$ with exactly the same words.

Lemma 3.6. *Suppose that (u, Ψ) is a simple rightmost pair over a domain U inside the* (SR)*-region of X. Then there exists a unique $y \in \mathcal{S}^{\mathrm{sm}}$ such that for any $\mathbf{t}_0^{(2)} \in U$, y respects (u, Ψ) over an open neighborhood of $\mathbf{t}_0^{(2)}$ with phase zero.*

Proof. Let $\hat{y}_{\mathbf{t}_0^{(2)}}(\mathbf{t}^{(2)}, \lambda)$ be the $\widehat{\nabla}^{\mathrm{sm}}$-flat section from Lemma 3.5, which is holomorphic near $(\mathbf{t}_0^{(2)}, u(\mathbf{t}_0^{(2)}))$ and satisfies $\hat{y}(\mathbf{t}^{(2)}, u(\mathbf{t}^{(2)})) = \Psi(\mathbf{t}^{(2)})$. Consider the Laplace transform

$$\bar{y}_{\mathbf{t}_0^{(2)}}(\mathbf{t}^{(2)}, z) := \frac{1}{z}\int_{u(\mathbf{t}^{(2)}) + \mathbb{R}_{\geq 0}\mathrm{e}^{\mathrm{i}0}} \hat{y}_{\mathbf{t}_0^{(2)}}(\mathbf{t}^{(2)}, \lambda)\mathrm{e}^{-\frac{\lambda}{z}}\mathrm{d}\lambda, \quad |\arg z| < \frac{\pi}{2}, \mathbf{t}^{(2)} \text{ near } \mathbf{t}_0^{(2)}.$$

Note that $\hat{y}_{\mathbf{t}_0^{(2)}}(\mathbf{t}^{(2)},\lambda)$ is holomorphic near $\lambda = u(\mathbf{t}^{(2)})$, and has moderate growth as $\lambda \to \infty$ since $\widehat{\nabla}^{\mathrm{sm}}$ has logarithmic singularities along the smooth divisor $\{\lambda = \infty\}$. Therefore the Laplace transform is well defined. Slightly changing the slope of the integration path, we can analytically continue $\bar{y}_{\mathbf{t}_0^{(2)}}(\mathbf{t}^{(2)},z)$ to $|\arg z| < \frac{\pi}{2} + \varepsilon$ for sufficiently small $\varepsilon > 0$. Using integration by parts, we see that $\bar{y}_{\mathbf{t}_0^{(2)}}(\mathbf{t}^{(2)},z)$ is $\widetilde{\nabla}^{\mathrm{sm}}$-flat and respects (u,Ψ) with phase zero over an open neighborhood of $\mathbf{t}_0^{(2)}$. When $\mathbf{t}_1^{(2)}$ is sufficiently close to $\mathbf{t}_0^{(2)}$, it follows from $\hat{y}_{\mathbf{t}_0^{(2)}}(\mathbf{t}^{(2)},\lambda) = \hat{y}_{\mathbf{t}_1^{(2)}}(\mathbf{t}^{(2)},\lambda)$ that $\bar{y}_{\mathbf{t}_0^{(2)}}(\mathbf{t}^{(2)},z) = \bar{y}_{\mathbf{t}_1^{(2)}}(\mathbf{t}^{(2)},z)$. This proves the existence part of the lemma.

For the uniqueness, suppose that $y_1(\mathbf{t}^{(2)},z)$ and $y_2(\mathbf{t}^{(2)},z)$ are two such $\widetilde{\nabla}^{\mathrm{sm}}$-flat sections, and $\mathbf{t}_0^{(2)} \in U$. Then we have $\mathrm{e}^{\frac{u(\mathbf{t}_0^{(2)})}{z}} y_1(\mathbf{t}_0^{(2)},z), \mathrm{e}^{\frac{u(\mathbf{t}_0^{(2)})}{z}} y_2(\mathbf{t}_0^{(2)},z) \to \Psi(\mathbf{t}_0^{(2)})$, as $z \to 0$ along the positive real axis. Then the functions $y_1(\mathbf{t}_0^{(2)},z)$ and $y_2(\mathbf{t}_0^{(2)},z)$ of z are in $\mathcal{A}(\mathbf{t}_0^{(2)})$ and equal to each other, since $\dim_{\mathbb{C}} \mathcal{A}(\mathbf{t}_0^{(2)}) = 1$. As a consequence, we have $y_1(\mathbf{t}^{(2)},z) = y_2(\mathbf{t}^{(2)},z)$. This proves the uniqueness part of the lemma. □

Theorem 3.7. *Suppose that U is a domain inside the* (SR)*-region of X. If X satisfies Gamma-I at some $\mathbf{t}_0^{(2)} \in U$, then X satisfies Gamma-I over U.*

Proof. For any $\mathbf{t}_1^{(2)} \in U$, let $U' \subset U$ be a simply connected domain containing both $\mathbf{t}_0^{(2)}$ and $\mathbf{t}_1^{(2)}$, and such that (u,Ψ) is a simple rightmost pair over U'. From Lemma 3.6, let y be the $\widetilde{\nabla}^{\mathrm{sm}}$-flat section respecting (u,Ψ) over U' with phase zero. Then $y \in \mathcal{A}(\mathbf{t}_0^{(2)})$. Since $\mathcal{Z}^{K,\mathrm{sm}}(\mathcal{O}) \in \mathcal{A}(\mathbf{t}_0^{(2)})$ and $\dim_{\mathbb{C}} \mathcal{A}(\mathbf{t}_0^{(2)}) = 1$, it follows that there exists a nonzero $c \in \mathbb{C}$ satisfying $\mathcal{Z}^{K,\mathrm{sm}}(\mathcal{O}) = cy$. Now $y \in \mathcal{A}(\mathbf{t}_1^{(2)})$ implies that $\mathcal{Z}^{K,\mathrm{sm}}(\mathcal{O}) \in \mathcal{A}(\mathbf{t}_1^{(2)})$. This finishes the proof of the theorem. □

3.3. **AEFS for $\widetilde{\nabla}^{\mathrm{sm}}$ and the strategy-type theorem.** Let $K_{ts} \subset H^2(X)$ be the subset of tame semisimple points in the Kähler moduli, i.e. for $\mathbf{t}^{(2)} \in K_{ts}$, $\hat{c}_1(\mathbf{t}^{(2)})$ has only simple eigenvalues. In this subsection, we assume that $K_{ts} \neq \varnothing$, in which case $H^2(X) \setminus K_{ts}$ is a divisor of $H^2(X)$.

For a sufficiently small domain $U \subset K_{ts}$, by the Implicit Function Theorem, there are holomorphic maps $U \xrightarrow{u_i} \mathbb{C}_\lambda$ and $U \xrightarrow{\psi_i} H^*(X)$ $(1 \le i \le s)$, such that for each $\mathbf{t}^{(2)} \in U$, $u_i(\mathbf{t}^{(2)})$ is an eigenvalue of $\hat{c}_1(\mathbf{t}^{(2)})$ and $\psi_i(\mathbf{t}^{(2)})$ is the idempotent generator of the associated one-dimensional eigenspace. The functions u_i's, and hence ψ_i's, are unique up to ordering. When the normalized idempotent $\Psi_i := (\psi_i,\psi_i)^{-\frac{1}{2}}\psi_i$ is a well-defined holomorphic map (which holds, for instance, if U is simply connected), we say that U is properly-chosen with respect to $\{(u_i,\Psi_i)\}$.

Lemma 3.8. *Suppose that $U \subset K_{ts}$ is properly-chosen with respect to $\{(u_i,\Psi_i)\}$, and $\mathbf{t}_0^{(2)} \in U$. Then near $P_i := (\mathbf{t}_0^{(2)}, u_i(\mathbf{t}_0^{(2)})) \in U \times \mathbb{C}_\lambda$, there is a unique holomorphic $\widehat{\nabla}^{\mathrm{sm}}$-flat section $\hat{y}_i(\mathbf{t}^{(2)},\lambda)$ satisfying $\hat{y}_i(\mathbf{t}^{(2)}, u_i(\mathbf{t}^{(2)})) = \Psi_i(\mathbf{t}^{(2)})$.*

Proof. This follows from Lemma 3.5 since each (u_i,Ψ_i) is a simple pair over U. □

Remark 3.9. Here we do not assume $\mathbf{t}_0^{(2)} \in B$, and so the proof does not use the assumption that $\{u_i\}$ is the canonical coordinate near $\mathbf{t}_0^{(2)}$.

Let $\phi \in \mathbb{R}$ be an admissible phase at $\mathbf{t}_0^{(2)}$. Now we follow [GGI16, Section 2] to use the Laplace transform of the above $\{\hat{y}_i\}$ to construct the AEFS $\{y_i^\phi\}$ of ∇^{sm} near $\mathbf{t}_0^{(2)}$

associated to the phase ϕ with respect to $\{\Psi_i\}$. Let $L_i(\mathbf{t}^{(2)},\phi) := u_i(\mathbf{t}^{(2)}) + \mathbb{R}_{\geq 0}\mathrm{e}^{\mathbf{i}\phi}$. Set

$$y_i^{\phi}(\mathbf{t}^{(2)},z) := \frac{1}{z}\int_{L_i(\mathbf{t}^{(2)},\phi)} \hat{y}_i(\mathbf{t}^{(2)},\lambda)\mathrm{e}^{-\frac{\lambda}{z}}\mathrm{d}\lambda, \quad |\arg z-\phi| < \frac{\pi}{2}, \mathbf{t}^{(2)} \text{ near } \mathbf{t}_0^{(2)}.$$

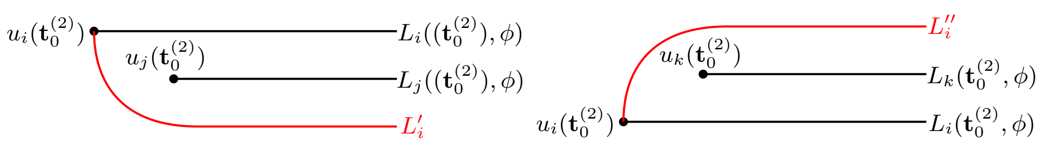
The AEFS $y_1^{\phi},\ldots,y_s^{\phi}$ near $\mathbf{t}_0^{(2)}$ are semiorthonormal with respect to the phase ϕ in the sense that $[y_i^{\phi},y_i^{\phi}) = 1$ and $[y_i^{\phi},y_j^{\phi}) = 0$ if $\mathrm{Im}(\mathrm{e}^{-\mathbf{i}\phi}u_i(\mathbf{t}_0^{(2)})) < \mathrm{Im}(\mathrm{e}^{-\mathbf{i}\phi}u_j(\mathbf{t}_0^{(2)}))$.

The above $L_i(\mathbf{t}_0^{(2)},\phi)$ is an admissible path at $\mathbf{t}_0^{(2)}$ starting from $u_i(\mathbf{t}_0^{(2)})$. For a regular embedding

$$\gamma : \mathbb{R}_{\geq 0} \to \mathbb{C}_{\lambda},$$

i.e. a smooth embedding with nowhere-vanishing $\gamma'(t)$, we say that it is **admissible at $\mathbf{t}_0^{(2)}$ with end-phase** $\phi \in \mathbb{R}$, if $\gamma(0) \in \mathrm{Spec}(\hat{c}_1(\mathbf{t}_0^{(2)}))$, $\gamma(\mathbb{R}_{>0}) \cap \mathrm{Spec}(\hat{c}_1(\mathbf{t}_0^{(2)})) = \varnothing$ and $\gamma(t)$ satisfies $\arg\gamma'(t) = \phi$ when $t \gg 0$. When $\gamma(0) = u_i(\mathbf{t}_0^{(2)})$, the following Laplace transform $y^{\gamma}(z) := \frac{1}{z}\int_{\gamma}\hat{y}_i(\mathbf{t}_0^{(2)},\lambda)\mathrm{e}^{-\frac{\lambda}{z}}\mathrm{d}\lambda$ when $|\arg z - \phi| < \frac{\pi}{2}$, is in $\mathcal{S}^{\mathrm{sm}}_{\mathbf{t}_0^{(2)}} := \{y(\mathbf{t}_0^{(2)},z) \mid \widetilde{\nabla}^{\mathrm{sm}}y(\mathbf{t}^{(2)},z) = 0\}$, and there is a unique $y^{\gamma}(\mathbf{t}^{(2)},z) \in \mathcal{S}^{\mathrm{sm}}$ such that $y^{\gamma}(\mathbf{t}_0^{(2)},z) = y^{\gamma}(z)$. Note that $\hat{y}_i(\mathbf{t}^{(2)},\lambda)$ is the unique $\widehat{\nabla}^{\mathrm{sm}}$-flat section holomorphic near $(\mathbf{t}_0^{(2)},u_i(\mathbf{t}_0^{(2)}))$ satisfying $\hat{y}_i(\mathbf{t}_0^{(2)},u_i(\mathbf{t}_0^{(2)})) = \Psi_i(\mathbf{t}_0^{(2)})$ by Lemma 3.8. We say that $y^{\gamma} \in \mathcal{S}^{\mathrm{sm}}$ is **represented by γ in $\mathbb{C}_{\lambda}$ at $\mathbf{t}_0^{(2)}$ with respect to** $(u_i(\mathbf{t}_0^{(2)}),\Psi_i(\mathbf{t}_0^{(2)}))$. Note that two $\widetilde{\nabla}^{\mathrm{sm}}$-flat sections represented by the same admissible path at $\mathbf{t}_0^{(2)}$ differ at most by a sign, due to sign ambiguity of normalized idempotents.

Mutations of $L_i(\mathbf{t}_0^{(2)},\phi)$ give mutations of y_i^{ϕ}. The right mutation of $L_i(\mathbf{t}_0^{(2)},\phi)$ with respect to $u_j(\mathbf{t}_0^{(2)})$ is depicted in Figure 1(A). During the deformation from $L_i(\mathbf{t}_0^{(2)},\phi)$ to L_i', the path is assumed to cross only one eigenvalue $u_j(\mathbf{t}_0^{(2)})$. Let y_i' be the corresponding right mutation of y_i^{ϕ}, which is determined by the Laplace transform of $\hat{y}_i(\mathbf{t}_0^{(2)},\lambda)$ over L_i'. Then $y_i' = y_i^{\phi} - [y_i^{\phi},y_j^{\phi})\cdot y_j^{\phi}$. Similarly, let y_i'' be the left mutation of y_i with respect to $u_k(\mathbf{t}_0^{(2)})$ as depicted in Figure 1(B), and we have $y_i'' = y_i^{\phi} - [y_k^{\phi},y_i^{\phi})\cdot y_k^{\phi}$. We refer the reader to [GGI16, Section 2] for details.



(A) Right mutation of $L_i((\mathbf{t}_0^{(2)}),\phi)$ with respect to $u_j(\mathbf{t}_0^{(2)})$. (B) Left mutation of $L_i(\mathbf{t}_0^{(2)},\phi)$ with respect to $u_k(\mathbf{t}_0^{(2)})$.

FIGURE 1. Mutations.

For completeness, we restate the following and includes its proof.

Theorem 3.10 (Strategy-Theorem)**.** *For a Fano manifold X, assume the followings.*

(1) The big quantum cohomology of X is convergent near the large radius limit.

(2) There exists $\mathbf{t}_0^{(2)} \in H^2(X)$ at which X satisfies property Gamma-I.

(3) There exists a domain U inside the (SR)*-region of X such that $\mathbf{t}_0^{(2)} \in U$ and $U \cap K_{ts} \neq \varnothing$.*

(4) Denote $V_1 := \mathcal{O}$. There exists $\mathbf{t}_1^{(2)} \in U \cap K_{ts}$ such that:

(a) there exist $V_2,\ldots,V_s \in \mathcal{D}^b_{\mathrm{coh}}(X)$ such that each V_i is obtained from $\mathcal{O}$ via the Galois action along some path $\gamma_i \subset H^2(X)$ starting at the same point $\mathbf{t}_1^{(2)}$;

(b) there exists a full exceptional collection $(\tilde{V}_1, \ldots, \tilde{V}_s)$ in $\mathcal{D}^b_{\mathrm{coh}}(X)$, obtained from the set $\{V_1, \ldots, V_s\}$ by "elementary operations";

(c) there exists an admissible phase ϕ_1 at $\mathbf{t}_1^{(2)}$ such that $\mathcal{Z}^{K,\mathrm{sm}}(\tilde{V}_i)$ respects the pair (u_i, Ψ_i) at $\mathbf{t}_1^{(2)}$ with phase ϕ_1 for all i, where the u_i are ordered so that $\mathrm{Im}(\mathrm{e}^{-\mathbf{i}\phi} u_1(\mathbf{t}_1^{(2)})) > \cdots > \mathrm{Im}(\mathrm{e}^{-\mathbf{i}\phi} u_s(\mathbf{t}_1^{(2)}))$.

Then Gamma conjecture II holds for X.

Proof. Let B be the domain of convergence of the big quantum cohomology of X and B_{ss} the semisimple locus, as in Section 2. Observe that $B \cap K_{ts} \subset B_{ss}$. Note that $B \cap H^2(X) \subset H^2(X)$ is a domain, and $H^2(X) \setminus K_{ts} \subset H^2(X)$ is a divisor. Therefore $B \cap K_{ts} \neq \varnothing$, implying $B_{ss} \neq \varnothing$. Let $\mathbf{t}_2^{(2)} \in B \cap K_{ts} \subset B_{ss}$, and let $\phi_2 \in \mathbb{R}$ be an admissible phase at $\mathbf{t}_2^{(2)}$.

We claim that, as $(\mathbf{t}^{(2)}, \phi)$ varies from $(\mathbf{t}_1^{(2)}, \phi_1)$ to $(\mathbf{t}_2^{(2)}, \phi_2)$ inside K_{ts}, the corresponding AEFS of ∇^{sm} changes by mutations, compatibly with the braid group action on full exceptional collections. To see this, let $\mathbf{t}^{(2)} \in K_{ts}$ be on the path connecting $\mathbf{t}_1^{(2)}$ and $\mathbf{t}_2^{(2)}$, and let a neighborhood of $\mathbf{t}^{(2)}$ be properly-chosen with respect to $\{(\bar{u}_i, \bar{\Psi}_i)\}$. Let $\phi, \phi' \in \mathbb{R}$ be admissible phases at $\mathbf{t}^{(2)}$, and suppose that $(E_1, \ldots, E_s)$ is full exceptional collections in $\mathcal{D}^b_{\mathrm{coh}}(X)$ that give AEFS of ∇^{sm} at $\mathbf{t}^{(2)}$ with respect to $\{\bar{\Psi}_i\}$ associated to phases ϕ. Suppose $\phi' > \phi$, and without loss of generality assume that they are close enough so that when we rotate $L_i(\mathbf{t}^{(2)}, \phi)$ counterclockwise around its endpoint $\bar{u}_i(\mathbf{t}^{(2)})$ by the angle $\phi' - \phi$ to get $L_i(\mathbf{t}^{(2)}, \phi')$ for $1 \leq i \leq s$, only the sector spanned by deformation of $L_{i_0}(\mathbf{t}^{(2)}, \phi)$ contains an eigenvalue of $\hat{c}_1(\mathbf{t}^{(2)})$; in this case, the eigenvalue is $\bar{u}_{i_0-1}(\mathbf{t}^{(2)})$. Now according to the discussion preceding Figure 1, the AEFS of ∇^{sm} at $\mathbf{t}^{(2)}$ with respect to $\{\bar{\Psi}_i\}$ associated to phases ϕ' is

$$
\begin{aligned}
&\mathcal{Z}^{K,\mathrm{sm}}(E_1), \ldots, \mathcal{Z}^{K,\mathrm{sm}}(E_{i_0-2}),\\
&\mathcal{Z}^{K,\mathrm{sm}}(E_{i_0}) - [\mathcal{Z}^{K,\mathrm{sm}}(E_{i_0-1}), \mathcal{Z}^{K,\mathrm{sm}}(E_{i_0})) \mathcal{Z}^{K,\mathrm{sm}}(E_{i_0-1}), \mathcal{Z}^{K,\mathrm{sm}}(E_{i_0-1}),\\
&\mathcal{Z}^{K,\mathrm{sm}}(E_{i_0+1}), \ldots, \mathcal{Z}^{K,\mathrm{sm}}(E_s).
\end{aligned}
$$

This is the AEFS given by the full exceptional collection obtained by applying $\sigma_{i_0-1}^{-1}$ on $(E_1, \ldots, E_s)$. Conversely, if $\phi' < \phi$ are close enough so that only the sector corresponding to $L_{i_0}(\mathbf{t}^{(2)}, \phi)$ contains an eigenvalue of $\hat{c}_1(\mathbf{t}^{(2)})$, then the rotation is clockwise. Similarly to the previous case, the AEFS mutation is given by applying σ_{i_0+1} to the same exceptional collection. This verifies the claim (see [GGI16, Section 4] for systematic discussions). The proof is completed by observing that AEFS of ∇^{sm} coincides with AEFS of ∇ near $\mathbf{t}_2^{(2)} \in K_{ts} \cap B$. □

4. Gamma conjecture II holds for del Pezzo surfaces

In this section, we investigate del Pezzo surfaces X_r ($1 \leq r \leq 8$), namely the blow-up of $\mathbb{P}^2$ at r general points. It is known that the big quantum cohomology converges near the large radius limit by [Iri07, Corollary 5.9], and is generically semisimple by [BM04, Theorem 3.6.1]. Therefore, Gamma conjecture II is well defined for X_r. The main result of this section is the following theorem, which we obtain by verifying all assumptions of the Strategy-Theorem in Section 4.2.

Theorem 4.1. *Gamma conjecture II holds for X_r for $1 \leq r \leq 8$.*

We start with some concrete information on the small quantum cohomology of X_r. Let H denote the pullback to X_r of the hyperplane class on $\mathbb{P}^2$, and let $E_1, ..., E_r$ be the exceptional divisors. Let $\mathbf{1} \in H^0(X_r)$ be the Poincaré dual of the fundamental class, and $[pt] \in H^4(X_r)$ the Poincaré dual of a point. Let

$$T_0 = \mathbf{1}, T_1 = H - E_1, \ldots, T_r = H - E_r, T_{r+1} = H, T_{r+2} = [pt].$$

Then these classes form a basis of $H^*(X_r)$, and $T_1, \ldots, T_{r+1}$ form a nef basis of $H^2(X_r, \mathbb{Z})$. We write $\mathbf{t} = \sum_{i=0}^{r+2} t_i T_i$, and set $q_i := \mathrm{e}^{t_i}(1 \le i \le r), q := \mathrm{e}^{t_{r+1}}$.

The dual basis of $\{T_1, \ldots, T_{r+1}\}$ in $H_2(X_r, \mathbb{Z})$ is given by $\{E_1, \ldots, E_r, H - \sum_i E_i\}$. So we can write $\mathbf{d} \in \mathrm{Eff}(X_r)$ as $\mathbf{d} = \sum_i k_i E_i + k(H - \sum_i E_i)$, and denote

$$\mathrm{e}^{\mathbf{t}^{(2)}(\mathbf{d})} = q_1^{k_1} \cdots q_r^{k_r} q^k =: \mathbf{q}^{\mathbf{d}}, \text{ where } k_i, k \in \mathbb{Z}_{\ge 0}. \tag{4.1}$$

For $\mathbf{d} \in \mathrm{Eff}(X_r)$ and $k \in \mathbb{Z}_{\ge 0}$, the invariant $\langle ([pt])^k \rangle_{\mathbf{d}}^{X_r}$ is a nonnegative integer [GP98, Section 4.1]. Moreover, we will use the following Gromov-Witten invariants (see [GP98, Lemma 3.3, Theorem 4.1], [BM04, Proposition 3.4.1]):

$$\langle \rangle_{E_i}^{X_r} = \langle \rangle_{H-E_i-E_j}^{X_r} = \langle [pt] \rangle_{H-E_i}^{X_r} = \langle ([pt])^2 \rangle_H^{X_r} = 1, \quad i \ne j. \tag{4.2}$$

4.1. **Distribution of the eigenvalues of $\hat{c}_1(\mathbf{t}^{(2)})$.** In this subsection, we study the distribution of the eigenvalues of $\hat{c}_1(\mathbf{t}^{(2)})$. We first assume the following proposition, whose proof is the most technically involved part and is deferred to Section 5. It is needed both in the formulation and proof of Proposition 4.8, as well as in the proof of Theorem 4.1.

Proposition 4.2. *There exists $\delta_r \in (0, \frac{1}{2}]$, such that the set*

$$V_{\delta_r} := \{\mathbf{t}^{(2)} \in H^2(X_r, \mathbb{R}) | 0 < q(\mathbf{t}^{(2)}) \le 1 \text{ and } 1 \le q_i(\mathbf{t}^{(2)}) \le 1 + \delta_r,\ i = 1, \ldots, r\}$$

is contained in the (SR)*-region of X_r.*

Let M_r be the matrix of the operator $\hat{c}_1(\mathbf{t}^{(2)})$ on the small quantum cohomology of X_r with respect to the ordered basis $[\mathbf{1}, H, E_1, ..., E_r, [pt]]$. Then the entries of M_r lie in $\mathbb{R}[q, q_1, \ldots, q_r]$. We study the asymptotic behavior of the spectrum of $\hat{c}_1(\mathbf{t}^{(2)})$, namely the roots of the characteristic polynomial $\chi_{M_r}(u) := \det(u \cdot I_{r+3} - M_r)$.

Example 4.3. From the data in [HKLY21, Section 2.2.3], the matrices M_1 and M_2 are respectively given by

$$\begin{bmatrix} 0 & 2q & 2q & 3qq_1 \\ 3 & 0 & 0 & 2q \\ -1 & 0 & -q_1 & -2q \\ 0 & 3 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2q(q_1+q_2) & 2qq_2 & 2qq_1 & 3qq_1q_2 \\ 3 & q & q & q & 2q(q_1+q_2) \\ -1 & -q & -q_1-q & -q & -2qq_2 \\ -1 & -q & -q & -q_2-q & -2qq_1 \\ 0 & 3 & 1 & 1 & 0 \end{bmatrix}.$$

Lemma 4.4. *Let $1 \leq r \leq 8$. Then modulo q^2, the matrix M_r has the form*

$$\begin{bmatrix} 0 & 2q(\sum_{k=1}^{r}\prod_{i\neq k} q_i) & 2q\prod_{i\neq 1} q_i & \cdots & 2q\prod_{i\neq r} q_i & 3q\prod_{i=1}^{r} q_i \\ 3 & q(\sum_{k<l}\prod_{i\neq k,l} q_i) & q(\sum_{k\neq 1}\prod_{i\neq 1,k} q_i) & \cdots & q(\sum_{k\neq 1}\prod_{i\neq r,k} q_i) & 2q(\sum_{k}\prod_{i\neq k} q_i) \\ -1 & -q(\sum_{k\neq 1}\prod_{i\neq 1,k} q_i) & -q_1 - q(\sum_{k\neq 1}\prod_{i\neq 1,k} q_i) & \cdots & -q(\prod_{i\neq 1,r} q_i) & -2q(\sum_{k\neq 1}\prod_{i\neq k} q_i) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & -q(\sum_{k\neq 1}\prod_{i\neq 1,k} q_i) & -q(\prod_{i\neq r,1} q_i) & \cdots & -q_r - q(\sum_{k\neq r}\prod_{i\neq r,k} q_i) & -2q(\sum_{k\neq r}\prod_{i\neq k} q_i) \\ 0 & 3 & 1 & \cdots & 1 & 0 \end{bmatrix},$$

where, for $1 \leq j, j_1, j_2 \leq r$ with $j_1 \neq j_2$, we have

$$(M_r)_{2+j,2+j} = -q_j - q(\sum_{k\neq j}\prod_{i\neq j,k} q_i) + O(q^2), \quad (M_r)_{2+j_1,2+j_2} = -q(\prod_{i\neq j_1,j_2} q_i) + O(q^2).$$

Furthermore, $\det M_r = (-1)^r 27 \left(\prod_{i=1}^{r} q_i\right)^2 q + O(q^2)$.

Proof. The matrix form follows from (4.2) and the divisor axiom of Gromov-Witten theory. Let $s = r+3$. We have

$$\det M_r = \sum_{(i_1 \ldots i_s)} \operatorname{sign}(i_1 \ldots i_s) \cdot (M_r)_{i_1,1} \cdots (M_r)_{i_s,s}.$$

Note that $q|(M_r)_{i,r+3}$ for all i. So for any term labeled by $(i_1 \ldots i_s)$ contributing to $\det M_r$ modulo q^2, we must have $i_2 = r+3$, which forces $i_j = j$ for $3 \leq j \leq r+2$. Then $i_1 = 2$ since $q|(M_r)_{11}$, implying that $i_{r+3} = 1$. Consequently,

$$\begin{aligned} \det M_r &\equiv (M_r)_{21}(M_r)_{r+3,2}(M_r)_{33}\cdots(M_r)_{r+2,r+2}(M_r)_{1,r+3} \mod q^2 \\ &\equiv 3\cdot 3\cdot(-q_1)\cdots(-q_r)\cdot 3q\prod_{i=1}^{r} q_i \mod q^2. \end{aligned}$$

This completes the proof. ∎

To analyze the spectrum of $\hat{c}_1(\mathbf{t}^{(2)})$ as $q \to 0$, we examine the roots of the characteristic polynomial $\chi_{M_r}(u)$. Since the coefficients of this polynomial depend polynomially on q, standard results on algebraic functions imply that its roots admit Puiseux expansions. These are generalized power series that allow fractional exponents in q.

The following Propositions 4.5 and 4.8 determine the leading terms of these root expansions. Finding this lowest-order behavior is a crucial first step: it gives us the analytic control needed to rigorously track the monodromy of the eigenvalues, which is the condition (3) for applying the Strategy-Theorem, as we move along paths $\gamma_i \subset H^2(X_r)$. Denote

$$\omega := \exp\left(\frac{2\pi\mathbf{i}}{3}\right). \tag{4.3}$$

Proposition 4.5. *For fixed $q_1, \ldots, q_r \in \mathbb{C}^\times$, the Puiseux expansions in q of the eigenvalues of M_r have the following asymptotic forms as $q \to 0$:*

$$\begin{aligned} u_k(q; q_1, \ldots, q_r) &= 3\omega^{k-1}(\prod q_i)^{\frac{1}{3}} q^{\frac{1}{3}} + o(q^{\frac{1}{3}}), & k &= 1,2,3, \\ u_{3+k}(q; q_1, \ldots, q_r) &= -q_k + o(1), & k &= 1, \ldots, r. \end{aligned}$$

Proof. By Lemma 4.4, we see that $\chi_{M_r}(u)|_{q=0} = u^3\prod(u+q_i)$. Hence the Puiseux expansions in q of the eigenvalues of M_r have the following asymptotic forms as $q\to 0$:

$$u_k(q) = C_k q^{w_k} + o(q^{w_k}), \quad k=1,2,3, (w_1\ge w_2\ge w_3>0, C_k \text{ is independent of } q)$$
$$u_{3+k}(q) = -q_k + o(1), \quad k=1,\dots,r.$$

Therefore, in the Puiseux expansion of the product $\prod_{k=1}^{3+r} u_k(q)$, the lowest-order exponent of q is $w_1+w_2+w_3$. Since $\prod_{k=1}^{3+r} u_k(q) = \det M_r$, Lemma 4.4 implies that

$$w_1+w_2+w_3 = 1. \tag{4.4}$$

Assume $w_1>w_2$. As a Puiseux expansion in q, the lowest-order exponent of $\sum_{i=1}^{r+3}\prod_{k\neq i}u_k(q)$ is w_2+w_3. Note that $\sum_{i=1}^{r+3}\prod_{k\neq i}u_k(q)$ can be given by minors of M_r, and hence is a polynomial in q. So $w_2+w_3\ge 1$, contradicting (4.4). Thus $w_1=w_2$. Similarly, by considering $\sum_{i<j}\prod_{k\neq i,j}u_k(q)$, we see $w_1=w_2=w_3=\frac{1}{3}$.

Consider the Puiseux expansions of the three terms $\sum_{i<j}\prod_{k\neq i,j}u_k(q)$, $\sum_{i=1}^{r+3}\prod_{k\neq i}u_i(q)$, $\prod_{k=1}^{3+r}u_k(q)$ in q. Note that these terms can be given by minors of M_r, and hence are polynomials in q. The powers of their lowest degree terms in q then respectively give

$$C_1+C_2+C_3=0, \quad C_1C_2+C_2C_3+C_1C_3=0, \quad C_1C_2C_3 = 27\prod q_i.$$

Hence C_1,C_2,C_3 are the three distinct roots of the cubic polynomial $C^3 = 27\prod q_i$. This proves the proposition. □

Lemma 4.6. *There exists $\delta_{r*}>0$, such that whenever $|q|\in(0,\delta_{r*})$, $|q_i|\in[1,1+\delta_r]$, $\theta\in\mathbb{R}$ and $1\le k\le 3$, the complex number $u_k^* := 3\omega^{k-1}q^{\frac13}(\prod q_i)^{\frac13}+\frac{1}{100}e^{\mathbf{i}\theta}q^{\frac13}$ is not an eigenvalue of M_r.*

Proof. Recall that the coefficients of $\chi_{M_r}(u)$ of M_r are in $\mathbb{R}[q,q_1,\cdots,q_r]$, and $\chi_{M_r}(u)|_{q=0} = u^3\prod(u+q_i)$. So we can write $\chi_{M_r}(u)$ in the following form:

$$u^{r+3}+\cdots+(\prod q_i + q\cdot a_3)u^3 + q\cdot a_2u^2 + q\cdot a_1 u + (-1)\cdot 27\cdot(\prod q_i)^2 q + q^2 a_0,$$

where $a_i\in\mathbb{R}[q,q_1,\dots,q_r]$. Note that $u_k^* = q^{\frac13}(3\omega^{k-1}(\prod q_i)^{\frac13}+\frac{1}{100}e^{\mathbf{i}\theta})$, and we see that $\chi_{M_r}(u_k^*)$ has the following form:

$$q\left[\left((3\omega^{k-1}(\prod q_i)^{\frac13}+\frac{1}{100}e^{\mathbf{i}\theta}\right)^3\prod q_i + (-1)\cdot 27\cdot(\prod q_i)^2\right] + q^{\frac43}Q_k, \tag{4.5}$$

where $Q_k = Q_k(q^{\frac13}, q_1^{\frac13},\dots,q_r^{\frac13}, e^{\mathbf{i}\theta})$ is a polynomial in $q^{\frac13}, q_1^{\frac13},\dots,q_r^{\frac13}, e^{\mathbf{i}\theta}$. Then, there exists $\delta_{r*}>0$ so that when $|q|<\delta_{r*}$, $|q_i|\in[1,1+\delta_r]$ and $\theta\in\mathbb{R}$, we have

$$|q^{\frac13}Q_k(q^{\frac13}, q_1^{\frac13},\dots,q_r^{\frac13}, e^{\mathbf{i}\theta})| < \frac{1}{100}.$$

Now we argue by contradiction and assume $\chi_{M_r}(u_k^*)=0$. Then the above form (4.5) gives

$$\left|\left(3\omega^{k-1}(\prod q_i)^{\frac13}+\frac{e^{\mathbf{i}\theta}}{100}\right)^3 - 27\prod q_i\right| < \frac{1}{|\prod q_i|}\cdot\frac{1}{100} < \frac{1}{100},$$

implying that

$$\left|3\cdot 3^2\omega^{2k-2}(\prod q_i)^{\frac23}e^{\mathbf{i}\theta} + 3\cdot 3\omega^{k-1}(\prod q_i)^{\frac13}\frac{e^{\mathbf{i}2\theta}}{100} + \frac{e^{\mathbf{i}3\theta}}{100^2}\right| < 1.$$

Comparing the norms of the three terms, we see that the inequality is impossible. This contradiction proves that $\chi_{M_r}(u_k^*) \neq 0$. $\square$

Lemma 4.7. *Let δ_{r*} be as in the preceding lemma, and $\delta_r \in (0, \frac{1}{2}]$ as in Proposition 4.2. When $|q| \in (0, \delta_{r*})$, $|q_i| \in [1, 1+\delta_r]$, we have*

$$|u_k(q; q_1, \ldots, q_r) - 3\omega^{k-1} q^{\frac{1}{3}} (\prod q_i)^{\frac{1}{3}}| < \frac{1}{100} |q^{\frac{1}{3}}|, \quad k = 1, 2, 3.$$

Proof. We argue by contradiction. Assume that there exists $k_0 \in \{1, 2, 3\}$ and $(q_0, q_{10}, \ldots, q_{r0}) \in (\mathbb{C}^\times)^{r+1}$ with $|q_0| \in (0, \delta_{r*})$, $|q_{i0}| \in [1, 1+\delta_r]$, such that

$$|u_{k_0}(q_0; q_{10}, \ldots, q_{r0}) - 3\omega^{k_0-1} q_0^{\frac{1}{3}} (\prod q_{i0})^{\frac{1}{3}}| \geq \frac{1}{100} |q_0^{\frac{1}{3}}|.$$

For these fixed q_{i0}, it follows from Proposition 4.5 that there exists $\delta_{r0} \in (0, |q_0|)$ such that when $|q'| \in (0, \delta_{r0})$, we have

$$|u_{k_0}(q'; q_{10}, \ldots, q_{r0}) - 3\omega^{k_0-1} (q')^{\frac{1}{3}} (\prod q_{i0})^{\frac{1}{3}}| < \frac{1}{100} |q'|^{\frac{1}{3}}.$$

By continuity of the function $|u_{k_0}(q; q_{10}, \ldots, q_{r0}) - 3\omega^{k_0-1} q^{\frac{1}{3}} (\prod q_{i0})^{\frac{1}{3}}| - \frac{1}{100}|q^{\frac{1}{3}}|$ in q, we see that there exists $q_0' \in \mathbb{C}^\times$ with $|q_0'| \in [\delta_{r0}, |q_0|) \subset (0, \delta_{r*})$ such that

$$|u_{k_0}(q_0'; q_{10}, \ldots, q_{r0}) - 3\omega^{k_0-1} (q_0')^{\frac{1}{3}} (\prod q_{i0})^{\frac{1}{3}}| = \frac{1}{100} |q_0'|^{\frac{1}{3}},$$

contradicting Lemma 4.6. This proves the lemma. $\square$

Let $\delta_r \in (0, \frac{1}{2}]$ be as in Proposition 4.2, and set $\bar{q}_i := 1 + \frac{i}{r}\delta_r (1 \leq i \leq r)$.

Proposition 4.8. *There exists $\bar{\delta}_r \in (0, \delta_r)$, such that for any $q \in (0, \bar{\delta}_r)$ and $t \in [0, 1]$:*

(1) the following inequalities hold:

$$|u_k(\mathrm{e}^{2\pi \mathrm{i} t} q; \bar{q}_1, \ldots, \bar{q}_r) - 3\omega^{k-1} (\prod \bar{q}_i)^{\frac{1}{3}} (\mathrm{e}^{2\pi \mathrm{i} t} q)^{\frac{1}{3}}| < \frac{1}{100} |q|^{\frac{1}{3}}, \quad k = 1, 2, 3,$$

$$|u_{3+k}(q; \bar{q}_1, \ldots, \bar{q}_r) + \bar{q}_k| < \frac{1}{100} \delta_r, \quad 1 \leq k \leq r;$$

(2) for $k_0 \in \{1, \ldots, r\}$, the following inequalities hold:

$$|u_k(\mathrm{e}^{-2\pi \mathrm{i} t} q; \bar{q}_1, \ldots, \mathrm{e}^{2\pi \mathrm{i} t} \bar{q}_{k_0}, \ldots, \bar{q}_r) - 3\omega^{k-1} (\prod \bar{q}_i)^{\frac{1}{3}} q^{\frac{1}{3}}| < \frac{1}{100} |q|^{\frac{1}{3}}, \quad k = 1, 2, 3,$$

$$|u_{3+k}(\mathrm{e}^{-2\pi \mathrm{i} t} q; \bar{q}_1, \ldots, \mathrm{e}^{2\pi \mathrm{i} t} \bar{q}_{k_0}, \ldots, \bar{q}_r) + \bar{q}_k| < \frac{1}{100} \delta_r, \quad 1 \leq k \leq r, k \neq k_0,$$

$$|u_{3+k_0}(\mathrm{e}^{-2\pi \mathrm{i} t} q; \bar{q}_1, \ldots, \mathrm{e}^{2\pi \mathrm{i} t} \bar{q}_{k_0}, \ldots, \bar{q}_r) + \mathrm{e}^{2\pi \mathrm{i} t} \bar{q}_{k_0}| < \frac{1}{100} \delta_r.$$

Proof. Part (1) follows immediately from Lemma 4.7 together with Proposition 4.5.

For case (2), we have $u_{3+k_0}(0; \bar{q}_1, \ldots, \mathrm{e}^{2\pi \mathrm{i} t} \bar{q}_{k_0}, \ldots, \bar{q}_r) = -\mathrm{e}^{2\pi \mathrm{i} t} \bar{q}_{k_0}$, and

$$u_{3+k}(0; \bar{q}_1, \ldots, \mathrm{e}^{2\pi \mathrm{i} t} \bar{q}_{k_0}, \ldots, \bar{q}_r) = -\bar{q}_k \quad (1 \leq k \leq r, k \neq k_0).$$

Recall that entries of M_r are in $\mathbb{R}[q, q_1, \ldots, q_r]$. So by the continuous dependence of roots on coefficients, for each $t \in [0, 1]$, there exists $\delta_r(t) \in (0, \frac{1}{10000}\delta_r)$ such that, when $|q'| < \delta_r(t)$ and $t' \in [0, 1] \cap (t - \delta_r(t), t + \delta_r(t))$, we have $|u_{3+k_0}(q'; \bar{q}_1, \ldots, \mathrm{e}^{2\pi \mathrm{i} t'} \bar{q}_{k_0}, \ldots, \bar{q}_r) + \mathrm{e}^{2\pi \mathrm{i} t} \bar{q}_{k_0}| < \frac{1}{10000}\delta_r$ and

$$|u_{3+k}(q'; \bar{q}_1, \ldots, \mathrm{e}^{2\pi \mathrm{i} t} \bar{q}_{k_0}, \ldots, \bar{q}_r) + \bar{q}_k| < \frac{1}{100} \delta_r \quad (1 \leq k \leq r, k \neq k_0),$$

which implies

$$\begin{aligned}&|u_{3+k_0}(q';\bar{q}_1,\dots,\mathrm{e}^{2\pi\mathbf{i}t'}\bar{q}_{k_0},\dots,\bar{q}_r)+e^{2\pi\mathbf{i}t'}\bar{q}_{k_0}|\\ \le\ &|u_{3+k_0}(q';\bar{q}_1,\dots,\mathrm{e}^{2\pi\mathbf{i}t'}\bar{q}_{k_0},\dots,\bar{q}_r)+e^{2\pi\mathbf{i}t}\bar{q}_{k_0}|+|e^{2\pi\mathbf{i}t'}-e^{2\pi\mathbf{i}t}|\cdot|\bar{q}_{k_0}|<\frac{1}{100}\delta_r.\end{aligned}$$

So by compactness of $[0,1]$, we can find $\delta_r'\in(0,\frac{1}{10000}\delta_r)$ such that when $|q'|<\delta_r'$ and $t\in[0,1]$, we have $|u_{3+k_0}(q';\bar{q}_1,\dots,\mathrm{e}^{2\pi\mathbf{i}t}\bar{q}_{k_0},\dots,\bar{q}_r)+\mathrm{e}^{2\pi\mathbf{i}t}\bar{q}_{k_0}|<\frac{1}{100}\delta_r$ and

$$|u_{3+k}(q';\bar{q}_1,\dots,\mathrm{e}^{2\pi\mathbf{i}t}\bar{q}_{k_0},\dots,\bar{q}_r)+\bar{q}_k|<\frac{1}{100}\delta_r\quad(1\le k\le r,k\ne k_0).$$

Together with Lemma 4.7, this proves case (2). □

4.2. **Gamma conjecture II for X_r.** We prove Theorem 4.1 by verifying all assumptions of the Strategy-Theorem.

Note that X_r is a Fano manifold and $H^*(X_r)$ is generated by $H^2(X_r)$. So the big quantum cohomology of X_r converges near the large radius limit by [Iri07, Corollary 5.9].

It was shown in [HKLY21, Theorem 1.2] that the original Gamma conjecture I of Galkin, Golyshev and Iritani holds for del Pezzo surfaces X_r. Hence, in the terminology of the present paper, we have

Proposition 4.9. *X_r satisfies property Gamma-I at $\mathbf{t}_0^{(2)}=\mathbf{0}\in H^2(X_r)$.*

We note that the quantum cohomology of X_r is non-semisimple at $\mathbf{0}$ when $r>4$ [BM04].

Let $\bar{\delta}_r,\bar{q}_1,\dots,\bar{q}_r$ be as in Proposition 4.8, and choose $\bar{q}\in(0,\min\{\bar{\delta}_r,10^{-6}\})$. Define

$$\bar{\mathbf{t}}^{(2)}:=\sum_{i=1}^r\log\bar{q}_i\cdot T_i+\log\bar{q}\cdot T_{r+1},\tag{4.6}$$

Then $\bar{\mathbf{t}}^{(2)}\in V_{\delta_r}$ and, in the present case, plays the role of $\mathbf{t}_1^{(2)}$ in the Strategy-Theorem. Denote by $\bar{u}_1,\dots,\bar{u}_{3+r}$ the eigenvalues of $\hat{c}_1(\bar{\mathbf{t}}^{(2)})$.

Proposition 4.10. *There exists a simply connected domain U inside the (SR)-region of X_r, such that $V_{\delta_r}\subset U$ and $\bar{\mathbf{t}}^{(2)}\in U\cap K_{ts}$.*

Proof. Let U' denote the connected component of $\mathbf{0}$ in the (SR)-region of X_r. The subset V_{δ_r} is connected by definition and is contained in the (SR)-region of X_r by Proposition 4.2. Thus $V_{\delta_r}\subset U'$ since $\mathbf{0}\in V_{\delta_r}$. Since V_{δ_r} is simply connected, we can further find a simply connected domain U such that $V_{\delta_r}\subset U\subset U'$.

By Proposition 4.8, the eigenvalues $\bar{u}_i$ are pairwise distinct from each other; they are depicted in Figure 2. Thus $\bar{\mathbf{t}}^{(2)}\in K_{ts}$, and hence $\bar{\mathbf{t}}^{(2)}\in V_{\delta_r}\cap K_{ts}\subset U\cap K_{ts}$. □

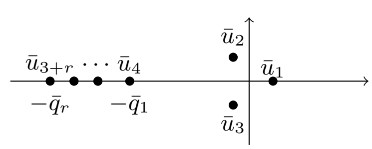


FIGURE 2. Eigenvalues $\bar{u}_1,\dots,\bar{u}_{3+r}$ of $\hat{c}_1(\bar{\mathbf{t}}^{(2)})$.

We will use the ordering of $[\bar{u}_1,\dots,\bar{u}_{3+r}]$ as in Figure 2 in the rest of this subsection.

Recall that we have abused notation by identifying an object of $\mathcal{D}^b_{\mathrm{coh}}(X)$ with its corresponding class in $K(X)$.

Lemma 4.11. *Let $(F_1, \ldots, F_s)$ be a full exceptional collection in $\mathcal{D}^b_{\mathrm{coh}}(X)$. Suppose that $V \in K(X) \otimes_{\mathbb{Z}} \mathbb{C}$ satisfies*

$$F_{j_1} + mF_{j_2} = \chi\left(F_{j_1} + F_{j_2}, V\right) \cdot V$$

for some $m \in \mathbb{Z}$ and $j_1 < j_2$. Assume further that $\chi(F_{j_1}, F_{j_2}) = -1$. Then $V = \pm(F_{j_1} + mF_{j_2})$.

Proof. Since $F_1, \ldots, F_s$ form a $\mathbb{Z}$-basis for $K(X)$, it follows that we can write $V = \sum_{i=1}^{s} a_i F_i$. Let $\xi := \chi(F_{j_1} + F_{j_2}, V)$. Then the condition becomes $F_{j_1} + mF_{j_2} = \xi \sum a_i F_i$. Comparing coefficients immediately yields

$$\xi \cdot a_{j_1} = 1,\ \xi \cdot a_{j_2} = m, \text{ and } \xi \cdot a_i = 0 \text{ for all } i \neq j_1, j_2.$$

This forces $a_i = 0$ when $i \neq j_1, j_2$, yielding $V = a_{j_1} F_{j_1} + a_{j_2} F_{j_2}$. Evaluating ξ via bilinearity gives

$$\xi = a_{j_1}\chi(F_{j_1}, F_{j_1}) + a_{j_1}\chi(F_{j_2}, F_{j_2}) + a_{j_2}\chi(F_{j_1}, F_{j_2}) + a_{j_2}\chi(F_{j_2}, F_{j_2}) = a_{j_1}.$$

Here in the second euality, we used the definition of an exceptional collection and the assumption $\chi(F_{j_1}, F_{j_2}) = -1$. Combining $\xi = a_{j_1}$ with the relation $\xi \cdot a_{j_1} = 1$ and $\xi \cdot a_{j_2} = m$, we obtain $a_{j_1} = \pm 1$ and correspondingly $a_{j_2} = \pm m$. This proves the lemma. □

Consider the paths $\phi_+(t), \phi_-(t), \phi_k(t)(1 \leq k \leq r) : [0, 1] \to H^2(X_r)$ given by

$$\phi_\pm(t) := \bar{\mathbf{t}}^{(2)} \pm 2\pi \mathbf{i} t \cdot H, \text{ and } \phi_k(t) := \bar{\mathbf{t}}^{(2)} - 2\pi \mathbf{i} t \cdot E_k.$$

We note that the images of these paths are contained in K_{ts} by Proposition 4.8.

Proposition 4.12. *Denote $V_1 := \mathcal{O}$. Let $V_2, \cdots, V_{3+r}$ be the objects in $D^b_{\mathrm{coh}}(X_r)$, such that they are obtained from $\mathcal{O}$ via Galois actions given by the paths $\phi_+(t), \phi_-(t), \phi_1(t), \ldots, \phi_r(t)$. Let $\tilde{V}_1 = V_3$, $\tilde{V}_2 = V_1 = \mathcal{O}$, $\tilde{V}_3 = V_2$. For each $1 \leq k \leq r$, let $\tilde{V}_{3+k} \in K(X) \otimes_{\mathbb{Z}} \mathbb{C}$ satisfy the equation*

$$\mathcal{Z}^{K,\mathrm{sm}}(V_2) = \mathcal{Z}^{K,\mathrm{sm}}(V_{3+k}) - [\mathcal{Z}^{K,\mathrm{sm}}(V_{3+k}), \mathcal{Z}^{K,\mathrm{sm}}(\tilde{V}_{3+k})) \cdot \mathcal{Z}^{K,\mathrm{sm}}(\tilde{V}_{3+k}).$$

Then $\tilde{V}_{3+k} = \pm L_{\mathcal{O}(-H)}\mathcal{O}(E_k - H)$. In particular, $(\tilde{V}_{3+r}, \ldots, \tilde{V}_4, \tilde{V}_3, \tilde{V}_2, \tilde{V}_1)$ form a full exceptional collection of $D^b_{\mathrm{coh}}(X_r)$.

Proof. It follows from Proposition 4.10 and Theorem 1.1 that X_r satisfies Gamma-I at $\bar{\mathbf{t}}^{(2)}$. Hence $\mathcal{Z}^{K,\mathrm{sm}}(\mathcal{O})$ can be represented at $\bar{\mathbf{t}}^{(2)}$ by the half line $L_1 := \bar{u}_1 + \mathbb{R}_{\geq 0} \mathrm{e}^{\mathbf{i}0}$.

Consider the movement of eigenvalues given by the path $\phi_+(t)$. Denote the eigenvalues of $\hat{c}_1(\phi_+(t))$ by $\{u_i^+(t)\}_i$ with $u_i^+(0) = \bar{u}_i$. By Proposition 4.8 (1), we see that $u_1^+(t), u_2^+(t), u_3^+(t)$ move almost along the circle centered at origin with radius $3(\prod \bar{q}_i)^{\frac{1}{3}} \bar{q}^{\frac{1}{3}}$ counterclockwise, and they do not collide with each other. In addition, $u_{3+k}^+(t)$ is nearly fixed and approximately equal to $-\bar{q}_k$ for $1 \leq k \leq r$. As $u_1^+(t)$ moves, the corresponding translation of L_1 does not pass through other eigenvalues $u_i^+(t)$ $(i \neq 1)$, and at $t = 1$, L_1 is translated to $L_2 := \bar{u}_2 + \mathbb{R}_{\geq 0} \mathrm{e}^{\mathbf{i}0}$ since $u_1^+(1) = \bar{u}_2$. So $\mathcal{Z}^{K,\mathrm{sm}}(\mathcal{O})$ can be represented by L_2 at $\phi_+(1) = \bar{\mathbf{t}}^{(2)} + 2\pi \mathbf{i} H$. From the Galois action (3.1) and its K-group framing interpretation (3.2), this implies that $\mathcal{Z}^{K,\mathrm{sm}}(\mathcal{O}(-H)) = G(-H)(\mathcal{Z}^{K,\mathrm{sm}}(\mathcal{O}))$ can be represented by L_2 at $\bar{\mathbf{t}}^{(2)}$. Thus $\mathcal{O}(-H) = V_2 = \tilde{V}_3$. Similarly, considering movement given by the path $\phi_-(t)$, we see that $\mathcal{Z}^{K,\mathrm{sm}}(\mathcal{O}(H))$ can be represented by the half line $L_3 := \bar{u}_3 + \mathbb{R}_{\geq 0} \mathrm{e}^{\mathbf{i}0}$ at $\bar{\mathbf{t}}^{(2)}$, and then $\mathcal{O}(H) = V_3 = \tilde{V}_1$.

For $k_0 = 1, \ldots, r$, we consider the movement of eigenvalues given by the path $\phi_{k_0}(t)$. Denote the eigenvalues of $\hat{c}_1(\phi_{k_0}(t))$ by $\{u_i^{(k_0)}(t)\}_i$ with $u_i^{(k_0)}(0) = \bar{u}_i$. By Proposition

4.8 (2), we see that for $1 \le j \le 3$, $u_j^{(k_0)}(t)$ is nearly fixed and approximately equal to $3\omega^{j-1}(\prod \bar{q}_i)^{\frac{1}{3}}\bar{q}^{\frac{1}{3}}$, and for $1 \le k \le r$ with $k \ne k_0$, $u_{3+k}^{(k_0)}(t)$ is nearly fixed and approximately equal to $-\bar{q}_k$. Besides, $u_{3+k_0}^{(k_0)}(t)$ moves almost along the circle centered at origin with radius $\bar{q}_{k_0}$ counterclockwise. As $u_{3+k_0}^{(k_0)}(t)$ moves, we correspondingly rotate and deform L_1 so that it does not pass through eigenvalues $u_i^{(k_0)}(t)$ for $i \ne 2$, and at $t = 1$ we get the (heavily bent) admissible path L'_{3+k_0} which starts from $\bar{u}_2$, goes around $\bar{u}_{3+k_0}$ and has zero end-phase (see Figure 3). So $\mathcal{Z}^{K,\mathrm{sm}}(\mathcal{O}(-H))$ can be represented by L'_{3+k_0} at $\phi_{k_0}(1) = \bar{\mathbf{t}}^{(2)} - 2\pi\mathbf{i}E_{k_0}$. From the Galois action (3.1) and its K-group framing interpretation (3.2), this implies that $\mathcal{Z}^{K,\mathrm{sm}}(\mathcal{O}(E_{k_0} - H)) = G(E_{k_0})(\mathcal{Z}^{K,\mathrm{sm}}(\mathcal{O}(-H)))$ can be represented by L'_{3+k_0} at $\bar{\mathbf{t}}^{(2)}$. Then $V_{3+k_0} = \mathcal{O}(E_{k_0} - H)$.

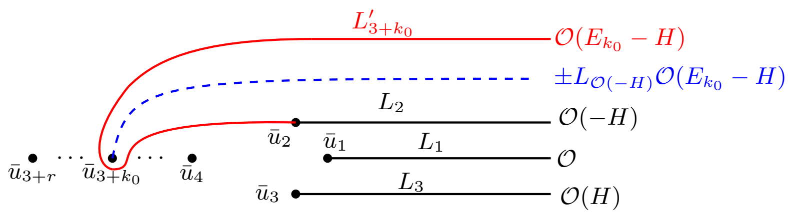


FIGURE 3. Admissible paths and corresponding K-classes. All paths have zero end-phase, and solid paths are given by monodromy transformation from L_1.

Take the simply connected domain U as in Proposition 4.10. Then $U \subset K_{ts}$, and suppose that U is properly-chosen with respect to $\{(u_i, \Psi_i)\}$, where $u_i(\bar{\mathbf{t}}^{(2)}) = \bar{u}_i$ for all i. Let $\hat{y}_i$ be the $\widehat{\nabla}^{\mathrm{sm}}$-flat section from Lemma 3.8. Choosing suitable signs of Ψ_1, Ψ_2, Ψ_3, we can suppose that the Laplace transforms of $\hat{y}_1, \hat{y}_2, \hat{y}_3$ over L_1, L_2, L_3 give $\mathcal{Z}^{K,\mathrm{sm}}(\mathcal{O})$, $\mathcal{Z}^{K,\mathrm{sm}}(\mathcal{O}(-H))$, $\mathcal{Z}^{K,\mathrm{sm}}(\mathcal{O}(H))$, respectively. Let y'_{3+k_0} be the Laplace transform of $\hat{y}_3$ over L'_{3+k_0}. Then we have a priory $y'_{3+k_0} = \pm\mathcal{Z}^{K,\mathrm{sm}}(\mathcal{O}(E_{k_0} - H))$ by the sign ambiguity of normalized idempotents. We will show that $y'_{3+k_0} = \mathcal{Z}^{K,\mathrm{sm}}(\mathcal{O}(E_{k_0} - H))$ in (4.10). Construct a dotted curve L''_{3+k_0} as in Figure 3 that starts from $\bar{u}_{3+k_0}$, does not touch L_2, L'_{3+k_0} nor eigenvalues $\bar{u}_i (i \ne 3 + k_0)$, and has zero end-phase. Let y_{3+k_0} be the Laplace transform of $\hat{y}_{3+k_0}$ over the dotted curve L''_{3+k_0}. Then $\mathcal{Z}^{K,\mathrm{sm}}(\mathcal{O}(-H))$ is the right mutation of y'_{3+k_0} with respect to $\bar{u}_{3+k_0}$, implying

$$\mathcal{Z}^{K,\mathrm{sm}}(\mathcal{O}(-H)) = y'_{3+k_0} - [y'_{3+k_0}, y_{3+k_0}) \cdot y_{3+k_0}. \tag{4.7}$$

We claim that for any $k_0 \in \{1, \dots, r\}$, $y_{3+k_0} = \pm\mathcal{Z}^{K,\mathrm{sm}}(L_{\mathcal{O}(-H)}\mathcal{O}(E_{k_0} - H))$; that is, $\tilde{V}_{3+k_0} = L_{\mathcal{O}(-H)}\mathcal{O}(E_{k_0} - H)$. Note that $\mathcal{O}(-H)$, $\mathcal{O}$, $\mathcal{O}(H)$ form a full exceptional collection on $\mathcal{D}^b_{\mathrm{coh}}(\mathbb{P}^2)$. Then their *augmentation*, $\mathcal{O}(-H), \mathcal{O}(E_r - H), \dots, \mathcal{O}(E_1 - H), \mathcal{O}, \mathcal{O}(H)$, form a full exceptional collection of $\mathcal{D}^b_{\mathrm{coh}}(X_r)$ (cf. [HP11]). Applying left mutations, we see that

$$L_{\mathcal{O}(-H)}\mathcal{O}(E_r - H), \dots, L_{\mathcal{O}(-H)}\mathcal{O}(E_1 - H), \mathcal{O}(-H), \mathcal{O}, \mathcal{O}(H)$$

also form a full exceptional collection. That is, $(\tilde{V}_{3+r}, \dots, \tilde{V}_1)$ form a full exceptional collection.

It remains to show our claim. Let $\tilde{V}_{3+k_0} \in K(X_r) \otimes_{\mathbb{Z}} \mathbb{C}$ satisfy $y_{3+k_0} = \mathcal{Z}^{K,\mathrm{sm}}(\tilde{V}_{3+k_0})$. Note that $y'_{3+k_0} = \pm\mathcal{Z}^{K,\mathrm{sm}}(\mathcal{O}(E_{k_0} - H))$. We first assume that $y'_{3+k_0} = -\mathcal{Z}^{K,\mathrm{sm}}(\mathcal{O}(E_{k_0} -$

$H)$), and then (4.7) together with (2.4) gives

$$\mathcal{O}(-H) = -\left(\mathcal{O}(E_{k_0} - H) - \chi(\mathcal{O}(E_{k_0} - H), \tilde{V}_{3+k_0}) \cdot \tilde{V}_{3+k_0}\right). \tag{4.8}$$

Recall the following relation in $K(X)$:

$$L_{\mathcal{O}(-H)}\mathcal{O}(E_{k_0} - H) = \mathcal{O}(E_{k_0} - H) - \mathcal{O}(-H). \tag{4.9}$$

Then

$$\chi(L_{\mathcal{O}(-H)}\mathcal{O}(E_{k_0} - H), \mathcal{O}(-H)) \overset{(4.9)}{=} \chi(\mathcal{O}(E_{k_0} - H) - \mathcal{O}(-H), \mathcal{O}(-H)) = -1,$$

where definition of exceptional collection is used in the second equality, and

$$\begin{aligned}\mathcal{O}(-H) + \mathcal{O}(E_{k_0} - H) &\overset{(4.9)}{=} L_{\mathcal{O}(-H)}\mathcal{O}(E_{k_0} - H) + 2\mathcal{O}(-H) \\ &\overset{(4.8)}{=} \chi(L_{\mathcal{O}(-H)}\mathcal{O}(E_{k_0} - H) + \mathcal{O}(-H), \tilde{V}_{3+k_0}) \cdot \tilde{V}_{3+k_0}.\end{aligned}$$

Apply Lemma 4.11 with $F_{j_1} = L_{\mathcal{O}(-H)}\mathcal{O}(E_{k_0} - H), F_{j_2} = \mathcal{O}(-H), m = 2$, we get $\tilde{V}_{3+k_0} = \pm\left(L_{\mathcal{O}(-H)}\mathcal{O}(E_{k_0} - H) + 2\mathcal{O}(-H)\right)$.

However, as part of AEFS near $\bar{\mathbf{t}}^{(2)}$ associated with a sufficiently small positive phase, we must have $[y_2, y_{3+k_0}) = 0$; on the other hand,

$$[y_2, y_{3+k_0}) \overset{(2.4)}{=} \chi(\mathcal{O}(-H), L_{\mathcal{O}(-H)}\mathcal{O}(E_{k_0} - H) + 2\mathcal{O}(-H)) = 2,$$

where definition of exceptional collection is used in the second equality. We get a contradiction. So we conclude that

$$y'_{3+k_0} = \mathcal{Z}^{K,\mathrm{sm}}(\mathcal{O}(E_{k_0} - H)) \tag{4.10}$$

A similar application of Lemma 4.11 gives $\tilde{V}_{3+k_0} = \pm L_{\mathcal{O}(-H)}\mathcal{O}(E_{k_0} - H)$. □

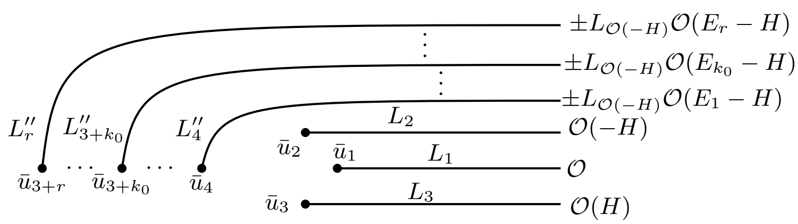


FIGURE 4. Admissible paths and corresponding K-classes. All paths have zero end-phase.

Proposition 4.13. *Let* $(\tilde{V}_{3+r}, \ldots, \tilde{V}_1)$ *be given as in Proposition 4.12. Set* $\phi := \frac{\pi}{8}$. *Then* $\mathrm{Im}(\mathrm{e}^{-\mathbf{i}\phi}\bar{u}_{r+3}) > \cdots > \mathrm{Im}(\mathrm{e}^{-\mathbf{i}\phi}\bar{u}_4) > \mathrm{Im}(\mathrm{e}^{-\mathbf{i}\phi}\bar{u}_2) > \mathrm{Im}(\mathrm{e}^{-\mathbf{i}\phi}\bar{u}_1) > \mathrm{Im}(\mathrm{e}^{-\mathbf{i}\phi}\bar{u}_3)$, *and* $\mathcal{Z}^{K,\mathrm{sm}}(\tilde{V}_i)$ *respects* (u_i, Ψ_i) *for* $4 \le i \le r+3$, $\mathcal{Z}^{K,\mathrm{sm}}(\tilde{V}_3)$ *respects* (u_2, Ψ_2), $\mathcal{Z}^{K,\mathrm{sm}}(\tilde{V}_2)$ *respects* (u_1, Ψ_1), *and* $\mathcal{Z}^{K,\mathrm{sm}}(\tilde{V}_1)$ *respects* (u_3, Ψ_3) *at* $\bar{\mathbf{t}}^{(2)}$ *with phase* ϕ.

Proof. For $k = 1, \ldots, r$, the Laplace transform of $\hat{y}_{3+k}$ over the curved path L''_{3+k} in Figure 4 starting from $\bar{u}_{3+k}$ gives the flat section $\mathcal{Z}^{K,\mathrm{sm}}(\tilde{V}_{3+k})$, and similarly $\mathcal{Z}^{K,\mathrm{sm}}(\mathcal{O}) = \mathcal{Z}^{K,\mathrm{sm}}(\tilde{V}_2)$, $\mathcal{Z}^{K,\mathrm{sm}}(\mathcal{O}(-H)) = \mathcal{Z}^{K,\mathrm{sm}}(\tilde{V}_3)$, $\mathcal{Z}^{K,\mathrm{sm}}(\mathcal{O}(H)) = \mathcal{Z}^{K,\mathrm{sm}}(\tilde{V}_1)$ for L_1, L_2, L_3 respectively.

Now we deform the paths $L_1, L_2, L_3, L''_4, \ldots, L''_{3+r}$ in Figure 4 to the parallel half lines in Figure 5, so that each path has argument ϕ. For $1 \le k \le r-1$, we have $(\bar{u}_{4+k} - \bar{u}_{3+k}) = -\frac{1}{r}\delta_r + (\bar{u}_{4+k} + \bar{q}_{k+1}) - (\bar{u}_{3+k} + \bar{q}_k)$, and consequently $\mathrm{Im}(\mathrm{e}^{-\mathbf{i}\phi}(\bar{u}_{4+k} - \bar{u}_{3+k})) = \frac{\delta_r}{r}\sin\phi + \mathrm{Im}((\bar{u}_{4+k} + \bar{q}_{k+1}) - (\bar{u}_{3+k} + \bar{q}_k)) > \frac{\delta_r}{r}\sin\phi - \frac{\delta_r}{50} > 0$. Here the first inequality follows from Proposition 4.8 (1), and the second inequality follows by noting $\phi = \frac{\pi}{8}$. The remaining inequalities $\mathrm{Im}(\mathrm{e}^{-\mathbf{i}\phi}\bar{u}_4) > \mathrm{Im}(\mathrm{e}^{-\mathbf{i}\phi}\bar{u}_2) > \mathrm{Im}(\mathrm{e}^{-\mathbf{i}\phi}\bar{u}_1) > \mathrm{Im}(\mathrm{e}^{-\mathbf{i}\phi}\bar{u}_3)$ follow from elementary estimates using Proposition 4.8 (1), since $\bar{q} < 10^{-6}$ is sufficiently small.

Following the discussion between Remark 3.9 and Theorem 3.10, the fact that the integration paths in Figure 5 are parallel half-lines ensures that $\mathcal{Z}^{K,\mathrm{sm}}(\tilde{V}_{3+r}), \dots, \mathcal{Z}^{K,\mathrm{sm}}(\tilde{V}_1)$ form an AEFS, and the statement follows. □

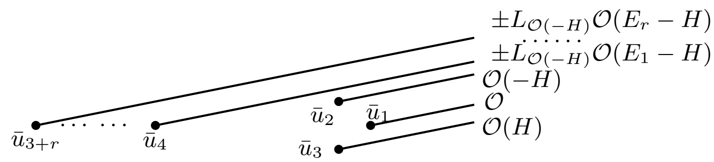


FIGURE 5. Admissible paths and corresponding K-classes. All paths are lines parallel to each other, with end-phase $\phi > 0$ small.

Proof of Theorem 4.1. Propositions 4.9, 4.10, 4.12 and 4.13 verify that all assumptions of Theorem 3.10 hold. Hence, Gamma conjecture II holds for X_r by Theorem 3.10. □

4.3. **Corollary for blow-ups of $\mathbb{P}^2$.** Let S_r be an r-fold blow-up of $\mathbb{P}^2$, i.e., S_r is obtained by $S_r \xrightarrow{b_r} S_{r-1} \xrightarrow{b_{r-1}} \cdots \xrightarrow{b_2} S_1 \xrightarrow{b_1} \mathbb{P}^2$, where each b_i is the blow-up at one point. In general, S_r need not be Fano.

Theorem 4.14. *Gamma conjecture II holds for S_r with $1 \le r \le 8$.*

Proof. Note that S_r can be deformed to X_r. By deformation invariance of Gromov-Witten invariants, Theorem 4.1 implies that, fixing $\mathbf{t}_0 \in B_{ss}(S_r)$ and admissible phase $\theta \in \mathbb{R}$ at $\mathbf{t}_0$, there exist $V_1, \dots, V_s \in K(S_r)$, such that $\{\mathcal{Z}^K(V_i)\}$ is an AEFS near $\mathbf{t}_0$ associated to phase θ, and $\{V_i\}$ is an exceptional basis in the pseudolattice $(K(S_r), \chi)$, i.e., it is a $\mathbb{Z}$-basis of $K(S_r)$ with $\chi(V_i, V_i) = 1$ and $\chi(V_j, V_i) = 0$ whenever $i < j$. Note that $\mathcal{D}^b(S_r)$ admits full exceptional collections by Orlov, and let $\{E_i\}$ be such a collection. By [Kra24, Theorem 1.2 (ii)], the two exceptional bases $\{V_i\}$ and $\{E_i\}$ in $(K(S_r), \chi)$ are related by mutations and sign changes, implying that $\{V_i\}$ arises from a full exceptional collection in $\mathcal{D}^b(S_r)$. This finishes the proof. □

5. CONDITION (SR) FOR DEL PEZZO SURFACES

This section is devoted to prove Proposition 4.2.

5.1. **Condition** (SR) **via Perron–Frobenius theorem.** A nonnegative square matrix M is **primitive** if M^m is positive for some $m \in \mathbb{Z}_{\ge 1}$ (see [Sen81, Definition 1.1]). We use the following standard form of the Perron–Frobenius theorem (see [Sen81, Theorem 1.1]).

Proposition 5.1. *Let M be a primitive matrix. Then M has a positive simple eigenvalue u such that for any other eigenvalue u', we have $u > |u'|$.*

To prove Proposition 4.2, we consider the matrix $M_r^{(a)} = \left((M_r^{(a)})_{ij}\right)_{1 \le i,j \le r+3}$ of $\hat{c}_1(\mathbf{t}^{(2)})$ with respect to the ordered basis $[\mathbf{1}, aH - \sum_i E_i, E_1, ..., E_r, [pt]]$, where $a > 0$. The dual basis with respect to Poincaré pairing is given by

$$\mathbf{1}^\vee = [pt], \quad (aH - \sum_i E_i)^\vee = \frac{H}{a}, \quad E_i^\vee = \frac{H}{a} - E_i, \quad [pt]^\vee = \mathbf{1}.$$

Then each entry $(M_r^{(a)})_{ij}$ is a polynomial in $q, q_1, ..., q_r$, and can be read off as a summation of three-point Gromov-Witten invariants over $\mathrm{Eff}(X_r)$. For instance,

$$(M_r^{(a)})_{32} = \sum_{\mathbf{d}} \langle c_1, E_1, (aH - \sum_i E_i)^\vee \rangle_{\mathbf{d}}^{X_r} \cdot \mathbf{q}^{\mathbf{d}}. \quad \text{(see (4.1) for } \mathbf{q}^{\mathbf{d}})$$

Example 5.2. By Example 4.3 and a change of bases, we have

$$M_1^{(a)} = \begin{bmatrix} 0 & (2a-2)q & 2q & 3qq_1 \\ \frac{3}{a} & 0 & 0 & \frac{2q}{a} \\ -1+\frac{3}{a} & q_1 & -q_1 & -2q+\frac{2q}{a} \\ 0 & -1+3a & 1 & 0 \end{bmatrix},$$

and the matrix $M_2^{(a)}$ is given by

$$\begin{bmatrix} 0 & (2a-2)q(q_1+q_2) & 2qq_2 & 2qq_1 & 3qq_1q_2 \\ \frac{3}{a} & q-\frac{2q}{a} & \frac{q}{a} & \frac{q}{a} & 2\frac{q}{a}(q_1+q_2) \\ -1+\frac{3}{a} & (2-a)(q-\frac{q}{a})+q_1 & -q+\frac{q}{a}-q_1 & -q+\frac{q}{a} & -2qq_2+2\frac{q}{a}(q_1+q_2) \\ -1+\frac{3}{a} & (2-a)(q-\frac{q}{a})+q_2 & -q+\frac{q}{a} & -q+\frac{q}{a}-q_2 & -2qq_1+2\frac{q}{a}(q_1+q_2) \\ 0 & -2+3a & 1 & 1 & 0 \end{bmatrix}.$$

We first assume the following proposition, and leave its proof to next subsection.

Proposition 5.3. *For $3 \le r \le 8$, there exist $\delta_r \in (0, \frac{1}{2}]$, $a_r > 0$ and $\epsilon_r > 0$, such that,*

$$\text{if either } q \in [\epsilon_r, 1],\ q_1 = \cdots = q_r = 1, \quad \text{or } q \in (0, \epsilon_r),\ q_1, \ldots, q_r \in [1, 1+\delta_r],$$

then every off-diagonal entry of $M_r^{(a_r)}$ is positive except $(M_r^{(a_r)})_{r+3,1} = 0$.

Proof of Proposition 4.2. Let I_m be the $m \times m$ identity matrix. By Proposition 5.1, it suffices to find $\delta_r, a_r, c_r > 0$ such that, when $0 < q \le 1$ and $1 \le q_i \le 1 + \delta_r$, $M_r^{(a_r)} + c_r I_{r+3}$ is nonnegative and $(M_r^{(a_r)} + c_r I_{r+3})^2$ is positive.

For $r = 1, 2$, take $\delta_r = \frac{1}{2}, a_r = 1, c_r = 2$. Then from Example 5.2, when $0 < q \le 1$ and $1 \le q_i \le \frac{3}{2}$ hold, sign pattern matrices of $M_1^{(1)} + c_1 I_4$ and $M_2^{(1)} + c_2 I_5$ are

$$\begin{bmatrix} + & 0 & + & + \\ + & + & 0 & + \\ + & + & + & 0 \\ 0 & + & + & + \end{bmatrix} \text{ and } \begin{bmatrix} + & 0 & + & + & + \\ + & + & + & + & + \\ + & + & + & 0 & + \\ + & + & 0 & + & + \\ 0 & + & + & + & + \end{bmatrix},$$

which imply that the square of them are positive.

For $3 \le r \le 8$, take a_r as in Proposition 5.3. Since $(M_r^{(a_r)})_{ij} \in \mathbb{R}[q, q_1, \ldots, q_r]$, there exists $c_r \gg 0$ such that, when q and q_i satisfy the conditions in Proposition 5.3, all entries of $M_r^{(a_r)} + cI_{r+3}$ are positive, except for the $(r+3, 1)$-entry, which is zero, and this sign pattern implies that $(M_r^{(a_r)} + c_r I_{r+3})^2$ is positive. In particular, this holds when $q \in [\epsilon_r, 1]$ and $q_i = 1$. Since $(M_r^{(a_r)} + c_r I_{r+3})^2$ being positive is an open condition on q and q_i, and $[\epsilon_r, 1]$ is compact, it follows that we may shrink δ_r from Proposition 5.3, if necessary, to ensure that$(M_r^{(a_r)} + c_r I_{r+3})^2$ is positive for $q \in (0, 1]$ and $q_i \in [1, 1+\delta_r]$. □

5.2. **Positivity of off-diagonal entries.** Note $3 \le r \le 8$ throughout this subsection but Lemma 5.4 (where the cases $r = 1, 2$ are included for completeness). We prove Proposition 5.3 at the end of the subsection; its proof relies on a series of estimates based on the following two key observations for del Pezzo surfaces X_r.

(o1) In the setting of $q_i = 1$ and $q \in [0, 1]$, there exists $a^* > 0$ such that the off-diagonal entries of $M_r^{(a^*)}$ are polynomials in q with **positive coefficients** (Proposition 5.9).

(o2) For q sufficiently small, estimating **only the terms of degree** 0 **and** 1 in q ensures the necessary positivity, provided q_i remain close to 1 (Proposition 5.10).

Denote $(M_r^*)_{ij} := (M_r^{(a_r^*)})_{ij}$ throughout. While the remaining entries are easier to treat as in the proof of Proposition 5.3, the following three types require detailed estimates:

$$
\begin{aligned}
P_{H,j}^{(a_r)}(q,q_i) &:= (M_r^{(a_r)})_{2,j+2} = \sum_{\mathbf{d}} \langle c_1, a_r H - \sum E_i, E_j^\vee \rangle_{\mathbf{d}}^{X_r} \mathbf{q}^{\mathbf{d}},\\
P_{b,j}^{(a_r)}(q,q_i) &:= (M_r^{(a_r)})_{b+2,j+2} = \sum_{\mathbf{d}} \langle c_1, E_b, E_j^\vee \rangle_{\mathbf{d}}^{X_r} \mathbf{q}^{\mathbf{d}},\\
P_{[pt],j}^{(a_r)}(q,q_i) &:= (M_r^{(a_r)})_{r+3,j+2} = \sum_{\mathbf{d}} \langle c_1, [pt], E_j^\vee \rangle_{\mathbf{d}}^{X_r} \mathbf{q}^{\mathbf{d}},
\end{aligned}
$$

where $1 \le b, j \le r$ and $b \ne j$. We prove positivities for $P_\bullet^* := P_\bullet^{(a_r^*)}$ first at $q_i = 1$ in Proposition 5.9 and then for q small with q_i close to 1 in Proposition 5.10.

To state and prove Proposition 5.9 and 5.10, we need the following Lemma 5.4. Set

$$
d_{rk} := \max\{H(\mathbf{d}) | \mathbf{d} \in \mathrm{Eff}(X_r) \text{ with } \left\langle \tau_0([pt])^{k-1} \right\rangle_{\mathbf{d}}^{X_r} \ne 0\}, \quad k = 1, 2,
$$

where $H(\mathbf{d}) = \int_{\mathbf{d}} H \in \mathbb{Z}$. Note that d_{rk} is well-defined since there are only finitely many effective curve classes $\mathbf{d}$ satisfying $\left\langle \tau_0([pt])^{k-1} \right\rangle_{\mathbf{d}}^{X_r} \ne 0$.

Lemma 5.4. *The values of d_{r1}, d_{r2} and $a_r^* := \frac{r d_{r1}}{3 d_{r1} - 1}$ are as listed in the following Table.*

TABLE 1. d_{r1}, d_{r2} and a_r^* for each X_r

r	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
d_{r1}	*0*	*1*	*1*	*1*	*2*	*2*	*3*	*6*
d_{r2}	*1*	*1*	*1*	*2*	*2*	*3*	*5*	*11*
a_r^*	*0*	*1*	*3/2*	*2*	*2*	*12/5*	*21/8*	*48/17*

Proof. By [GP98, Section 5.2], any contributing curve class $\mathbf{d}$ must have non-negative arithmetic genus, i.e. $(H(\mathbf{d})-1)(H(\mathbf{d})-2) - \sum_{i=1}^r E_i(\mathbf{d})(E_i(\mathbf{d})-1) \ge 0$. Using dimension constraint for equality and Cauchy-Schwarz for inequality, we have

$$
(3H(\mathbf{d})-k) + \sum_{i=1}^r E_i(\mathbf{d})(E_i(\mathbf{d})-1) = \sum_{i=1}^r E_i(\mathbf{d})^2 \ge \frac{1}{r}(\sum_{i=1}^r E_i(\mathbf{d}))^2 = \frac{1}{r}(3H(\mathbf{d})-k)^2 .
$$

Combining the two gives $\frac{1}{r}(3H(\mathbf{d})-k)^2 - (3H(\mathbf{d})-k) \le (H(\mathbf{d})-1)(H(\mathbf{d})-2)$. This yields the quadratic constraint in d_{rk}

$$
9d_{rk}^2 - 6d_{rk}k + k^2 - r(3d_{rk}-k) \le r d_{rk}^2 - 3r d_{rk} + 2r.
$$

For $k=2$, the constraint gives

$$d_{r2} \leq \frac{6+2\sqrt{r}}{9-r} \Longrightarrow d_{r2} \leq \lfloor \frac{6+2\sqrt{r}}{9-r} \rfloor \quad (\text{use } d_{r2} \in \mathbb{Z}_{\geq 0}).$$

One can check that $\lfloor \frac{6+2\sqrt{r}}{9-r} \rfloor$ is exactly the number listed in the third row for $r = 1, \ldots, 8$ (e.g. $\lfloor \frac{6+2\sqrt{r}}{9-r} \rfloor = 11$ when $r = 8$). To prove $d_{r2} = \lfloor \frac{6+2\sqrt{r}}{9-r} \rfloor$, it suffices to find $\mathbf{d} \in \mathrm{Eff}(X_r)$ satisfying $\langle [pt] \rangle^{X_r}_{\mathbf{d}_r} \neq 0$ and $H(\mathbf{d}) = \lfloor \frac{6+2\sqrt{r}}{9-r} \rfloor$. Note that $\langle [pt] \rangle^{X_r}_{H-E_1} \neq 0$, and we can apply Cremona transformation (see [GP98, Section 5.1]) repeatedly to $H - E_1$ to find such $\mathbf{d} \in \mathrm{Eff}(X_r)$. For instance $\mathbf{d} = 11H - \sum_{i=1}^{7} 4E_i - 3E_8$ when $r = 8$. This proves the case $k = 2$. The case $k = 1$ is similar. □

For $d, N \in \mathbb{Z}_{\geq 1}$, let $\mathrm{Eff}_{d,N}(X_r)$ (resp. $\mathrm{Eff}^{[pt]}_{d,N}(X_r)$) denote the subset of $\mathrm{Eff}(X_r)$ consisting of effective classes $\mathbf{d}$ satisfying $H(\mathbf{d}) = d$ and $\langle\rangle^{X_r}_{\mathbf{d}} = N$ (resp. $\langle [pt] \rangle^{X_r}_{\mathbf{d}} = N$).

The following lemma relies on the symmetry of the exceptional divisors E_i and plays a central role throughout our subsequent arguments.

Lemma 5.5. *For $1 \leq b \leq r$, we have $\sum_{\mathbf{d} \in \mathrm{Eff}_{d,N}(X_r)} E_b(\mathbf{d}) = \sum_{\mathbf{d} \in \mathrm{Eff}_{d,N}(X_r)} \frac{3d-1}{r}$ and $\sum_{\mathbf{d} \in \mathrm{Eff}^{[pt]}_{d,N}(X_r)} E_b(\mathbf{d}) = \sum_{\mathbf{d} \in \mathrm{Eff}^{[pt]}_{d,N}(X_r)} \frac{3d-2}{r}$*

Proof. For $\mathbf{d} \in \mathrm{Eff}_{d,N}(X_r)$, we have $c_1(\mathbf{d}) = 3H(\mathbf{d}) - \sum_{i=1}^{r} E_i(\mathbf{d}) = 1$ by the dimension constraint. Moreover, by symmetry among divisors E_i's (see [GP98, (P3) before Remark 3.4]), for each b the quantity $\sum_{\mathbf{d} \in \mathrm{Eff}_{d,N}(X_r)} E_b(\mathbf{d})$ is the same. Therefore

$$\sum_{\mathbf{d} \in \mathrm{Eff}_{d,N}(X_r)} (3d-1) = \sum_{\mathbf{d} \in \mathrm{Eff}_{d,N}(X_r)} \sum_{i=1}^{r} E_i(\mathbf{d}) = \sum_{i=1}^{r} \sum_{\mathbf{d} \in \mathrm{Eff}_{d,N}(X_r)} E_i(\mathbf{d}) = r \sum_{\mathbf{d} \in \mathrm{Eff}_{d,N}(X_r)} E_b(\mathbf{d}),$$

where we exchanged the order of summation since $\mathrm{Eff}_{d,N}(X_r)$ is a finite set. This proves the first equality. The proof for the second equality is similar. □

We proceed by determining conditions on a_r that ensure $P^{(a_r)}_{\bullet}|_{q_i=1} \geq 0$.

Lemma 5.6. *Let $3 \leq r \leq 8$. Set $\Lambda_1(d) := \min\{\frac{3d-1}{d}, \frac{rd}{3d-1}\}$ and $\Lambda_2(d) := \max\{\frac{3d-1}{d}, \frac{rd}{3d-1}\}$ for $1 \leq d \leq d_{r1}$. If*

$$a_r \in \bigcap_{1 \leq d \leq d_{r1}} [\Lambda_1(d), \Lambda_2(d)],$$

then for any $q \in (0,1]$, we have $P^{(a_r)}_{H,j}(q,1) > 0$ with $1 \leq j \leq r$.

Proof. The divisor axiom, the dimension constraint, and Lemma 5.5 give

$$\begin{aligned} P^{(a_r)}_{H,j}(q,1) &= \sum_{\mathbf{d}} \langle\rangle^{X_r}_{\mathbf{d}} \left((a_r - 3)H(\mathbf{d}) + 1\right) \left(\frac{H(\mathbf{d})}{a_r} - E_j(\mathbf{d})\right) q^{H(\mathbf{d})} \\ &= \sum_{\mathbf{d}} \langle\rangle^{X_r}_{\mathbf{d}} (a_r H(\mathbf{d}) - 3H(\mathbf{d}) + 1) \frac{1}{r} \left(\frac{rH(\mathbf{d})}{a_r} - 3H(\mathbf{d}) + 1\right) q^{H(\mathbf{d})} \text{ (Lemma 5.5).} \end{aligned}$$

For $H(\mathbf{d}) = d$ and $1 \leq d \leq d_{r1}$, the assumed range of a_r ensures the nonnegativity of the degree-d coefficient of $P^{(a_r)}_{H,j}(q,1)$: $(a_r d - 3d + 1)(\frac{rd}{a_r} - 3d + 1) \geq 0$. Therefore

$$P^{(a_r)}_{H,j}(q,1) \geq \sum_{H(\mathbf{d})=0} \langle\rangle^{X_r}_{\mathbf{d}} \frac{1}{r} = \sum_{1 \leq i \leq r} \langle\rangle^{X_r}_{E_i} \frac{1}{r} = 1 > 0. \qquad \square$$

Lemma 5.7. *Let $3 \le r \le 8$ and $a_r \in (0, \frac{rd_{r1}}{3d_{r1}-1}]$. If $q \in (0,1]$ and $q_i = 1$, then we have*

$$P_{b,j}^{(a_r)}(q,1) > 0, \quad 1 \le b, j \le r \text{ with } b \neq j.$$

Proof. Let $b \neq j$. By divisor axiom and dimension constraint, $P_{b,j}^{(a_r)}(q,1)$ is equal to $\sum_{\mathbf{d}} \langle\rangle_{\mathbf{d}}^{X_r} E_b(\mathbf{d})(\frac{H(\mathbf{d})}{a_r} - E_j(\mathbf{d}))q^{H(\mathbf{d})}$. Note that the terms with $H(\mathbf{d}) = 0$ do not contribute to the sum, since then $\mathbf{d} = E_i$ and $b \neq j$ implies $E_b(\mathbf{d})(\frac{H(\mathbf{d})}{a_r} - E_j(\mathbf{d})) = 0$. Hence

$$P_{b,j}^{(a_r)}(q,1) = \sum_{d,N\ge 1} \sum_{\mathbf{d}\in \mathrm{Eff}_{d,N}(X_r)} N \left(E_b(\mathbf{d})\frac{d}{a_r} - E_b(\mathbf{d})E_j(\mathbf{d}) \right) q^d.$$

For $\mathrm{Eff}_{d,N}(X_r) \neq \varnothing$, set $\sigma := \left(\sum_{\mathbf{d}\in \mathrm{Eff}_{d,N}(X_r)} (E_b(\mathbf{d})E_j(\mathbf{d}))\right) \cdot |\mathrm{Eff}_{d,N}(X_r)|^{-1}$, which is the average over summation and independent of b, j by symmetry among divisors E_i's. Using $\sum_{\mathbf{d}\in \mathrm{Eff}_{d,N}(X_r)} \sigma = \sum_{\mathbf{d}\in \mathrm{Eff}_{d,N}(X_r)} E_b(\mathbf{d})E_j(\mathbf{d})$, we get

$$\begin{aligned}
\sum_{\mathbf{d}\in \mathrm{Eff}_{d,N}(X_r)} (3d-1)^2 &= \sum_{\mathbf{d}\in \mathrm{Eff}_{d,N}(X_r)} \left(\sum_{i=1}^r E_i(\mathbf{d})\right)^2 \quad (\text{use } c_1(\mathbf{d}) = 1) \\
&= \sum_{\mathbf{d}\in \mathrm{Eff}_{d,N}(X_r)} \left(\sum_{i=1}^r E_i(\mathbf{d})^2 + r(r-1)\sigma\right) \\
&\ge \sum_{\mathbf{d}\in \mathrm{Eff}_{d,N}(X_r)} \left(\frac{(3d-1)^2}{r} + r(r-1)\sigma\right). \qquad (5.1)
\end{aligned}$$

In the last inequality we have used $c_1(\mathbf{d}) = 1$ and the Cauchy-Schwarz inequality, which states that $r(\sum_{i=1}^r E_i(\mathbf{d})^2) \ge (\sum_{i=1}^r E_i(\mathbf{d}))^2 = (3d-1)^2$, with "=" holding if and only if $E_1(\mathbf{d}) = \cdots = E_r(\mathbf{d})$. Therefore we can conclude that $(\frac{3d-1}{r})^2 \ge \sigma$. Hence

$$\sum_{\mathbf{d}\in \mathrm{Eff}_{d,N}(X_r)} E_b(\mathbf{d})\frac{3d-1}{r} \overset{\text{Lemma 5.5}}{=} \sum_{\mathbf{d}\in \mathrm{Eff}_{d,N}(X_r)} \left(\frac{3d-1}{r}\right)^2 \ge \sum_{\mathbf{d}\in \mathrm{Eff}_{d,N}(X_r)} E_b(\mathbf{d})E_j(\mathbf{d}).$$

Then we obtain the following inequality on the degree-d coefficient of $P_{b,j}^{(a_r)}(q,1)$:

$$\sum_{\mathbf{d}\in \mathrm{Eff}_{d,N}(X_r)} E_b(\mathbf{d})\frac{d}{a_r} - E_b(\mathbf{d})E_j(\mathbf{d}) \ge \sum_{\mathbf{d}\in \mathrm{Eff}_{d,N}(X_r)} \frac{E_b(\mathbf{d})}{r}\frac{rd}{a_r} - 3d + 1.$$

Since $rd/(3d-1)$ is decreasing about d and $a_r \le rd_{r1}/(3d_{r1}-1)$, the coefficients are nonnegative for $1 \le d \le d_{r1}$, then so is $P_{b,j}^{(a_r)}(q,1)$. Strict positivity of $P_{b,j}^{(a_r)}(q,1)$ follows from that $\langle\rangle_{H-E_1-E_2}^{X_r} = 1 \neq 0$ and the Cauchy–Schwarz inequality in (5.1) is strict for this $H - E_1 - E_2$ when $r \ge 3$. □

Lemma 5.8. *Let $3 \le r \le 8$, and set*

$$D := \begin{cases} (0,3), & r = 3, \\ (0, \frac{rd_{r2}}{3d_{r2}-2}], & r \ge 4. \end{cases} \qquad (5.2)$$

For $a_r \in D$, if $q \in (0,1]$, then $P_{[pt],j}^{(a_r)}(q,1) > 0$ with $1 \le j \le r$.

Proof. Observe that, by (4.2), $H - E_1 \in \mathrm{Eff}_{1,1}^{[pt]}(X_r)$, so the sum over $Eff_{d,N}^{[pt]}(X_r)$ is nonempty. By divisor axiom, dimension constraint, and Lemma 5.5, $P_{[pt],j}^{(a_r)}(q,1)$ is equal to

$\sum_{\mathbf{d}} 2\langle[pt]\rangle^{X_r}_{\mathbf{d}} \frac{1}{r}\left(\frac{rH(\mathbf{d})}{a_r} - 3H(\mathbf{d}) + 2\right) q^{H(\mathbf{d})}$. Recall that $\langle[pt]\rangle^{X_r}_{\mathbf{d}} \in \mathbb{Z}_{\geq 0}$, and $\langle[pt]\rangle^{X_r}_{\mathbf{d}} \neq 0$ implies $H(\mathbf{d}) \geq 1$. So we have

$$P^{a_r}_{[pt],j}(q,1) = \sum_{d,N\geq 1} \sum_{\mathbf{d}\in \mathrm{Eff}^{[pt]}_{d,N}(X_r)} 2N\frac{1}{r}\left(\frac{rd}{a_r} - 3d + 2\right) q^d.$$

For $r = 3$, 1 gives $d_{r2} = 1$, hence the factor $\frac{rd}{a_r} - 3d + 2$ reduces to $\frac{1}{a_3} - \frac{1}{3}$. Since $a_3 \in (0,3)$ by (5.2), we obtain $P^{(a_3)}_{[pt],j}(q,1) > 0$.

For $r \geq 4$, Table 1 gives $d_{r2} \geq 2$. Since $\frac{rd}{3d-2} = \frac{r}{3} + \frac{2r}{3(3d-2)}$ is strictly decreasing in d, (5.2) implies the coefficients of $P^{(a_3)}_{[pt],j}(q,1)$ are nonnegative, and

$$\frac{rd}{a_r} - 3d + 2 > 0 \quad (1 \leq d \leq d_{r2} - 1), \qquad \frac{rd}{a_r} - 3d + 2 \geq 0 \quad (d = d_{r2}).$$

Moreover, $H - E_1 \in \mathrm{Eff}^{[pt]}_{1,1}(X_r)$, so its contribution is strictly positive, while all remaining contributions with $1 \leq d \leq d_{r2}$ are nonnegative. Therefore $P^{(a_r)}_{[pt],j}(q,1) > 0$ for $r \geq 4$, this proves the lemma. □

Proposition 5.9. *Let $3 \leq r \leq 8$. Then for any $q \in (0,1]$, we have*

$$P^*_{H,j}(q,1) > 0, \quad P^*_{[pt],j}(q,1) > 0, \quad P^*_{b,j}(q,1) > 0, \quad 1 \leq b,j \leq r \text{ with } b \neq j.$$

Proof. By Lemmata 5.6, 5.7, and 5.8, and with the notation therein, it remains to show that

$$a^*_r \in \bigcap_{1\leq d\leq d_{r1}} [\Lambda_1(d), \Lambda_2(d)] \cap (0, \frac{rd_{r1}}{3d_{r1}-1}] \cap D.$$

Recall that $\frac{rd}{3d-2}$ is strictly decreasing in $d \geq 1$, and so is $\frac{rd}{3d-1}$. Checking with Table 1, we have $\frac{rd_{r1}}{3d_{r1}-1} < \frac{rd_{r2}}{3d_{r2}-2}$ for $r = 3$, and $\frac{rd_{r1}}{3d_{r1}-1} \leq \frac{rd_{r2}}{3d_{r2}-2}$ for $r \geq 4$, which implies that

$$a^*_r = \frac{rd_{r1}}{3d_{r1}-1} \in (0, \frac{rd_{r1}}{3d_{r1}-1}] \cap D.$$

It remains to show that $a^*_r \in \bigcap_{1\leq d\leq d_{r1}} [\Lambda_1(d), \Lambda_2(d)]$. For $r = 3,4$, Table 1 gives $d_{r1} = 1$, so that $a_r = \frac{rd_{r1}}{3d_{r1}-1} \in \bigcap_{1\leq d\leq d_{r1}} [\Lambda_1(d), \Lambda_2(d)]$ is automatically satisfied. For $r \geq 5$, Table 1 gives $d_{r1} \geq 2$, and moreover one can check case by case that

$$\frac{3(d_{r1}-1)-1}{d_{r1}-1} \leq \frac{rd_{r1}}{3d_{r1}-1} = a^*_r.$$

Notice that $\frac{3d-1}{d}$ is strictly increasing in $d \geq 1$. Therefore for all $1 \leq d \leq d_{r1} - 1$,

$$\frac{3d-1}{d} \leq \frac{3(d_{r1}-1)-1}{d_{r1}-1} \leq a = \frac{rd_{r1}}{3d_{r1}-1} < \frac{rd}{3d-1}.$$

Moreover $a^*_r = \frac{rd_{r1}}{3d_{r1}-1}$ is one of $\Lambda_1(d_{r1}), \Lambda_2(d_{r1})$, thus $a^*_r \in \bigcap_{1\leq d\leq d_{r1}} [\Lambda_1(d), \Lambda_2(d)]$ as required, which concludes the proof. □

As noted at the beginning of this subsection, the sign of $P^*_\bullet(q, q_i)$ for sufficiently small $q > 0$ is governed by its leading term. This observation yields the following result.

Proposition 5.10. *Let $3 \leq r \leq 8$. There exist positive numbers $\delta_r \in (0, \frac{1}{2}]$ and ϵ_r, such that, for any $q \in (0, \epsilon_r]$ and $q_i \in [1, 1 + \delta_r]$, we have*

$$P^*_{H,j}(q,q_i) > 0, \quad P^*_{[pt],j}(q,q_i) > 0, \quad P^*_{b,j}(q,q_i) > 0, \quad 1 \leq b,j \leq r \text{ with } b \neq j.$$

Proof. For $\bullet = (H,j),(b,j),([pt],j)$, write $P^*_\bullet$ as the sum of its leading term and higher order terms in q, namely

$$P^*_\bullet(q,q_i) = q^k \cdot L_\bullet(q_i) + q^{k+1}\cdot R_\bullet(q,q_i),$$

where $k\in\mathbb{Z}_{\geq 0}$, $L_\bullet$ is a nonzero polynomial in q_i, and $R_\bullet$ is a polynomial in q, q_i. If $L_\bullet$ has a positive lower bound on a box $[1,1+\delta]^r$, then boundedness of $R_\bullet$ on $0\leq q\leq 1$, $1\leq q_i\leq \frac{3}{2}$ gives $P^*_\bullet > 0$ on $(0,\epsilon]\times[1,1+\delta]^r$ for sufficiently small ϵ.

Now it remains to prove that there exists $\delta_r\in(0,\frac{1}{2}]$ such that $L_\bullet(q_i)$ has a positive lower bound when $(q_i)\in[1,1+\delta_r]^r$.

For $L_{H,j}$ with $r\geq 3$, note that $P^*_{H,j}(q,q_i) = \langle\rangle^{X_r}_{E_j} q_j\cdot q^0 + R_{H,j}(q,q_i)q^1$. With $\langle\rangle^{X_r}_{E_j}$, we have $L_{H,j}(q_i) = q_j\geq 1$ for any $(q_i)\in[1,1+\frac{1}{2}]^r$.

For $L_{b,j}$ with $r\geq 3$, we have

$$P^*_{b,j}(q,q_i) = \sum_{\substack{1\leq k\leq r\\ k\neq b}} \langle c_1, E_b, E_j^\vee\rangle^{X_r}_{H-E_b-E_k}\left(\prod_{i\neq b,k} q_k\right) q + \sum_{\mathbf{d}:H(\mathbf{d})\geq 2}\langle c_1,E_b,E_j^\vee\rangle^{X_r}_{\mathbf{d}}\mathbf{q}^{\mathbf{d}}.$$

So together with $\langle\rangle^{X_r}_{H-E_b-E_k}=1$, the divisor equation gives

$$L_{b,j}(q_i) = (\frac{1}{a_r^*}-1)\prod_{i\neq b,j} q_i + \sum_{k\neq b,j}\frac{1}{a_r^*}\prod_{i\neq b,k} q_i.$$

By Table 1, for $r\geq 3$ one has $1 < a_r^* = \frac{rd_{r1}}{3d_{r1}-1} < r-1$, and hence $\frac{1}{a_r^*}-1+\frac{r-2}{a_r^*} > 0$. Therefore, for all $(q_i)\in[1,1+\delta^*_{r1}]^r$with choosing $\delta^*_{r1}>0$ sufficiently small, we have

$$L_{b,j}(q_i)\geq (\frac{1}{a_r^*}-1)(1+\delta_{r1})^{r-2} + \sum_{k\neq b,j}\frac{1}{a_r^*} = (\frac{1}{a_r^*}-1)(1+\delta_{r1})^{r-2} + \frac{r-2}{a_r^*} > 0.$$

Thus $L_{b,j}(q_i)$ has a positive lower bound on $[1,1+\delta^*_{r1}]^r$.

For $L_{[pt],j}$ with $r\geq 3$, we have

$$P^*_{[pt],j} = \sum_{1\leq k\leq r}\langle c_1,[pt],E_j^\vee\rangle^{X_r}_{H-E_k}\left(\prod_{i\neq k} q_i\right) q + \sum_{\mathbf{d}:H(\mathbf{d})\geq 2}\langle c_1,[pt],E_j^\vee\rangle^{X_r}_{\mathbf{d}}\mathbf{q}^{\mathbf{d}}.$$

So together with $\langle[pt]\rangle_{H-E_k}=1$, the divisor equation gives

$$L_{[pt],j} = 2(\frac{1}{a_r^*}-1)\prod_{i\neq j} q_i + \sum_{k\neq j} 2\frac{1}{a_r^*}\prod_{i\neq k} q_i.$$

By Table 1, for $r\geq 3$ one has $1<a_r^*<r-1<r$, and hence $\left(\frac{1}{a_r^*}-1\right)+\frac{r-1}{a_r^*} = \frac{r-a_r^*}{a_r^*} > 0$. Therefore, for all $(q_i)\in[1,1+\delta^*_{r2}]^r$ with choosing $\delta^*_{r2}>0$ sufficiently small, we have

$$L_{[pt],j}(q_i)\geq 2\left(\frac{1}{a_r^*}-1\right)(1+\delta^*_{r2})^{r-1} + \sum_{k\neq j}\frac{2}{a_r^*} = 2\left(\frac{1}{a_r^*}-1\right)(1+\delta^*_{r2})^{r-1} + \frac{2(r-1)}{a_r^*} > 0.$$

Thus $L_{[pt],j}(q_i)$ has a positive lower bound on $[1,1+\delta^*_{r2}]^r$.

Finally, taking $\delta_r = \min\{\frac{1}{2},\delta^*_{r1},\delta^*_{r2}\}$ completes the proof. □

Now we end this subsection with the proof of Proposition 5.3.

Proof of Proposition 5.3. Take $a_r = a_r^*$ and denote $(M_r^*)_{ij} := (M_r^{(a_r^*)})_{ij}$. By Propositions 5.9 and 5.10, the required positivity of $P^*_\bullet(q,q_i)$ for $r \geq 3$ is ensured, and it remains to consider the following four cases. We use T to denote an arbitrary class in the basis $[\mathbf{1}, a_r^* H - \sum_{i=1}^r E_i, E_1, ..., E_r, [pt]]$.

(1) The **first column** is given by $\langle c_1, \mathbf{1}, T^\vee\rangle_{\mathbf{d}}$. We have

$$(M_r^*)_{11} = 0, \quad (M_r^*)_{21} = \frac{3}{a_r^*}, \quad (M_r^*)_{31} = \cdots = (M_r^*)_{r+2,1} = \frac{3}{a_r^*} - 1, \quad (M_r^*)_{r+3,1} = 0.$$

By Table 1, we see $(M_r^*)_{j1} > 0$ for $j = 2, \cdots, r+2$.

(2) The **last row** is given by $\langle c_1, T, [pt]^\vee\rangle_{\mathbf{d}}^{X_r}$. We have

$$(M_r^*)_{r+3,1} = 0, \quad (M_r^*)_{r+3,2} = 3a_r^* - r, \quad (M_r^*)_{r+3,3} = \cdots = (M_r^*)_{r+3,r+2} = 1, \quad (M_r^*)_{r+3,r+3} = 0.$$

By Table 1, we see $(M_r^*)_{r+3,2} > 0$.

(3) The **first row** is given by $\langle c_1, T, \mathbf{1}^\vee\rangle_{\mathbf{d}}^{X_r}$. The dimension constraint and divisor axiom give

$$(M_r^*)_{12} = \sum_{\mathbf{d}} \langle c_1, a_r^* H - \sum E_i, \mathbf{1}^\vee\rangle_{\mathbf{d}}^{X_r} \mathbf{q}^{\mathbf{d}} = \sum_{\mathbf{d}} 2\left((a_r^* - 3)H(\mathbf{d}) + 2\right)\langle [pt]\rangle_{\mathbf{d}}^{X_r} \mathbf{q}^{\mathbf{d}},$$

$$(M_r^*)_{1,b+2} = \sum_{\mathbf{d}} \langle c_1, E_b, \mathbf{1}^\vee\rangle_{\mathbf{d}}^{X_r} \mathbf{q}^{\mathbf{d}} = \sum_{\mathbf{d}} 2E_b(\mathbf{d})\langle [pt]\rangle_{\mathbf{d}}^{X_r} \mathbf{q}^{\mathbf{d}} \quad \text{for any } 1 \leq b \leq r,$$

$$(M_r^*)_{1,r+3} = \sum_{\mathbf{d}} \langle c_1, [pt], \mathbf{1}^\vee\rangle_{\mathbf{d}}^{X_r} \mathbf{q}^{\mathbf{d}} = \sum_{\mathbf{d}} 3\langle [pt]^2\rangle_{\mathbf{d}}^{X_r} \mathbf{q}^{\mathbf{d}}.$$

For $(M_r^*)_{12}$, note that $a_r^* < 3$, and then for $1 \leq d \leq d_{r2}$, we have

$$(a_r^* - 3)d + 2 \geq (a_r^* - 3)d_{r2} + 2 \overset{\text{Table 1}}{=} \begin{cases} 0, & r = 4, 5, \\ \frac{1}{3d_{r1} - 1} > 0, & r = 3, 6, 7, 8, \end{cases}$$

implying that $\left((a_r^* - 3)H(\mathbf{d}) + 2\right)\langle [pt]\rangle_{\mathbf{d}}^{X_r} \geq 0$ for all $\mathbf{d}$. In particular, considering $\mathbf{d} = H - E_1$ for $r \geq 3$, we get $(M_r^*)_{12} > 0$. Note that $E_b(\mathbf{d})\langle [pt]\rangle_{\mathbf{d}}^{X_r} \geq 0$ for all $\mathbf{d}$, considering $\mathbf{d} = H - E_b$, we get $(M_r^*)_{1,b+2} > 0$. Finally, $\langle [pt]^2\rangle_H^{X_r} > 0$ gives $(M_r^*)_{1,r+3} > 0$.

(4) The **second row** is given by $\langle c_1, T, (aH - \sum E_i)^\vee\rangle_{\mathbf{d}}^{X_r}$, the same argument gives

$$(M_r^*)_{2,b+2} = \sum_{\mathbf{d}} \langle c_1, E_b, (a_r^* H - \sum E_i)^\vee\rangle_{\mathbf{d}}^{X_r} \mathbf{q}^{\mathbf{d}} = \sum_{\mathbf{d}} E_b(\mathbf{d}) \frac{H(\mathbf{d})}{a_r^*} \langle\rangle_{\mathbf{d}}^{X_r} \mathbf{q}^{\mathbf{d}} \quad \text{for any } 1 \leq b \leq r,$$

$$(M_r^*)_{2,r+3} = \sum_{\mathbf{d}} \langle c_1, [pt], (a_r^* H - \sum E_i)^\vee\rangle_{\mathbf{d}}^{X_r} \mathbf{q}^{\mathbf{d}} = \sum_{\mathbf{d}} 2\frac{H(\mathbf{d})}{a_r^*} \langle [pt]\rangle_{\mathbf{d}}^{X_r} \mathbf{q}^{\mathbf{d}}.$$

Note that $E_b(\mathbf{d})H(\mathbf{d})\langle\rangle_{\mathbf{d}}^{X_r} \geq 0$ for all $\mathbf{d}$, considering $\mathbf{d} = H - E_b - E_1$, we get $(M_r^*)_{2,b+2} > 0$. Similarly, considering $\mathbf{d} = H - E_1$, we get $(M_r^*)_{2,r+3} > 0$. □

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