

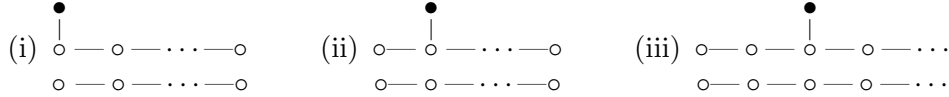
A-D-E DIAGRAMS, HODGE–TATE HYPERPLANE SECTIONS AND SEMISIMPLE QUANTUM COHOMOLOGY

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ABSTRACT. It is known that the semisimplicity of quantum cohomology implies the vanishing of off-diagonal Hodge numbers (Hodge–Tateness). We investigate which hyperplane sections of homogeneous varieties possess either of the two properties. We provide a new efficient criterion for non-semisimplicity of the small quantum cohomology ring of Fano manifolds that depends only on the Fano index and Betti numbers. We construct a bijection between Dynkin diagrams of types A , D or E , and complex Grassmannians with Hodge–Tate smooth hyperplane sections. By applying our criteria and using monodromy action, we completely characterize the semisimplicity of the small quantum cohomology of smooth hyperplane sections in the case of complex Grassmannians, and verify a conjecture of Benedetti and Perrin in the case of (co) adjoint Grassmannians.

HIGHLIGHTS

- Hyperplane sections of complex Grassmannians $Gr(k, n)$ are Hodge–Tate if and only if there is a Dynkin diagram of type A_n ($k = 1$), D_n ($k = 2$) or E_n ($k = 3$ and $n \leq 8$) obtained by adding a node adjacent to the Dynkin diagram of type A_{n-1} . Namely,



- Quite often knowledge of Betti numbers and Fano index is sufficient to witness non-semisimplicity of small quantum cohomology, even for Hodge–Tate Fano manifolds. Complete characterizations for the case of hyperplane sections of generalized Grassmannians are proposed in Conjecture 1.7, with partial cases confirmed.
- Quantum Lefschetz hyperplane principle admits user-friendly formulations, namely Proposition 1.10 after [GI], directly applicable by non-specialists in Gromov–Witten theory. We illustrate this advantage in the case of $Gr(3, 8)$, an avatar of E_8 .

1. INTRODUCTION

The big quantum cohomology ring $BQH^*(X)$ of a Fano manifold X encodes genus-zero Gromov–Witten invariants, and is canonically equipped with a Frobenius manifold structure. It is important to investigate the semisimplicity of $BQH^*(X)$. For instance by Givental’s conjecture [Giv01] proved by Teleman [Te12], all higher genus Gromov–Witten invariants of X are determined by the genus-zero ones, provided that $BQH^*(X)$ is generically semisimple. The remarkable Dubrovin’s conjecture [Du98] with clarification [GMS15] on part 1, together with the refinement of its part 3 known as Gamma conjecture II [GGI16] and independently formulated in [Du13, CDG], also concerned with the semisimplicity of the big quantum cohomology.

The (small) quantum cohomology ring $QH^*(X) = (H^*(X) \otimes \mathbb{C}[\mathbf{q}], \star)$ is a deformation of the classical cohomology $H^*(X) = H^*(X, \mathbb{C})$ by incorporating genus-zero, three-point Gromov–Witten invariants. It is relatively more accessible than $BQH^*(X)$, and if $QH^*(X)$ is semisimple at some specialization of the quantum variables $\mathbf{q}$, then $BQH^*(X)$ is generically semisimple.

Some important classes of Fano manifolds have generically semisimple small quantum cohomology, including toric Fano manifolds [Ba93, OsTy09], complete flag manifolds [Ko96] and (co)minuscule Grassmannians [CMP10]. Nevertheless, there are some examples with nice geometry such as most of (co)adjoint Grassmannians [CP11, PS21] that have non-semisimple small quantum cohomology, whilst having its big quantum cohomology generically semisimple. It is known that semisimplicity of either big or small quantum cohomology $QH^*(X)$ puts some constraints on the classical cohomology $H^*(X)$. The first one was given by Bayer and Manin, and was strengthened by Hertling, Manin and Teleman.

Proposition 1.1 ([BM04, HMT09]). *The even part of big quantum cohomology $BQH^{\text{ev}}(X)$ is generically semisimple only if $H^*(X)$ is of Hodge–Tate type, i.e. $H^*(X) = \bigoplus_p H^{p,p}(X)$.*

There is another criterion for generic semi-simplicity of small quantum cohomology given by Chaput and Perrin [CP11, Theorem 4] in terms of the degree of defining equations for a presentation of $H^*(X)$ when X is of Picard number one.

Recall that the Fano index of X is defined by

$$r = r_X := \max\{k \in \mathbb{Z} \mid \frac{c_1(X)}{k} \in H^2(X, \mathbb{Z})\}.$$

Define *index-periodic even Betti numbers*

$$\tilde{b}_{\bar{i}} = \tilde{b}_{\bar{i}}(X) := \sum_{j \equiv i \pmod{r_X}} b_{2j}(X),$$

with indices $\bar{i}$ running over residue classes modulo r_X . As a very simple observation, we obtain the following theorem.

Theorem 1.2. *The even quantum cohomology $QH^{\text{ev}}(X)$ is generically semisimple only if both of the following hold.*

- (1) $\tilde{b}_{\bar{i}}(X) \leq \tilde{b}_{\overline{di}}(X)$ for all integer i and d ,
- (2) $\tilde{b}_{\bar{i}}(X) = \tilde{b}_{-\bar{i}}(X)$ for all integer i .

Here part (2) is a direct consequence of part (1) by taking $d = r - 1$. We specify this property as it is already very useful in many cases. For instance, we consider a generalized Grassmannian G/P_k , where G is a connected complex simple Lie group and P_k is the maximal parabolic subgroup that corresponds to the complement of the k -th simple root in the base of simple roots. Here we follow the label of the Dynkin diagram for G as in [Bou], see also [Bel]. Therein we can see that 9 (resp. 14) among all the 27 Grassmannians of exceptional Lie type have semisimple (resp. non-semisimple) small quantum cohomology. As first applications, we will reprove in **Theorem 2.3** that the non-semisimplicity of all the 14 known cases but E_8/P_4 follow directly from Theorem 1.2 (2).

As one main aim of this paper, we investigate the semisimplicity of $QH^*(Y)$ for a smooth hyperplane section Y of a generalized Grassmannian G/P_k (with respect to its minimal embedding). It is natural to start with G of type A , i.e. the complex Grassmannian $A_{n-1}/P_k = Gr(k, n) = \{V \leq \mathbb{C}^n \mid \dim V = k\}$. When $k = 1$, $Y = \mathbb{P}^{n-2}$ and $QH^*(Y)$ is generically semisimple. When $k = 2$ and $n = 2m$ with $m > 1$, $Y \simeq C_n/P_2$ and $QH^*(Y)$ is non-semisimple [CP11]. When $k = 2$ and $n = 2m + 1$ with $m > 1$, Y is a quasi-homogeneous variety and $QH^*(Y)$ is generically semisimple [Pec13, Per14]. When $k > 2$ and $n > 3k$, $H^*(Y)$ is not of Hodge–Tate type by [BeFaM21, Theorem 3]. As one main result of this paper, we provide the following complete characterization, which is a combination of **Theorem 3.8** and **Theorem 3.22**.

Theorem 1.3. *Let Y be a smooth hyperplane section of $Gr(k, n) \simeq Gr(n - k, n)$ where $n \geq 2k$.*

(1) $H^*(Y)$ is of Hodge–Tate type if and only if one of the following holds:

- (i) $k = 1$; (ii) $k = 2$; (iii) $k = 3$ and $n \in \{6, 7, 8\}$.

(2) $QH^*(Y)$ is generically semisimple if and only if one of the following holds:

- (i) $k = 1$; (ii) $k = 2$ and $n \in \{4\} \cup \{2m + 1 \mid m \in \mathbb{Z}_{\geq 2}\}$; (iii) $k = 3$ and $n \in \{7, 8\}$.

Remark 1.4. The classification in Theorem 1.3 (1) is in bijection with ADE Dynkin diagrams in the sense that the diagram obtained by adding a node adjacent to the k -th node of type A_{n-1} is a simply-laced Dynkin diagram. Moreover, by [BaFuM20, SK77, Sc06], the following are all equivalent:

- (1) $H^*(Y)$ is of Hodge–Tate type.
- (2) Y is locally rigid.
- (3) The fundamental representation $\Lambda^k \mathbb{C}^n$ of GL_n has a Zariski open orbit.
- (4) The cluster algebra of A_{n-1}/P_k is of finite type.

However, we do not have a conceptual explanation for the equivalence.

Besides the cases in Theorem 1.3 (1), (co)adjoint Grassmannians G/P_k (in Table 1) provide another family of generalized Grassmannians such that the cohomology of a smooth hyperplane section Y is of Hodge–Tate type, as shown by Benedetti and Perrin [BP22]. Such hyperplane

type	B	C	D	E_6	E_7	E_8	F_4	G_2
adjoint	B_n/P_2	C_n/P_1	D_n/P_2	E_6/P_2	E_7/P_1	E_8/P_8	F_4/P_1	G_2/P_2
coadjoint	B_n/P_1	C_n/P_2	D_n/P_2	E_6/P_2	E_7/P_1	E_8/P_8	F_4/P_4	G_2/P_1

TABLE 1. (Co)Adjoint Grassmannians G/P_k

sections Y admit a uniform characterization. By Conjecture 1.10 (2) in loc. cit., $QH^*(Y)$ is generically semisimple if and only if so is $QH^*(G/P_k)$. Now this conjecture holds true by combining the study of the semisimplicity of $QH^*(G/P_k)$ in [CP11, PS21] and the following theorem.

Theorem 1.5. *Let Y be a smooth hyperplane section of G/P_k of (co)adjoint type. Then $QH^*(Y)$ is generically semi-simple if and only if G/P_k is adjoint and not coadjoint, i.e. $G/P_k \in \{B_n/P_2, C_n/P_1, F_4/P_1, G_2/P_2\}$.*

The semisimplicity of $QH^*(Y)$ has been precisely determined except for Y in C_n/P_2 or D_n/P_2 (see [BP22, Theorem 1.11]). As another main result of this paper, we obtain the following, which is a combination of **Theorem 4.1** and **Theorem 4.5** and fills in the last piece of the proof of Theorem 1.5.

Theorem 1.6. *Let Y be a smooth hyperplane section of C_n/P_2 or D_n/P_2 . Then $QH^*(Y)$ is not semi-simple.*

For Y in $Gr(3, 6) = A_5/P_3$ or the coadjoint Grassmannian C_n/P_2 , we achieve the non-simplicity of $QH^*(Y)$ by a direct application of Theorem 1.2 (2). We remark that for Y in a coadjoint Grassmannian G/P_k of type B_n, F_4 or G_2 , the non-simplicity of $QH^*(Y)$ can be simply verified by using Theorem 1.2 (2) as well.

The monodromy action plays a surprising role in the proof (resp. disproof) of the semisimplicity of quantum cohomology for Y in A_7/P_3 (resp. D_n/P_2). More precisely, a hyperplane section depends on a choice of global sections, and thus can be viewed as a fiber of a universal family (see Sections 3.3 and 4.2 for precise descriptions). The fundamental group of the smooth locus acts on $H^*(Y)$ by monodromy, and preserves the quantum product. We

obtain the semisimplicity of (the radical part of) $QH^*(Y)$ in case A_7/P_3 by using Deligne invariant cycle theorem. We obtain the non-semisimplicity in case D_n/P_2 by carrying out a perverse sheaf-theoretic study of the universal family and applying Springer theory in geometric representation theory. This will lead to an extra $\mathbb{Z}_2$ -graded algebra structure of the quantum cohomology, so that Theorem 1.2 (1) (strictly speaking, Lemma 2.1) can be applied. We remark that our proof works straightforwardly for Y in a coadjoint Grassmannian of type E as well. We also notice that the monodromy action method has been used to study Gromov–Witten invariants [Hu15, Mi19] before.

In our proof of Theorem 1.3 (1), the key ingredient is the combinatorial characterization of nonvanishing cohomology by Snow [Sn86] (see Proposition 3.1) from the parabolic Borel–Weil Theorem by Bott [Bo57]. The hypotheses on (k, n) in Theorem 1.3 (1) is equivalent to the following condition (see Lemma 3.5)

$$\dim Gr(k, n) = k(n - k) < 2n = 2r_{Gr(k, n)}.$$

For any G/P_k of (co)adjoint type, the inequality $\dim G/P_k < 2r_{G/P_k}$ holds by direct calculations, and $H^*(Y)$ is always of Hodge–Tate type [BP22]. These observations, together with our computations in examples of general Lie type, lead us to the following conjecture. We remark that the inequality $\dim X < 2r_X$ also appeared in the early study of Fano complete intersections in projective spaces [Be95, TX97]. Although the cases for G/P_k of type A or (co)adjoint type have been classified in Theorems 1.3 and 1.5, we include them in the statement below for completeness.

Conjecture 1.7. *Let Y be a smooth hyperplane section of G/P_k . The following should hold.*

- (1) *$H^*(Y)$ is of Hodge–Tate type if and only if $\dim G/P_k < 2r_{G/P_k}$, namely G/P_k is given by one of the cases:*
 - (a) *a complex Grassmannian in Theorem 1.3 (1);*
 - (b) *a (co)adjoint Grassmannian;*
 - (c) (i) B_n/P_1 or D_n/P_1 , (ii) C_3/P_3 , (iii) $B_{n-1}/P_{n-1} \simeq D_n/P_n \simeq D_n/P_{n-1}$ for $n \in \{4, 5, 6, 7\}$, (iv) $E_6/P_1 \simeq E_6/P_6$ or (v) E_7/P_7 .
- (2) *$QH^*(Y)$ is generically semisimple if and only if G/P_k is given by one of the cases:*
 - (a) *a complex Grassmannian in Theorem 1.3 (2);*
 - (b) *an adjoint Grassmannian of type B, C, F or G .*
 - (c) (i) B_n/P_1 or D_n/P_1 , (ii) C_3/P_3 , (iii) $B_{n-1}/P_{n-1} \simeq D_n/P_n \simeq D_n/P_{n-1}$ for $n \in \{4, 5\}$.

Remark 1.8. A hyperplane section Y in Case (c) (i) is a quadratic hypersurface, so $H^*(Y)$ is of Hodge–Tate type and $QH^*(Y)$ is generically semisimple. For the remaining cases in Conjecture 1.7 (1)(c), we provide an outline as follows, which would require more work in practice. The cohomology $H^*(Y)$ of such Y are of Hodge–Tate type by direct calculations. Among them, case (ii) and case (iii) with $n \in \{4, 5\}$ has semisimple quantum cohomology by using similar arguments to Y in $Gr(3, 7)$; the rest have non-semisimple quantum cohomology by using Theorem 1.2.

Remark 1.9. There are in total 5 generalized Grassmannians G/P_k with $\dim G/P_k = 2r_{G/P_k}$, given by A_7/P_4 , $A_8/P_3 = A_8/P_6$, $B_7/P_7 = D_8/P_7 = D_8/P_8$, C_4/P_4 and C_4/P_3 . They are not N -Calabi–Yau in the sense of Bernardara, Fatighenti and Manivel [BeFaM21]. For general hyperplane section Y in these cases, the Hodge number $h^{r_Y+1, r_Y}(Y)$ is nonzero. As pointed out by Laurent Manivel, such nonzero Hodge number in the first 4 cases is the genus of an algebraic curve that can be attached to Y . See [GSW13, BBFM25] and the references therein for the incorporation of higher genus curves with representations in Vinberg theory.

Finally, let us describe a convenient and powerful variation [GI] of a quantum Lefschetz hyperplane principle that originally organized our investigation for the cases A_6/P_3 and A_7/P_3 , and helped to lead it through. Information about quantum cohomology of homogeneous varieties is relatively easy to obtain and organize, in particular tables of [GG05], based on Peterson’s quantum Chevalley formula often turn out to be sufficient. In turn, hyperplane sections of homogeneous varieties usually lack sufficient homogeneity, with a notable exception of (co)adjoint varieties, as observed in [BP22]. So ways to transfer of information about quantum cohomology from an ambient space to its hypersurface or other way around are valuable. Quantum Lefschetz Hyperplane Principles is a variety of such ways. Some of them more obviously can be related to classical Lefschetz hyperplane theorems, and others (e.g. some spectral formulations below) may look very different from classical counterparts, but more useful in practice. One feature that may look surprising in contrast to the classical case, is that in many non-trivial situations the passage from ambient space to hyperplane section is invertible, up to some minor constants fitting.

Now we let Y be a smooth ample hypersurface of X with natural inclusion $j : Y \hookrightarrow X$. The genus-zero Gromov–Witten theory of Y can be related with the (twisted) Gromov–Witten theory of X by the quantum Lefschetz principle theorems of [Ki99, Le01, CG07]. In these works it is mainly phrased as a relation either between the virtual fundamental classes of moduli spaces of stable maps or between the Givental’s J -functions when passing from an ambient space X to its hypersurface Y . There have been various extensions to other situations, such as [CCIT09, Ts10, IMM16], and we refer to [OhTh24] and references therein for more progress in recent years. Note that the induced ring homomorphism $j^* : H^*(X) \rightarrow H^*(Y) = H^*_{\text{amb}}(Y) \oplus H^*_{\text{prim}}(Y)$ has image $j^*(H^*(X)) = H^*(Y)_{\text{amb}}$. All the various versions of the aforementioned quantum Lefschetz principle relate the information of X to the ambient part of the corresponding information of Y . By the classical Lefschetz hyperplane theorem, the restriction $j^*|_{H^m(X)} : H^m(X) \rightarrow H^m(Y)$ is an isomorphism of abelian groups for $0 \leq m < \dim Y$.

These original quoted versions of quantum Lefschetz principle turn out to be cumbersome for some applications in practice, especially when important information is encoded differently. See [Go07, Section 6.6] for a formulation that relates quantum D -modules, [GM11, Section 7.3] for a relevant example that shows its convenience. A version of a quantum Lefschetz principle on the level of quantum D -modules was in [IMM16, Corollary 1.2].

A statement of a quantum Lefschetz hyperplane principle that directly relates $QH^*(X)$ with $QH^*(Y)$ on the level of algebras is being developed by Galkin and Iritani. For the purposes of this article the preliminary form available in [GI] would suffice.

For simplicity, here we restrict to Fano manifolds X of Picard number 1. In this case, $QH^*(X) = H^*(X) \otimes \mathbb{C}[q_X]$ is a $\mathbb{C}[q_X]$ -module, and we consider the linear operator $\hat{\alpha}$ on $QH^\bullet(X) := QH^*(X)|_{q_X=1}$ induced by the quantum multiplication: $\beta \mapsto (\alpha \star \beta)|_{q_X=1}$. Make $H^*(X) = \bigoplus_i \mathcal{A}^{\bar{i}}$ into a $\mathbb{Z}_r$ -graded vector space, where

$$\mathcal{A}^{\bar{i}} := \bigoplus_{j \equiv i \pmod r} H^{2j}(X).$$

Note, in particular, that periodic Betti numbers $\tilde{b}_i = \dim \mathcal{A}^{\bar{i}}$ are dimensions of graded pieces. Define $Rad(X) := \{\alpha \in H^*(X) \mid (c_1(X))^{*m} \star \alpha = 0 \text{ for some } m > 0\}$, and consider its orthogonal complement

$$\mathcal{A}^0_{\perp}(X) := \{\gamma \in \mathcal{A}^0(X) \mid \int_X \alpha \cup \gamma = 0 \text{ for any } \alpha \in Rad(X)\}.$$

For $\dim Y > 2$, by Lefschetz hyperplane theorem, the induced map $\text{Pic}(X) \rightarrow \text{Pic}(Y)$ is an isomorphism of free cyclic groups $\text{Pic}(X) = \mathbb{Z}\mathcal{O}_X(1)$ and $\text{Pic}(Y) = \mathbb{Z}\mathcal{O}_Y(1)$ that sends an ample generator $\mathcal{O}_X(1)$ to an ample generator $\mathcal{O}_Y(1)$. Denote by $h_X = c_1(\mathcal{O}_X(1))$ and $h_Y = j^*h_X = c_1(\mathcal{O}_Y(1))$, so that $c_1(X) = r_X h_X$ and $c_1(Y) = r_Y h_Y$.

Proposition 1.10 (Quantum Lefschetz hyperplane; Galkin–Iritani [GI]). *Let $j : Y \hookrightarrow X$ be the natural inclusion of a smooth Fano hypersurface Y in a Fano manifold X of Picard number 1. Assume $\dim Y > 2$.*

- (1) *There is a ring homomorphism $j_q^* : QH^*(X) \rightarrow QH^*(Y)$ that fits into a commutative diagram*

$$\begin{array}{ccc} QH^*(X) & \xrightarrow{j_q^*} & QH^*(Y) \\ \pi_X \downarrow & & \downarrow \pi_Y \\ QH^*(X)/(q_X) = H^*(X) & \xrightarrow{j^*} & H^*(Y) = QH^*(Y)/(q_Y) \end{array}$$

and $j_q^(q_X) \neq 0$. Moreover, $j_q^*(q_X) = q_Y h_Y$ if both $r_X - 1 = r_Y$ and $r_Y > 1$ hold.*

- (2) *When $r_Y > 1$, the set of nonzero eigenvalues of $\widehat{h}_X^{r_X}$ coincides with that of $\widehat{h}_Y^{r_Y}$, up to dilation by a constant if $r_X - r_Y > 1$.*

Remark 1.11. The stability of spectra in part (2) of the above proposition could have been observed independently more than once. We know it was observed and used in early 2000s by mirror symmetry research group in Moscow, that included Golyshev, Galkin, and Przyjalkowski, not later than 2004, and is implicitly used in supplementary materials [GG05] for [GG06]. Some of these developments were announced in Golyshev’s report [Go08] on spectra and strains, including various definitions of spectra, and a claim of spectral stability. We note that the case $r_Y = 1$ will further require a shift of $\text{Spec}(\widehat{h}_Y)$ by a constant.

Remark 1.12. In [GI], Galkin and Iritani studied the relationship between $QH^*(X)$ and the Euler-twisted quantum cohomology $QH_{\text{tw}}^*(X, L)$ (twisted by a nef line bundle L), and showed a ring homomorphism $QH_{\text{tw}}^*(X, L) \rightarrow QH^*(Y)$. As a consequence, they can obtain the current Proposition 1.10. In specific cases, Proposition 1.10 can be verified more directly. Indeed, the cases of Fano complete intersections in projective spaces, follow immediately from [Giv96, Corollaries 9.3 and 10.9] and [Ke24, Lemmas 3.2 and 4.2]. In Theorem 3.19, we also provide a direct verification for the cases of hyperplane sections in A_6/P_3 or A_7/P_3 .

We may further compare $\mathcal{A}_\perp^0(X)$ and $\mathcal{A}_\perp^0(Y)$ as follows.

Conjecture 1.13. *With the same notation in Proposition 1.10, there is a natural injective morphism of algebras $\mathcal{A}_\perp^0(X) \rightarrow \mathcal{A}_\perp^0(Y)$ that intertwines operators $\widehat{h}_X^{r_X}$ and $\widehat{h}_Y^{r_Y}$.*

Conjecture 1.13 tells us that the semisimplicity of $QH^*(X)$ could be related to the semisimplicity of the subalgebra of $QH^*(Y)$ generated by $\mathcal{A}_\perp^0(Y)$ and $c_1(Y)$. This could be useful when $\text{Rad}(Y)$ is of small dimension. Indeed, we were guided from this philosophy when investigating $QH^*(Y)$ for Y in A_6/P_3 or A_7/P_3 . We succeeded to verify all the expected properties, providing evidences for Conjecture 1.13 and achieving the semisimplicity of $QH^*(Y)$.

This paper is organized as follows. In Section 2, we provide necessary conditions for the semisimplicity of small quantum cohomology. In Section 3, we completely characterize the semisimplicity of $QH^*(Y)$ for smooth hyperplane sections Y in A_{n-1}/P_k . Finally in Section 4, we show the non-semisimplicity of $QH^*(Y)$ for Y in C_n/P_2 or D_n/P_2 .

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2. CRITERIONS FOR THE SEMISIMPLICITY

2.1. Semisimple commutative algebra. Let $\mathcal{A}$ be a finite-dimensional commutative (unital) $\mathbb{C}$ -algebra. The algebra $\mathcal{A}$ is called semisimple if for any $\alpha \in \mathcal{A}$, the induced linear operator $\hat{\alpha} : \mathcal{A} \rightarrow \mathcal{A}; \beta \mapsto \alpha \cdot \beta$ is semisimple; or equivalently, every nonzero α is not nilpotent. Suppose that $\mathcal{A} = \bigoplus_{\chi \in M} \mathcal{A}^\chi$ is equipped an M -graded algebra structure, where M is an abelian group.

Lemma 2.1. *Suppose that there exists $(d, \chi) \in \mathbb{Z}_{>0} \times M$ such that $\dim \mathcal{A}^\chi > \dim \mathcal{A}^{d\chi}$. Then there exists $\varepsilon \in \mathcal{A} \setminus \{0\}$ such that $\varepsilon^d = 0$.*

Proof. Take a basis $\{e_i\}_{i=1}^n$ (resp. $\{\hat{e}_j\}_{j=1}^m$) of $\mathcal{A}^\chi$ (resp. $\mathcal{A}^{d\chi}$). Then $0 = \varepsilon^d = (\sum_{i=1}^n a_i e_i)^d = \sum_{j=1}^m f_j(\mathbf{a}) \hat{e}_j$ holds if and only if $\mathbf{a} \in \mathbb{C}^n$ is a common root of the m homogeneous polynomials $f_j(y_1, \dots, y_n)$ of degree $d > 0$. Since $n > m$, there exists a nonzero common root. $\square$

Lemma 2.2. *Take any $m > 0$ and any $\alpha \in \mathcal{A}$ with $\hat{\alpha}$ invertible. Then $\mathcal{A}$ is semisimple if and only if $\mathcal{A}[y]/(y^m - \alpha)$ is semisimple.*

Proof. If $\mathcal{A}$ is semisimple, we have the decomposition $\mathcal{A} = \bigoplus_i A_i$ into one-dimensional subalgebras A_i . The map $\alpha \mapsto \alpha_i$ by taking the A_i -component of α induces a ring homomorphism $\mathcal{A}[y] \rightarrow A_i[y]$, and it further induces a ring isomorphism $\mathcal{A}[y]/(y^m - \alpha) \cong \bigoplus_i A_i[y]/(y^m - \alpha_i)$. As each A_i is isomorphic to the complex field, the algebra $A_i[y]/(y^m - \alpha_i)$ is semisimple if and only if every root of $y^m - \alpha_i$ is of multiplicity one. Note that α is invertible if and only if every component α_i is invertible. Hence, $y^m - \alpha_i$ does not have multiple roots.

Conversely, the subalgebra $\mathcal{A}$ of the semisimple algebra $\mathcal{A}[y]/(y^m - \alpha)$ is semisimple. $\square$

2.2. Quantum cohomology. We refer to [CoKa] for more details of Gromov–Witten theory. Throughout the paper, we assume X to be a Fano manifold with even cohomology only. Let $\overline{\mathcal{M}}_{0,m}(X, \mathbf{d})$ denote the moduli space of stable maps to X of degree $\mathbf{d} \in H_2(X, \mathbb{Z})$, and $\text{ev}_i : \overline{\mathcal{M}}_{0,m}(X, \mathbf{d}) \rightarrow X$ denote the i th evaluation map. For $\gamma_1, \dots, \gamma_m \in H^*(X)$, we consider the genus-zero, m -point Gromov–Witten invariant defined by

$$\langle \gamma_1, \dots, \gamma_m \rangle_{\mathbf{d}}^X := \int_{[\overline{\mathcal{M}}_{0,m}(X, \mathbf{d})]^{\text{vir}}} \text{ev}_1^*(\gamma_1) \cup \dots \cup \text{ev}_m^*(\gamma_m). \quad (1)$$

Here the virtual fundamental class $[\overline{\mathcal{M}}_{0,m}(X, \mathbf{d})]^{\text{vir}} \in H_{2\text{expdim}}(\overline{\mathcal{M}}_{0,m}(X, \mathbf{d}), \mathbb{Q})$ can be defined from some variety of dimension

$$\text{expdim} = \dim X + \int_{\mathbf{d}} c_1(X) + m - 3. \quad (2)$$

Take a basis $\{[C_1], \dots, [C_b]\}$ of the Mori cone $\overline{\text{NE}}(X)$ of effective curve classes. Each generator $[C_i]$ associates with an indeterminate q_i . For $\mathbf{d} = \sum_j d_j [C_j]$, we denote $\mathbf{q}^{\mathbf{d}} := \prod_j q_j^{d_j}$. The (small) quantum cohomology $QH^*(X) = (H^*(X) \otimes \mathbb{C}[q_1, \dots, q_b], \star)$ is an associative commutative algebra with unit $1 \in H^0(X) \otimes \mathbb{C}$, with the quantum product defined by

$$\alpha_i \star \alpha_j := \sum_i \sum_{\mathbf{d} \in \overline{\text{NE}}(X)} \langle \alpha_i, \alpha_j, \alpha_k^\vee \rangle_{\mathbf{d}}^X \alpha_k \mathbf{q}^{\mathbf{d}};$$

Here $\{\alpha_i\}_i$ denotes a basis of $H^*(X)$, and $\{\alpha_i^\vee\}_i$ denotes its dual basis with respect to Poincaré pairing: $(\alpha_i, \alpha_j^\vee)_X = \int_{[X]} \alpha_i \cup \alpha_j^\vee = \delta_{i,j}$. The quantum cohomology $QH^*(X)$ is naturally a $\mathbb{Z}$ -graded algebra with respect to the grading

$$\deg q_i := \int_{[C_i]} c_1(X), \quad \deg \alpha := j, \quad \forall \alpha \in H^{2j}(X) \setminus \{0\}.$$

Proof of Theorem 1.2. Notice that r equals the greatest common divisor of $\deg q_1, \dots, \deg q_b$. Thus for any specialization $\mathbf{q} = \eta \in \mathbb{C}^b$, the $\mathbb{Z}$ -graded algebra $QH^*(X)$ naturally induces an $\mathbb{Z}_r$ -graded algebra $(H^*(X) = QH^*(X)|_{\mathbf{q}=\eta} = \bigoplus_{\bar{i} \in \mathbb{Z}_r} \mathcal{A}^{\bar{i}}, \star_\eta)$, where $\mathcal{A}^{\bar{i}} = \bigoplus_{j \equiv \bar{i}} H^{2j}(X)$.

If $QH^*(X)$ is semisimple at some $\mathbf{q} = \eta$, then $(H^*(X), \star_\eta)$ has no nonzero nilpotent element, and hence statement (1) follows from Lemma 2.1.

Consequently any i , we have $\dim \mathcal{A}^{\bar{i}} \leq \dim \mathcal{A}^{(r-1)\bar{i}} \leq \dim \mathcal{A}^{-\bar{i}}$. Since i is arbitrary, we then have $\dim \mathcal{A}^{\bar{i}} = \dim \mathcal{A}^{-\bar{i}}$. That is, statement (2) holds. $\square$

2.3. First applications. Given a Lie type $\mathcal{D}_n$ and an integer $1 \leq k \leq n$, we denote by $\mathcal{D}_n/P_k$ the quotient of a simply-connected, complex simple Lie group G of Lie type $\mathcal{D}_n$ by the maximal parabolic subgroup P_k of G that corresponds to the subset $\Delta \setminus \{\alpha_k\}$. Here $\Delta = \{\alpha_1, \dots, \alpha_n\}$ is a base of simple roots for G with the same ordering as in [Bel], and $\mathcal{D}_n/P_k$ is called a generalized Grassmannian (of type $\mathcal{D}_n$). In particular, the Grassmannian of type A_{n-1} ,

$$A_{n-1}/P_k = \{V \leq \mathbb{C}^n \mid \dim V = k\} =: Gr(k, n),$$

is known as a complex Grassmannian. There are in total 27 Grassmannians of exceptional Lie type: 9 of which are known to have semisimple quantum cohomology, 14 are known to have non-semisimple quantum cohomology, and the other 4 cases are unknown [Bel]. Here we provide a proof for 13 non-semisimple cases by Theorem 1.2. Unfortunately, the remaining non-semisimple case E_8/P_4 cannot be checked by Theorem 1.2, neither it gives any obstruction for the 4 unknown cases E_7/P_2 , E_7/P_4 , E_7/P_5 , E_8/P_6 .

Theorem 2.3. *For any $X \in \{E_6/P_2, E_6/P_4, E_7/P_1, E_7/P_3, E_7/P_6, E_8/P_1, E_8/P_2, E_8/P_3, E_8/P_5, E_8/P_7, E_8/P_8, F_4/P_3, F_4/P_4\}$, the quantum cohomology $QH^*(X)$ is not semisimple.*

Proof. Simply denote $r = r_X$ and $b_{2j} = b_{2j}(X)$. Then we can read off the data of $\dim X, r, b_{2j}$ of each X from [Bel] directly, where $b_{2j} = 0$ if $j > \dim X$. Then we can calculate $\dim \mathcal{A}^{\pm \bar{i}}$ for F_4/P_4 and $\dim \mathcal{A}^{\pm \bar{i}}$ for the other 12 cases. For instance for E_7/P_6 , we have $\dim \mathcal{A}^{\bar{1}} = b_2 + b_{2 \cdot 14} + b_{2 \cdot 27} + b_{2 \cdot 40} = 1 + 26 + 29 + 2 = 58$, and $\dim \mathcal{A}^{-\bar{1}} = b_{2 \cdot 12} + b_{2 \cdot 25} + b_{2 \cdot 38} + b_{2 \cdot 51} = 21 + 34 + 4 + 0 = 59$.

X	$\dim X$	r	i	m	$\dim \mathcal{A}^{\bar{i}} = \sum_{j=0}^m b_{2(i+jr)}$	$\dim \mathcal{A}^{-\bar{i}} = \sum_{j=0}^m b_{2(m-i+jr)}$
E_6/P_2	21	11	1	1	6	7
E_6/P_4	29	7		4	102	104
E_7/P_1	33	17		1	7	8
E_7/P_3	47	11		4	183	184
E_7/P_6	42	13		3	58	59
E_8/P_1	78	23		3	94	95
E_8/P_2	92	17		5	1016	1017
E_8/P_3	98	13		7	5317	5318
E_8/P_5	104	11		10	21993	21992
E_8/P_7	83	19		4	354	355
E_8/P_8	57	29		1	8	9
F_4/P_3	20	7		2	13	14
F_4/P_4	15	11	4	2	3	2

As from the table, none of the 13 cases satisfies $\dim \mathcal{A}^{\bar{i}} = \dim \mathcal{A}^{-\bar{i}}$ for the given i . Hence, none of them has semisimple quantum cohomology by Theorem 1.2 (2). $\square$

3. SMOOTH HYPERPLANE SECTIONS OF $Gr(k, n)$

In this section, we let $X = Gr(k, n) = \{V \leq \mathbb{C}^n \mid \dim V = k\}$, which is a closed subvariety in $\mathbb{P}^{\binom{n}{k}-1}$ via the Plücker embedding. The intersection of X with a general hyperplane of $\mathbb{P}^{\binom{n}{k}-1}$ gives a smooth hyperplane section Y of the complex Grassmannian X . We will investigate the semisimplicity of $QH^*(Y)$. Since $Gr(k, n) \cong Gr(n-k, n)$, we can always assume $n \geq 2k$.

3.1. Characterization of $H^*(Y)$ of Hodge–Tate type. Due to Proposition 1.1, we start with the study of the Hodge diamond of Y , which has its own interest in classical algebraic geometry. One key ingredient is the combinatorial characterization of nonvanishing cohomology by Snow [Sn86] from the parabolic Borel–Weil Theorem by Bott [Bo57].

Denote by $\mathcal{P}_{k,n} := \{\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{Z}^k \mid n-k \geq \lambda_1 \geq \dots \geq \lambda_k \geq 0\}$. Denote $|\lambda| := \sum_i \lambda_i$. The hook length of a cell in the Young diagram of a partition λ is defined to be the total number of cells which are either directly to the right or directly below the cell together with the cell.

Proposition 3.1 ([Sn86, Section 3.1 (2)]). *For $\ell \geq 0$, we have*

$$H^j(Gr(k, n), \Omega^p(\ell)) \neq 0$$

if and only if there exists $\lambda \in \mathcal{P}_{k,n}$ of p cells with no cell of hook length ℓ such that j equals the number of cells in λ of hook length larger than ℓ .

Remark 3.2. Recall a smooth projective variety X is said to satisfy Bott vanishing if

$$H^j(X, \Omega_X^p \otimes L) = 0 \quad j > 0, p \geq 0$$

for any ample line bundle L . However, Bott vanishing fails for complex Grassmannians other than projective spaces, see [BTLM97, Section 4.3]. We refer to [Be25, Fo25] for the failure of Bott vanishing for certain generalized Grassmannians of general Lie type.

We call a partition λ a j -core partition, if j does not appear as the hook length of cells in λ . We first assume Lemma 3.3, which will be proved in Section 3.4 in purely combinatorial way.

Lemma 3.3. *For $3 \leq k \leq n/2$, there exists $(\lambda, i) \in \mathcal{P}_{k,n} \times [1, n-1]$ such that $|\lambda| \geq k(n-k) - i$ and λ is an $(n-i)$ -core partition if and only if $(k, n) \in \{(3, 6), (4, 8), (3, 9)\}$.*

Example 3.4. Below are partitions $(3, 2, 1)$, $(4, 3, 2, 1)$, $(6, 4, 2)$, $(4, 4, 2, 2)$, whose cells are put their hook length. For example, the partition $(6, 4, 2)$ has 12 cells with no cell of hook length 3, and has 6 cells of hook length larger than 3. Thus $H^6(Gr(3, 9), \Omega^{12}(3)) \neq 0$ by Proposition 3.1

λ	<table> <tr><td>5</td><td>3</td><td>1</td></tr> <tr><td>3</td><td>1</td><td></td></tr> <tr><td>1</td><td></td><td></td></tr> </table>	5	3	1	3	1		1			<table> <tr><td>7</td><td>5</td><td>3</td><td>1</td></tr> <tr><td>5</td><td>3</td><td>1</td><td></td></tr> <tr><td>3</td><td>1</td><td></td><td></td></tr> <tr><td>1</td><td></td><td></td><td></td></tr> </table>	7	5	3	1	5	3	1		3	1			1				<table> <tr><td>8</td><td>7</td><td>5</td><td>4</td><td>2</td><td>1</td></tr> <tr><td>5</td><td>4</td><td>2</td><td>1</td><td></td><td></td></tr> <tr><td>2</td><td>1</td><td></td><td></td><td></td><td></td></tr> </table>	8	7	5	4	2	1	5	4	2	1			2	1					<table> <tr><td>7</td><td>6</td><td>3</td><td>2</td></tr> <tr><td>6</td><td>5</td><td>2</td><td>1</td></tr> <tr><td>3</td><td>2</td><td></td><td></td></tr> <tr><td>2</td><td>1</td><td></td><td></td></tr> </table>	7	6	3	2	6	5	2	1	3	2			2	1		
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(k, n, i)	$(3, 6, 4)$	$(4, 8, 6)$	$(3, 9, 6)$	$(4, 8, 4)$																																																											

As we will see from the proof of Lemma 3.3, they are the only $(n - i)$ -core partitions.

Lemma 3.5. *For $n \geq 2k$, $k(n - k) > 2n$ holds if and only if one of the following holds:*

- (i) $k \geq 5$; (ii) $k = 4$ and $n \geq 9$; (iii) $k = 3$ and $n \geq 10$.

Proof. For $k \geq 5$, $k(n - k) - 2n = (k - 2)n - k^2 \geq (k - 2)2k - k^2 = k^2 - 4k > 0$. For $k \leq 4$, by direct calculation, $k(n - k) > 2n$ if and only if $(k = 3, n \geq 10)$ or $(k = 4, n \geq 9)$ holds. $\square$

Lemma 3.6. *Assume $\dim X > 2n$. Then for any $0 \leq p \leq n$, we have*

$$H^s(X, \Omega_X^{\dim X - j}(p - j)) = 0 \quad \text{for all } s \text{ and } 1 \leq j < p.$$

Proof. Assume $H^s(X, \Omega_X^{\dim X - j}(p - j)) \neq 0$ for some (s, j, p) . Then by Proposition 3.1, there exists $\lambda \in \mathcal{P}_{k, n}$ with $|\lambda| = k(n - k) - j$, having no cell of hook length $p - j$ and having s cells of hook length larger than $p - j$. Since $1 \leq p - j < n$, then for $i := n - p + j \in [j, n - 1]$, $|\lambda| \geq k(n - k) - i$ and λ is an $(n - i)$ -corn partition. Thus by Lemma 3.3, $(k, n) \in \{(3, 6), (3, 9), (4, 8)\}$. This contradicts to the assumption $\dim X = k(n - k) > 2n$. $\square$

We have the following proposition, whose proof is a refined version of Griffiths' theory ([Gr69], see also [Voi, Section 6.1.2]) with more precise control of cohomology vanishing. A similar argument appears in [DV10, Theorem 2.2].

Proposition 3.7. *Assume $k(n - k) > 2n$. Then*

$$h^{k(n-k)-p, p-1}(Y) = \begin{cases} 0, & p < n, \\ 1, & p = n. \end{cases}$$

In particular, $H^(Y)$ is not of Hodge-Tate type.*

Proof. Let $U = X \setminus Y$ be the complement of Y . Let $\Omega_X^p(\log Y)$ be the sheaf of logarithmic p -forms, i.e. meromorphic differential p -forms ϕ such that ϕ and $d\phi$ both have a pole of order at most 1 along Y . By the proof of [Voi, Theorem 6.5], the Hodge filtration of $H^{\dim X}(U, \mathbb{C})$ is given by

$$F^{\dim X - p + 1} H^{\dim X}(U, \mathbb{C}) = H^{p-1}(X, \Omega_X^{\dim X - p + 1}(\log Y)_{\text{closed}}).$$

Note that by definition, $\Omega_X^{\dim X - p + 1}(\log Y)_{\text{closed}} = \Omega_X^{\dim X - p + 1}(Y)_{\text{closed}}$. We have the following exact sequence for $p < \dim X$

$$0 \rightarrow \Omega_X^{\dim X - p + 1}(Y)_{\text{closed}} \xrightarrow{\subseteq} \Omega_X^{\dim X - p + 1}(Y) \xrightarrow{d} \Omega_X^{\dim X - p + 2}(2Y) \xrightarrow{d} \cdots \xrightarrow{d} K_X(pY) \rightarrow 0.$$

By the vanishing in Lemma 3.6, we can shift the degree, and hence for $1 \leq p \leq n$,

$$F^{\dim X - p + 1} H^{\dim X}(U) \cong H^0(X, K_X(pY)).$$

Then by the assumption $\dim X > 2n$, we have

$$F^{\dim X - p} H^{\dim X - 1}(F) \cong H^0(X, K_X(pY)).$$

It has dimension 0 when $p < n$ and 1 when $p = n$. This gives the Hodge number. $\square$

Theorem 3.8. *Let Y be a smooth hyperplane section of $Gr(k, n)$ where $n \geq 2k$. Then $H^*(Y)$ is of Hodge–Tate type if and only if either (i) $k \in \{1, 2\}$ or (ii) $k = 3$ and $n \in \{6, 7, 8\}$ holds.*

Proof. By Proposition 3.7, it remains to check the case when $k(n - k) \leq 2n$. For $k = 1, 2$, the hyperplane sections are known to be Hodge–Tate. Thus by Lemma 3.5, it remains to check the cases when (k, n) belongs to $\{(3, 6), (3, 7), (3, 8), (3, 9), (4, 8)\}$.

To compute the Hodge numbers, we consider the λ -class $\lambda_y(Y) = \sum_{p \geq 0} y^p [\Omega_Y^p] \in K(Y)[y]$ in the polynomial ring of y with coefficients in the K -theory of Y [Hir]. We have

$$(1 + y\mathcal{O}(-Y)) \cdot \lambda_y(Y) = \lambda_y(X)|_Y.$$

So

$$\chi_y(Y) := \sum_{p \geq 0} y^p \chi(Y, \Omega_Y^p) = \chi(Y, \lambda_y(Y)) = \chi\left(X, \lambda_y(X) \frac{1 - \mathcal{O}(-Y)}{1 + y\mathcal{O}(-Y)}\right) \in \mathbb{Z}[y].$$

This can be obtained by using Bott–Lefschetz localization formula and taking non-equivariant limit. For $X = Gr(3, 9)$, we have

$$\begin{aligned} \chi_y(Y) = & -y^{17} + y^{16} - 2y^{15} + 3y^{14} - 4y^{13} + 5y^{12} - 7y^{11} + 7y^{10} - 6y^9 \\ & + 6y^8 - 7y^7 + 7y^6 - 5y^5 + 4y^4 - 3y^3 + 2y^2 - y + 1. \end{aligned}$$

That is,

p	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\chi(Y, \Omega_Y^p)$	1	-1	2	-3	4	-5	7	-7	6	-6	7	-7	5	-4	3	-2	1	-1

By hard Lefschetz theorem, we have

- (1) $h^{i,j}(Y) = 0$ unless $i = j$ or $i + j = \dim Y = 17$;
- (2) when $i < \dim Y = 17$, $h^{i,j}(Y) = h^{i,j}(X)$.

Comparing them with the Poincaré polynomial of $Gr(3, 9)$, we conclude when $i + j = 17$,

$$h^{8,9} = h^{9,8} = 2, \quad h^{i,j} = 0 \text{ otherwise.}$$

In particular, $H^*(Y)$ is not of Hodge–Tate type. By the same methods, we can work out the other cases. The following list their Hodge diamonds.

$Y \subset Gr(3,6)$	$Y \subset Gr(3,7)$	$Y \subset Gr(3,8)$	$Y \subset Gr(3,9)$	$Y \subset Gr(4,8)$
$\begin{array}{c} 1 \\ 1 \\ 2 \\ 3 \\ 4 \\ 3 \\ 2 \\ 1 \\ 1 \end{array}$	$\begin{array}{c} 1 \\ 1 \\ 2 \\ 3 \\ 4 \\ 4 \\ 4 \\ 3 \\ 2 \\ 1 \\ 1 \end{array}$	$\begin{array}{c} 1 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 6 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \\ 1 \end{array}$	$\begin{array}{c} 1 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 7 \\ 7 \\ 8 \\ 8 \\ 28 \\ 7 \\ 7 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \\ 1 \end{array}$	$\begin{array}{c} 1 \\ 1 \\ 2 \\ 3 \\ 5 \\ 5 \\ 7 \\ 7 \\ 8 \\ 7 \\ 34 \\ 7 \\ 5 \\ 5 \\ 3 \\ 2 \\ 1 \\ 1 \end{array}$

Hence, for $k \geq 3$, $H^*(Y)$ is of Hodge–Tate type if and only if $k = 3$ and $n \in \{6, 7, 8\}$. $\square$

Remark 3.9. The method in the proof is well-known and works for any $Gr(k, n)$ in principle, while the computation of $\chi_y(Y)$ is not efficient in practice. The polynomial $\chi_y(Y)$ can be efficiently computed via the Pieri rule of motivic Chern classes of Schubert cells over Grassmannian [FGSX24]. The readers can try it online: <https://cubicbear.github.io/PluckerHodge.html>.

3.2. Quantum Pieri rule for Hodge–Tate hyperplane sections.

3.2.1. *Quantum Pieri rule for X .* Here we review some facts on $QH^*(X)$ (see e.g. [Bu03]).

For $\lambda \in \mathcal{P}_{k,n}$, the Schubert subvariety $X_\lambda \subset X$ of codimension $|\lambda|$, associated to a fixed complete flag $E_\bullet$ of $\mathbb{C}^n$, is defined by $X_\lambda = \{V \in X \mid \dim V \cap E_{n-k+i-\lambda_i} \geq i, \quad i = 1, \dots, k\}$. The Schubert classes $\sigma_\lambda := P.D.([X_\lambda]) \in H^{2|\lambda|}(X, \mathbb{Z})$ form an additive basis of $H^*(X, \mathbb{Z})$. The dual basis is given by $\{(\sigma_\lambda)^\vee = \sigma_{\lambda^\vee}\}_\lambda$, where $\lambda^\vee = (n-k-\lambda_k, \dots, n-k-\lambda_1)$ is the dual partition. We simply denote the special partitions $p = (p, 0, \dots, 0)$ and $1^p = (1, \dots, 1, 0, \dots, 0)$ where there are p copies of 1. There is an exact sequence

$$0 \rightarrow \mathcal{S} \rightarrow \mathbb{C}^n \rightarrow \mathcal{Q} \rightarrow 0$$

of tautological vector bundles over X . The fiber of the tautological subbundle $\mathcal{S}$ at a point $V \in X$ is given by the vector space V . The Schubert classes σ_p (resp. σ_{1^p}) coincide with the p -th Chern classes $c_p(\mathcal{Q})$ (resp. $(-1)^p c_p(\mathcal{S})$). Hence, they are related by $c(\mathcal{S}) \cup c(\mathcal{Q}) = 1$, i.e.

$$\sum_{i=0}^m (-1)^i \sigma_{1^i} \sigma_{m-i} = 0 \quad (3)$$

for all $m \geq 1$. Here we take the convention $\sigma_\lambda = 0$, whenever $\lambda \notin \mathcal{P}_{k,n}$. There is a canonical ring isomorphism, where σ_a 's are polynomials in σ_{1^b} 's read off from (3),

$$QH^*(X) \cong \mathbb{C}[\sigma_1, \dots, \sigma_{1^k}, q_X] / (\sigma_{n-k+1}, \dots, \sigma_{n-1}, \sigma_n + (-1)^k q_X). \quad (4)$$

The quantum multiplications by σ_p 's (or σ_{1^p} 's) are known as the quantum Pieri rule, and were first provided by Bertram. We refer to [BCFF99, Proposition 4.2] for the following form.

Proposition 3.10 (Quantum Pieri rule). *Let $1 \leq p \leq k$ and $\lambda \in \mathcal{P}_{k,n}$. In $QH^*(X)$, we have*

$$\sigma_{1^p} \star \sigma_\lambda = \sum_{\mu} \sigma_\mu + q_X \sum_{\nu} \sigma_\nu,$$

the first sum over μ obtained by adding p cells to λ with no two in the same row, and the second sum over ν with $|\nu| = |\lambda| + p - n$ such that $\tilde{\lambda}_1 - 1 \geq \tilde{\nu}_1 \geq \tilde{\lambda}_2 - 1 \geq \tilde{\nu}_2 \geq \dots \geq \tilde{\lambda}_k - 1 \geq \tilde{\nu}_k \geq 0$, which occurs only if $\lambda_1 = n - k$ and where $\tilde{\lambda} \in \mathcal{P}_{n-k,n}$ denotes the transpose of λ .

3.2.2. *Quantum Pieri rule for Y .* Denote by $j : Y \hookrightarrow X$ the natural inclusion. It induces an algebra homomorphism $j^* : H^*(X) \rightarrow H^*(Y)$, with $j^*|_{H^{2i}(X)}$ an isomorphism of vector spaces for $0 \leq i < \dim Y$. Taking a line $\mathbb{P}^1 \subset Y$, we have $H_2(Y, \mathbb{Z}) = \mathbb{Z}[\mathbb{P}^1]$ and $H_2(X, \mathbb{Z}) = \mathbb{Z}j_*[\mathbb{P}^1]$, and simply denote by $d \in \mathbb{Z}$ a curve class under this identification.

Proposition 3.11 ([BP22, Proposition 5.13 and Theorem 5.11 (1)]). *Assume $n \geq 2k > 3$.*

- (1) *For $d \in \{1, 2\}$, $\overline{\mathcal{M}}_{0,n}(Y, d)$ is irreducible of expected dimension.*
- (2) *For any $\alpha, \alpha' \in H^*(Y)$ and $\gamma \in H^{2i}(X)$ with $i < n - 1$, we have*

$$\langle \alpha, \alpha', j^* \gamma \rangle_1^Y = \langle j_* \alpha, j_* \alpha', \gamma \rangle_1^X.$$

Remark 3.12. Part (1) with $d = 1$ is a consequence of the result in [LM03] as explained [BP22, Corollary 5.8]. Then part (2) is obtained by a canonical argument with projection formula as in [BP22, Lemma 5.12]. Part (1) with $d = 2$ was proved in [BP22, Theorem 5.11 (1)] for adjoint or quasi-minuscule Grassmannians (excluding type G_2), and also holds in our situation by exactly the same arguments therein.

Extend the morphism $j^* : H^*(X) \rightarrow H^*(Y)$ to a $\mathbb{C}$ -linear map $j^* : QH^*(X) \rightarrow QH^*(Y)$ by defining $j^*(q_X^m) = q_Y^m$, which is distinct from the conjectural lifting j_q^* .

Proposition 3.13. *Let $\lambda \in \mathcal{P}_{k,n}$, $1 \leq p \leq k$ and $\beta \in H_{\text{prim}}(Y)$. In $QH^*(Y)$, we have*

$$\begin{aligned} j^* \sigma_{1^p} \star j^* \sigma_\lambda &\equiv j^*(\sigma_{1^p} \cup \sigma_\lambda) + j^*(\sigma_{1^p} \star (\sigma_\lambda \cup \sigma_1) - \sigma_{1^p} \cup (\sigma_\lambda \cup \sigma_1)) \mod q_Y^2, \\ j^* \sigma_{1^p} \star \beta &\equiv 0 \mod q_Y^2. \end{aligned}$$

Proof. Denote $m := p + |\lambda| - (n - 1)$. Take an ordering $\{\sigma_{\mu(1)}, \dots, \sigma_{\mu(b)}\}$ of the Schubert basis of $H^{2m}(X)$ such that $H^{2m}(Y)$ has a basis $\{j^* \sigma_{\mu(i)}\}_{1 \leq i \leq a} \cup \{\beta_i\}_{1 \leq i \leq c}$ for some $1 \leq a \leq b$ and primitive classes β_i 's which appear only if $\dim Y = 2m$. Write

$$j^* \sigma_{1^p} \star j^* \sigma_\lambda \equiv j^* \sigma_{1^p} \cup j^* \sigma_\lambda + q_Y \sum_{i=1}^a m_i j^* \sigma_{\mu(i)} + q_Y \sum_{i=1}^c \tilde{m}_i \beta_i \mod q_Y^2.$$

Then for any $1 \leq i \leq c$, $\tilde{m}_i = \langle j^* \sigma_{1^p}, j^* \sigma_\lambda, \beta_i^\vee \rangle_1^Y = \langle \sigma_{1^p}, j_*(j^* \sigma_\lambda), j_*(\beta_i^\vee) \rangle_1^X = 0$ by using Proposition 3.11 and noting $j_*(\beta_i^\vee) = 0$ (since $\beta_i^\vee$ is again a primitive class). For $1 \leq i \leq a$,

$$m_i = \langle j^* \sigma_{1^p}, j^* \sigma_\lambda, (j^* \sigma_{\mu(i)})^\vee \rangle_1^Y = \langle \sigma_{1^p}, j_*(j^* \sigma_\lambda), j_*((j^* \sigma_{\mu(i)})^\vee) \rangle_1^X.$$

Since Y is a hyperplane section of X , by projection formula we have $j_*(j^* \sigma_\lambda) = j_*(j^* \sigma_\lambda \cup 1) = \sigma_\lambda \cup j_* 1 = \sigma_\lambda \cup \sigma_1$. For any $\nu \in \mathcal{P}_{k,n} \setminus \{\mu^{(s)}\}_{a < s \leq b}$, we have

$$\int_{[X]} j_*((j^* \sigma_{\mu(i)})^\vee) \cup \sigma_\nu = \int_{[Y]} (j^* \sigma_{\mu(i)})^\vee \cup j^* \sigma_\nu = \delta_{\mu(i), \nu}$$

For $a < s \leq b$, writing $j^* \sigma_{\mu(s)} = \sum_{t=1}^a c_{st} j^* \sigma_{\mu(t)}$, we have $\int_{[X]} j_*((j^* \sigma_{\mu(i)})^\vee) \cup \sigma_{\mu(s)} = \int_{[Y]} (j^* \sigma_{\mu(i)})^\vee \cup (\sum_t c_{st} j^* \sigma_{\mu(t)}) = c_{si}$. It follows that $j_*((j^* \sigma_{\mu(i)})^\vee) = \sigma_{\mu(i)}^\vee + \sum_s c_{si} \sigma_{\mu(s)}^\vee$. Hence,

$$\begin{aligned} q_Y \sum_{i=1}^a m_i j^* \sigma_{\mu(i)} &= q_Y \sum_{i=1}^a (\langle \sigma_{1^p}, \sigma_\lambda \cup \sigma_1, \sigma_{\mu(i)}^\vee \rangle_1^X j^* \sigma_{\mu(i)} + \sum_{s=a+1}^b \langle \sigma_{1^p}, \sigma_\lambda \cup \sigma_1, \sigma_{\mu(s)}^\vee \rangle_1^X c_{si} j^* \sigma_{\mu(i)}) \\ &= q_Y \sum_{i=1}^b (\langle \sigma_{1^p}, \sigma_\lambda \cup \sigma_1, (\sigma_{\mu(i)}^\vee)^\vee \rangle_1^X j^* \sigma_{\mu(i)}^\vee) \\ &= j^*(\sigma_{1^p} \star (\sigma_\lambda \cup \sigma_1) - \sigma_{1^p} \cup (\sigma_\lambda \cup \sigma_1)). \end{aligned}$$

Here the last equality follows from the quantum Pieri rule for X , which shows that the quantum part of $\sigma_{1^p} \star (\sigma_\lambda \cup \sigma_1)$ consists of the degree-one part of the quantum product. $\square$

The *span* $\text{Span}(Z)$ of a subvariety $Z \subset X$ is the smallest vector subspace of $\mathbb{C}^n$ that contains all the k -dimensional spaces given by points of Z . It has been used to study $QH^*(X)$ in [Bu03].

Proposition 3.14. *Let $k \geq 3$ and Z be a closed subvariety of Y of dimension m . If $n > 2k + m$, then for any $\alpha \in H^*(Y)$, we have*

$$\langle P.D.[\text{pt}], P.D.[Z], \alpha \rangle_2^Y = 0.$$

Proof. It suffices to show the case when α is represented by a closed subvariety of dimension $\tilde{m} = 2 \dim Y - m - 2(n - 1)$. Assume $\langle [\text{pt}], [Z], \alpha \rangle_d^Y \neq 0$, then for any point $P \in Y$ and any closed subvariety Z' of dimension $\tilde{m}$, there exists a conic C passing through P , Z and Z' by Proposition 3.11 (1). Say $\hat{P} \in C \cap Z$, then we have $\dim \text{Span}(P) = \dim(\hat{P}) = k$. Hence, $\text{Span}(P) \cap \text{Span}(\hat{P}) \neq 0$, by noting that they are both vector subspaces of $\text{Span}(C)$ while $\dim \text{Span}(C) \leq k + 2$ by [Bu03, Lemma 1].

Consider the configuration space

$$B_Z := \{(V_1, V_k, \bar{V}_k) \mid V_1 \leq \bar{V}_k, V_1 \leq V_k \in Z, \dim V_1 = 1, \dim V_k = \dim \bar{V}_k = k\}$$

as well as the natural projections π_i by sending $(V_1, V_k, \bar{V}_k)$ to the i th vector space. Note that B_Z is a closed variety, which is a fibration over Z via π_2 with fiber at V_k being a $Gr(k-1, n-1)$ -bundle over $\mathbb{P}(V_k)$. Thus $\dim B_Z = \dim Z + (k-1) + (k-1)(n-k)$. Hence,

$$\text{codim}_X \pi_3(B_Z) = \dim X - \pi_3(B_Z) \geq \dim X - \dim B_Z = n - 2k - m + 1.$$

Since Y is codimension 1 in X , $\text{codim}_Y(Y \cap \pi_3(B_Z)) \geq n - 2k - m > 0$ by the hypothesis. Therefore, there exists $\bar{V}_k \in Y \setminus \pi_3(B_Z)$. Then for any $V_k \in Z$ and any 1-dimensional vector subspace V_1 , either $V_1 \not\leq \bar{V}_k$ or $V_1 \leq V_k$ holds. That is, $\bar{V}_k \cap V_k = 0$, saying that the span of the point $\bar{V}_k$ in Y and the span of the point $V_k \in Z \subset Y$ has zero intersection and resulting in a contradiction. $\square$

Definition 3.15. We call Y a Hodge–Tate hyperplane section, if $H^*(Y)$ is of Hodge–Tate type.

Proposition 3.16 (Quantum Pieri rule for Y). *Let Y be a Hodge–Tate hyperplane section. Take any $1 \leq p \leq k$, $\lambda \in \mathcal{P}_{k,n}$ and $\beta \in H_{\text{prim}}(Y)$. In $QH^*(Y)$, we have*

$$j^* \sigma_{1^p} \star j^* \sigma_\lambda = j^*(\sigma_{1^p} \cup \sigma_\lambda) + j^*(\sigma_{1^p} \star (\sigma_\lambda \cup \sigma_1) - \sigma_{1^p} \cup (\sigma_\lambda \cup \sigma_1)); \quad j^* \sigma_{1^p} \star \beta = 0.$$

Proof. Denote $q := q_Y$. For $k \in \{1, 2\}$, we have $H_{\text{prim}}(Y) = 0$ and $p + |\lambda| \leq k + \dim Y = k + k(n-k) - 1 < 2(n-1) = 2 \deg q$. Thus there are no q^d -terms with $d \geq 2$ in $j^* \sigma_{1^p} \star j^* \sigma_\lambda$. By Theorem 3.8, it remains to consider the case when $k = 3$ and $n \in \{6, 7, 8\}$.

By degree counting again, there are no q^d -terms with $d \geq 3$ (resp. $d \geq 2$) in $j^* \sigma_{1^p} \star j^* \sigma_\lambda$ (resp. $j^* \sigma_{1^p} \star \beta$). Thus by Proposition 3.13, we have $j^* \sigma_{1^p} \star \beta = 0$.

For $p = 1$, we further conclude that there are no q^2 -terms in $j^* \sigma_1 \star j^* \sigma_\lambda$. Indeed, when $n = 6$, this holds by degree counting. When $n = 7$, a q^2 -term occurs only if $12 = 2 \deg q \leq 1 + |\lambda| \leq 1 + \dim Y = 12$, implying $\lambda = (4, 4, 3)$. The coefficient of $q^2 \cdot 1$ in $j^* \sigma_1 \star j^* \sigma_{(4,4,3)}$ equals $\langle j^* \sigma_1, P.D.[\text{pt}], P.D.[\text{pt}] \rangle_2^Y$, and hence equals 0 by Proposition 3.14. When $n = 8$, there are at most two possibilities by degree counting, being the coefficient of $q^2 \cdot 1$ (resp. $q^2 j^* \sigma_1$) in $j^* \sigma_1 \star j^* \sigma_{(5,5,3)}$ (resp. $j^* \sigma_1 \star j^* \sigma_{(5,5,4)}$). They both equal $\langle j^* \sigma_1, P.D.[\text{pt}], P.D.[\mathbb{P}^1] \rangle_2^Y$, and hence equal 0 by Proposition 3.14 again. Hence, the first equality in the statement holds for $p = 1$.

For $p \in \{2, 3\}$, it follows from the associativity of quantum products that q^2 -terms do not occur. For instance for $p = 2$ and $n = 7$, q^2 -terms occur at most in $j^* \sigma_{(1,1,0)} \star j^* \sigma_{(4,4,2)}$ and $j^* \sigma_{(1,1,0)} \star j^* \sigma_{(4,4,3)}$, by degree counting. By direct calculations, we have

$$\begin{aligned} j^* \sigma_1 \star j^* \sigma_{(4,4,1)} &= j^* \sigma_{(4,4,2)} + q j^* \sigma_{(3,1,0)}, \\ j^* \sigma_{(1,1,0)} \star j^* \sigma_{(4,4,1)} \star j^* \sigma_1 &= q(j^* \sigma_{(3,2,0)} + j^* \sigma_{(4,1,0)}) \star j^* \sigma_1 = q(j^* \sigma_{(3,2,0)} + j^* \sigma_{(4,1,0)}) \cup j^* \sigma_1 + q \cdot q, \\ q j^* \sigma_{(3,1,0)} \star j^* \sigma_{(1,1,0)} &= q j^* \sigma_{(3,1,0)} \cup j^* \sigma_{(1,1,0)} + q \cdot q. \end{aligned}$$

Hence, $j^* \sigma_{(1,1,0)} \star j^* \sigma_{(4,4,2)} = q(j^* \sigma_{(3,2,0)} + j^* \sigma_{(4,1,0)}) \cup j^* \sigma_1 - q j^* \sigma_{(3,1,0)} \cup j^* \sigma_{(1,1,0)}$, which does not contain q^2 -term. The arguments for the remaining a few cases are similar. $\square$

3.2.3. Applications. Denote by $p_{\alpha,V}(x)$ the characteristic polynomial of the induced operator $\hat{\alpha}$ on an algebra V containing α . Using the quantum Pieri rules, we have the following.

Lemma 3.17. *Let Y be a smooth hyperplane section of $X = Gr(3, n)$, where $n \in \{7, 8\}$.*

- (1) *For $n = 7$, $p_{\sigma_1^7, A^0(X)}(x) = p_{(j^* \sigma_1)^6, A^0(Y)}(x) = (128 - 13x + x^2)(1 - 57x - 289x^2 + x^3)$.*
 - (2) *For $n = 8$, we have $p_{\sigma_1^8, A^0(X)}(x) = (1 - x)^3(1 - 1154x + x^2)(6561 - 34x + x^2)$ and $p_{\sigma_1^6 \star \sigma_{1^2}, A^0(X)}(x) = (1 - x)(1 + 478x - x^2)(1 + x^2)(2187 + 6x + x^2)$. Moreover,*
- $$p_{(j^* \sigma_1)^7, A^0(Y)}(x) = x^2 p_{\sigma_1^8, A^0(X)}(x), \quad p_{(j^* \sigma_1)^5 \star j^* \sigma_{1^2}, A^0(Y)}(x) = x^2 p_{\sigma_1^6 \star \sigma_{1^2}, A^0(X)}(x)$$

Proof. For $n = 7$, $\mathcal{A}^0(X) = \mathbb{C}\{1, \sigma_{(4,3,0)}, \sigma_{(4,2,1)}, \sigma_{(3,3,1)}, \sigma_{(3,2,2)}\}$ and $\mathcal{A}^0(Y) = \mathbb{C}\{1, j^*\sigma_{(4,1,1)}, j^*\sigma_{(3,3,0)}, j^*\sigma_{(3,2,1)}, j^*\sigma_{(2,2,2)}\}$ with $j^*\sigma_{(4,2,0)} = 2j^*\sigma_{(4,1,1)} + j^*\sigma_{(3,3,0)} - j^*\sigma_{(3,2,1)} + j^*\sigma_{(2,2,2)}$.

For $n = 8$, $\mathcal{A}^0(X) = \mathbb{C}\{1, \sigma_{(5,3,0)}, \sigma_{(5,2,1)}, \sigma_{(4,4,0)}, \sigma_{(4,3,1)}, \sigma_{(4,2,2)}, \sigma_{(3,3,2)}\}$ and $\mathcal{A}^0(Y) = \mathbb{C}\{1, j^*\sigma_{(5,2,0)}, j^*\sigma_{(5,1,1)}, j^*\sigma_{(4,2,1)}, j^*\sigma_{(4,3,0)}, j^*\sigma_{(3,2,2)}, j^*\sigma_{(3,3,1)}, j^*\sigma_{(5,5,4)}, \beta\}$, where β is a primitive class and we have $j^*\sigma_1 \star \beta = 0$.

Then the statements follow from direct calculations by using Proposition 3.16 $\square$

Proposition 3.18. *Let Y be a smooth hyperplane section of $X = Gr(3, n)$ with $n \in \{7, 8\}$. There is an isomorphism of algebras*

$$\mathcal{A}^0(X) \xrightarrow{\cong} \mathcal{A}_\perp^0(Y)$$

with $\sigma_1^{r_X} \mapsto ((j^*\sigma_1)^{r_Y})_\perp$.

Proof. Denote $\alpha_X := \sigma_1^{r_X}$ and $\alpha_Y := (j^*\sigma_1)^{r_Y}$. For $n = 7$, by direct calculations, $\{1, \alpha_X, \dots, \alpha_X^4\}$ (resp. $\{1, \alpha_Y, \dots, \alpha_Y^4\}$) form an additive basis of $\mathcal{A}^0(X)$ (resp. $\mathcal{A}^0(Y)$). Therefore the minimal polynomial of α_X (resp. α_Y) is given by the characteristic polynomial of it. Therefore, by Lemma 3.17, $\mathcal{A}^0(X)$ (resp. $\mathcal{A}^0(Y)$) is isomorphic to $\mathbb{C}[x]/(p_{\alpha_X, \mathcal{A}^0(X)}(x))$ by sending α_X (resp. α_Y) to x .

Now we consider $n = 8$ and denote $\tilde{\alpha}_X := \sigma_1^{r_X-2} \star \sigma_{1^2}$, $\tilde{\alpha}_Y := (j^*\sigma_1)^{r_Y-2} \star j^*\sigma_{1^2}$. By exactly the same argument for $n = 7$, we conclude that $\mathcal{A}^0(X)$ (resp. $\mathcal{A}_\perp^0(Y)$) is isomorphic to $\mathbb{C}[x]/(p_{\tilde{\alpha}_X, \mathcal{A}^0(X)}(x))$ by sending $\tilde{\alpha}_X$ (resp. $(\tilde{\alpha}_Y)_\perp$) to x . By directly calculations, we see that α_X and $(\alpha_Y)_\perp$ have exactly the same linear expansion in terms of $\{\tilde{\alpha}_X^i\}_i$ and $\{(\tilde{\alpha}_Y)_\perp^i\}_i$ respectively. Hence α_X is sent to $(\alpha_Y)_\perp$ under the isomorphism. $\square$

Theorem 3.19. *Let Y be a smooth hyperplane section of $X = Gr(3, n)$ with $n \in \{7, 8\}$. Then both Proposition 1.10 and Conjecture 1.13 hold for Y .*

Proof. Conjecture 1.13 and Part (2) of Proposition 1.10 follow directly from Lemma 3.17 and Proposition 3.18 respectively.

Recall $QH^*(Gr(3, n)) \cong \mathbb{C}[\sigma_1, \sigma_{1^2}, \sigma_{1^3}, q_X]/(\sigma_{n-2}, \sigma_{n-1}, \sigma_n - q_X)$, where σ_j can be obtained by the relation (3). Denote by $e_1 = j^*\sigma_1, e_2 = j^*\sigma_{1^2}$ and $e_3 = j^*\sigma_{1^3}$. Note $j_q^*(\sigma_{n-2}) = 0$, as $n - 2 < r_Y$. To verify part (1) of Proposition 1.10, it suffices to show that

$$j_q^*(\sigma_{n-1}) = 0, \quad j_q^*(\sigma_n) = j_q^*(q_X) = q_Y e_1,$$

where $j_q^*(\sigma_{n-1}) = h_6, j_q^*(\sigma_n) = h_7$ for $n = 7$ (resp. $j_q^*(\sigma_{n-1}) = h_7, j_q^*(\sigma_n) = h_8$ for $n = 8$) with

$$h_6 = e_1^6 - 5e_1^4 e_2 + 6e_1^2 e_2^2 + 4e_1^3 e_3 - e_2^3 - 6e_1 e_2 e_3 + e_3^2,$$

$$h_7 = e_1^7 - 6e_1^5 e_2 + 10e_1^3 e_2^2 + 5e_1^4 e_3 - 4e_1 e_2^3 - 12e_1^2 e_2 e_3 + 3e_2^2 e_3 + 3e_1 e_3^2,$$

$$h_8 = e_1^8 - 7e_1^6 e_2 + 15e_1^4 e_2^2 + 6e_1^5 e_3 - 10e_1^2 e_2^3 - 20e_1^3 e_2 e_3 + e_2^4 + 12e_1 e_2^2 e_3 + 6e_1^2 e_3^2 - 3e_2 e_3^2.$$

Then we are done by direct calculations using Proposition 3.16. $\square$

3.3. Semisimplicity of $QH^*(Y)$. Here we investigate the semisimplicity of $QH^*(Y)$ for a smooth hyperplane section Y of $X = Gr(k, n)$. We will need to apply the following proposition in the case $(k, n) = (3, 8)$.

Let $\pi : \mathcal{Y} \rightarrow U$ be a projective smooth morphism. Let $p \in U$ be any point and denote $\mathcal{Y}_p := \pi^{-1}(p)$ the fiber of π at p . There is a *monodromy action* (see e.g. [Voi, Chapter 3]) of the fundamental group $\pi_1(U, p)$ on the classical cohomology $H^*(\mathcal{Y}_p)$.

Proposition 3.20 ([LT98, Theorem 4.3], see also [Hu15, Corollary 3.2]). *The monodromy action of $\pi_1(U, p)$ on $H^*(\mathcal{Y}_p)$ naturally extends to an action on $QH^*(\mathcal{Y}_p)$ and preserves the quantum product.*

For Y in $Gr(3, 8)$, we consider the family $\pi|_U : \mathcal{Y}^\circ \rightarrow U$ obtained by restricting the natural projection

$$\pi : \mathcal{Y} = \{(V, \tau) \in Gr(3, 8) \times \Lambda^3(\mathbb{C}^8)^* \mid V|_\tau = 0\} \longrightarrow \Lambda^3(\mathbb{C}^8)^*.$$

to the unique Zariski open GL_8 -orbit U of $\Lambda^3(\mathbb{C}^8)^*$. More precisely, the above projection is a vector bundle over $Gr(3, 8)$ whose fiber at V is the space $\ker[\Lambda^3(\mathbb{C}^8)^* \rightarrow \Lambda^3 V^*]$. Therefore, the smooth hyperplane section Y of $Gr(3, 8)$ can be realized as a fiber over any $p \in U$. Notice $H_{\text{prim}}(Y) = \mathbb{C}\{\beta\}$.

Lemma 3.21. *For Y in $Gr(3, 8)$, there exists $\sigma \in \pi_1(U)$ such that $\sigma\beta = -\beta$ under monodromy action.*

Proof. By Deligne invariant cycle theorem [Voi, Theorem 4.24], the image of the restriction

$$H^*(X, \mathbb{Q}) \cong H^*(\mathcal{Y}, \mathbb{Q}) \rightarrow H^*(Y, \mathbb{Q})$$

is the monodromy invariant subalgebra. In particular, the restriction of $\sigma_1 \in H^2(X, \mathbb{Q})$ is invariant, so the primitive space $\mathbb{Q}\beta$ is preserved. Since β is not in the image, there exists an $\sigma \in \pi_1(U)$ such that $\beta \neq \sigma\beta \in \mathbb{Q}\beta$. Note that $\int_Y \beta^2 = \int_Y \sigma\beta^2 \neq 0$, the only possibility is $\sigma\beta = -\beta$. $\square$

Theorem 3.22. *Let Y be a smooth hyperplane section of $Gr(k, n)$ where $n \geq 2k$. Then $QH^*(Y)$ is generically semisimple if and only if one of the following holds:*

- (a) $k = 1$; (b) $k = 2$ and either $n = 4$ or n is odd; (c) $k = 3$ and $n \in \{7, 8\}$.

Proof. Let $X = Gr(k, n)$ with $n \geq 2k$. For $k = 1$, Y is a projective space and hence $QH^*(Y)$ is generically semisimple. For $X = Gr(2, 4)$, X is a quadric in $\mathbb{P}^5$, and hence Y is a quadric in $\mathbb{P}^3$ with generically semisimple quantum cohomology. For $k = 2$ and $n = 2m$ with $m > 1$, $Y \cong SG(2, 2m)$ and $QH^*(Y)$ is known to be non-semisimple [CP11]. For $k = 2$ and $n = 2m + 1$ with $m > 1$, $Y \cong SG(2, 2m + 1)$ is a quasi-homogeneous variety with generically semisimple quantum cohomology [Pec13, Per14]. For $k > 3$ or ($k = 3$ and $n > 8$), $QH^*(Y)$ is not semisimple by Theorem 3.8 and Proposition 1.1.

Now we assume $k = 3$. It remains to investigate the cases $n \in \{6, 7, 8\}$. For $n = 6$, we notice that $\dim \mathcal{A}^1 = b_2(Y) + b_{12}(Y) = 1 + 2 \neq 4 = b_8(Y) = \dim \mathcal{A}^{-1}$. Hence, $QH^*(Y)$ is not semisimple in this case.

Now we consider $n \in \{7, 8\}$. Notice $\mathcal{A}_\perp^0(Y) = \mathcal{A}^0(Y)$ and $Rad(Y) = 0$ if $n = 7$. It is well known that $QH^*(X)$ is semisimple, so does the subalgebra $\mathcal{A}^0(X)$. By Proposition 3.18, $\mathcal{A}_\perp^0(Y) \cong \mathcal{A}^0(X)$ is semisimple. By Lemma 3.17, the restriction of $(j^*\sigma_1)^{r_Y}$ to the complement of $Rad(Y)$ in $QH^*(Y)$ is invertible, so is $j^*\sigma_1$. Hence, $\mathcal{A}_\perp^0(Y)[j^*\sigma_1] \cong \mathcal{A}_\perp^0(Y)[y]/(y^{r_Y} - (j^*\sigma_1)^{r_Y})$ is semisimple by Lemma 2.2, and it is of dimension 49. Now we have $QH^*(Y) = Rad(Y) \oplus \mathcal{A}_\perp^0(Y)[j^*\sigma_1]$, where $Rad(Y) = \mathbb{C}\{\beta, \gamma\}$ with

$$\gamma := 2\sigma_{(5,5,4)} - \sigma_{(3,2,2)} + \sigma_{(3,3,1)} + \sigma_{(4,2,1)} - 2\sigma_{(4,3,0)} - 3\sigma_{(5,1,1)} + 2\sigma_{(5,2,0)} \in H^*(Gr(3, 8)).$$

Then we are done by showing that $Rad(Y)$ is semisimple. Indeed, we notice that $j^*\gamma$ is not nilpotent, as $\int_Y j^*\gamma = 6 \neq 0$. Since $Rad(Y)$ is two-dimensional, the nilpotent elements of $Rad(Y)$ form a subspace of dimension at most 1 and any nilpotent element has nilpotent index no more than 2. If $a\beta + bj^*\gamma$ is nilpotent, then so is $-a\beta + bj^*\gamma$ by applying the monodromy action by Proposition 3.20 and Lemma 3.21. This forces $b = 0$. Then it follows from $\int_Y \beta^2 \neq 0$ that $a = 0$. $\square$

3.4. Proof of Lemma 3.3. We discuss all the possible $n - i$.

Case I: $n - i \in \{1, 2\}$. If $n - i = 1$, only the empty partition is a 1-core partition, but our assumption implies $k(n - k) \geq k^2 > k \geq i$. Thus it is impossible.

If $n - i = 2$, then the 2-core partition must be a staircase. The maximal 2-core partition in $\mathcal{P}_{k,n}$ is $(k, \dots, 2, 1)$ has $\frac{k(k+1)}{2}$ cells. By the assumption, we have

$$\frac{k(k+1)}{2} \geq k(n - k) - i.$$

Hence, $\frac{k(k+1)}{2} > k(n - k) - n$, implying $\frac{k(k+1)}{2} + k^2 > (k - 1)n > 2(k - 1)k$ and consequently $k < 5$. Combining the above inequality with $3 \leq k \leq \frac{n}{2}$, we obtain $(k, n, i) \in \{(3, 6, 4), (4, 8, 6)\}$, see the first two diagrams in Example 3.4.

To proceed in the remaining cases, we use the bijection $\mathcal{P}_{k,n} \rightarrow \binom{[n]}{k}$ defined by $\lambda \mapsto A = \{a_1, \dots, a_k\}$ with $a_i = \lambda_i + k - i + 1$ for all i . Under this identification, we have

- (i) $|\lambda| \geq k(n - k) - i \iff \sum_{i=1}^k a_i \geq n + (n - 1) + \dots + (n - k + 1) - i$;
- (ii) λ is $(n - i)$ -core $\iff$ if $a \in A$ and $a - (n - i) \geq 1$ then $a - (n - i) \in A$.

Here the equivalence in (ii) follows from the fact that (mult-)set of hook lengths of λ is given by $\bigcup_{i=1}^k (\{1, \dots, a_i\} \setminus \bigcup_{j>i} \{a_i - a_j\})$, see [St99, proof of Lemma 7.21.1].

Case II: $n - i \geq 3$ and $i < n - i$. Let us consider

$$B = \{a \in A \mid n - i < a \leq n\}.$$

Since $n - i < n < 2(n - i)$, we have the disjoint union

$$A = B \cup B' \cup C$$

where $B' = \{b - (n - i) \mid b \in B\}$ and $C = A \setminus (B \cup B')$. Assume $|B| = b$. Note that

$$k = |B| + |B'| + |C| \geq 2b, \quad k - b = |B'| + |C| \leq n - i$$

and

$$\#(A \cap \{1, \dots, i\}) \geq b, \quad \#(A \cap \{1, \dots, n - i\}) \geq k - b.$$

That is,

$$\begin{aligned} a_b &< \dots < a_2 < a_1 \leq n \\ a_{k-b} &< \dots < a_{b+2} < a_{b+1} \leq n - i \\ a_k &< \dots < a_{k-b} < a_{k-b+1} \leq i. \end{aligned}$$

This implies

$$\begin{aligned} a_1 + \dots + a_k &\leq n + (n - 1) + \dots + (n - b + 1) \\ &\quad + (n - i) + (n - i - 1) + \dots + (n - i + 1 - k + 2b) \\ &\quad + i + (i - 1) + \dots + (i + 1 - b) \\ &= n + (n - 1) + \dots + (n - k + 1) - (k - 2b)(i - b) - b(n - k + b - i). \end{aligned}$$

By the assumption (i), we have

$$0 \geq (k - 2b)(i - b) + b(n - k + b - i) - i, \tag{5}$$

implying $b > 0$ (otherwise we would have $0 \geq ki - i > 0$). If $n - k + b - i \leq 1$, then we have $k > k - b \geq n - i - 1 > \frac{n}{2} - 1$, contradicting to $k \leq \frac{n}{2}$. Thus we can assume $n - k + b - i \geq 2$.

If $k - 2b > 0$, then (5) $\geq (i - b) + 2b - i = b > 0$, a contradiction. It follows that $k = 2b$. Now (5) $= b(n - b) - (b + 1)i$. Since $b = \frac{k}{2} \leq \frac{n}{4}$, $b(n - b) \geq \frac{3n^2}{16}$. Since $i < \frac{n}{2}$, $(b + 1)i < (\frac{n}{4} + 1)\frac{n}{2}$. So

$$(5) > \frac{3n^2}{16} - \left(\frac{n}{4} + 1\right)\frac{n}{2} = \frac{n(n - 8)}{16}.$$

This implies $n < 8$. Hence $k \leq n/2 < 4$, so $k = 3$ by our assumption. This contradicts $k = 2b$.

Case III: $n - i \geq 3$ and $i \geq n - i$. Let us consider

$$M = \{a \in A \mid i < a \leq n\}, \quad m = |M|.$$

By (ii), we have an injective map $\phi : M \rightarrow \{1, \dots, n - i\} \cap A$ such that $\phi(a) = a - \ell(n - i)$ for some $\ell \geq 0$. Moreover since $i \geq n - i$, we have $a - (n - i) \geq a - i > 0$, hence $\ell \geq 1$ and in particular $a \neq \phi(a)$. Now $M \cap \phi(M) = \emptyset$, we can conclude that

$$k \geq 2m, \quad \#(A \cap \{1, \dots, n - i\}) \geq m.$$

Subcase III.1. When $|M| = 1$, i.e. $M = \{a_1\}$, we have

$$a_k \leq n - i, \quad a_{k-1} < \dots < a_3 < a_2 \leq i, \quad a_1 \leq n.$$

Then

$$\begin{aligned} a_1 + \dots + a_k &\leq n + i + (i - 1) + \dots + (i - k + 3) + (n - i) \\ &= n + (n - 1) + (n - 2) + \dots + (n - k + 1) - (k - 2)(n - i - 1) - (i - k + 1). \end{aligned}$$

By assumption (i) and noting $n - i \geq 3$, we have

$$0 \geq (k - 2)(n - i - 1) - k + 1 \geq 2(k - 2) - k + 1 = k - 3 \geq 0.$$

So $k = 3$, and all equalities should be achieved at each step, i.e.

$$n - i = 3, \quad a_3 = n - i, \quad a_2 = i, \quad a_1 = n.$$

Then $a_3 = 3$. Since $a_2 > a_3 = n - i$, by (ii), this forces $a_2 - (n - i) = a_3$, i.e. $i = 6$ and $n = 9$. So the only case is $(k, n, i) = (3, 9, 6)$, see the third diagram in Example 3.4.

Subcase III.2. When $|M| = m \geq 2$. We have

$$a_k < a_{k-1} < \dots < a_{k-m+1} \leq n - i \quad \text{and} \quad a_{k-m} < \dots < a_{m+2} < a_{m+1} \leq i.$$

Then

$$\begin{aligned} a_1 + \dots + a_k &\leq n + (n - 1) + \dots + (n - m + 1) \\ &\quad + i + (i - 1) + \dots + (i - (k - 2m) + 1) \\ &\quad + (n - i) + (n - i - 1) + \dots + (n - i - m + 1) \\ &= n + (n - 1) + \dots + (n - k + 1) - (k - 2m)(n - m - i) - m(i + m - k). \end{aligned}$$

By (i), we have

$$0 \geq (k - 2m)(n - m - i) + m(i + m - k) - i \tag{6}$$

If $n - m - i = 0$, then

$$(6) = m(n - k) - i \geq 2(n - k) - i \geq n - i > 0,$$

a contradiction. So $n - m - i > 0$, and

$$(6) \geq (k - 2m) + 2(i + m - k) - i = i - k \geq n/2 - n/2 \geq 0.$$

This shows $i = k$ and all equalities should be achieved at each step, i.e. Then $i = k = n/2$, $m = 2$ (since $i + m - k = m > 0$) and $k - 2m = 0$ (since if $n - m - i = i - 2 = 1$, we can solve $(k, n, i) = (3, 6, 3)$ contradicting to $k \geq 2m$). So the only possibility is $(k, n, i) = (4, 8, 4)$ with $a_1 = n = 8$, $a_2 = a_1 - 1 = 7$, $a_3 = n - i = 4$ and $a_4 = a_3 - 1 = 3$, see the last diagram in Example 3.4. Now the proof of Lemma 3.3 is completed.

4. SMOOTH HYPERPLANE SECTIONS OF (CO)ADJOINT GRASSMANNIANS

In this section, we consider a (co)adjoint Grassmannian $X = G/P_k$, as shown in Table 1. There is an embedding $X \hookrightarrow \mathbb{P}(V_{\omega_k})$, where ω_k is the k -th fundamental weight and V_{ω_k} denotes the corresponding fundamental representation of G . Let Y be a smooth hyperplane section of X , given by the intersection of a general hyperplane in $\mathbb{P}(V_{\omega_k})$ with X .

4.1. Hyperplane sections of C_n/P_2 . The coadjoint Grassmannian G/P_2 of type C_n is the symplectic Grassmannian $SG(2, 2n) = \{V \leq \mathbb{C}^{2n} \mid \dim V = 2, \omega(V, V) = 0\}$, where ω is a symplectic form on $\mathbb{C}^{2n}$ and $n \geq 3$. It can be realized as a smooth hyperplane section of $Gr(2, 2n)$. In particular, $r_X = 2n - 1$ and $\dim X = 4n - 5$. The following theorem verifies [BP22, Conjecture 1.10 (2)] for type C , and provides a new proof of the non-semisimplicity of $QH^*(SG(2, 2n))$ as well.

Theorem 4.1. *Let $n \geq 3$. For a smooth hyperplane section Y of $X = SG(2, 2n)$, both $QH^*(X)$ and $QH^*(Y)$ are non-semisimple.*

Proof. Note that the odd Betti numbers $b_{2j+1}(X) = \dim H^{2j+1}(X)$ all vanish. Since the two-step flag variety $SFl(1, 2; 2n)$ of type C is a $\mathbb{P}^1$ -bundle over $SG(2, 2n)$, and also a $\mathbb{P}^{2n-3}$ -bundle over $\mathbb{P}^{2n-1}$, we have the Poincaré polynomial (with respect to complex degree)

$$\begin{aligned} p_X(t) &= \frac{1-t^{2n}}{1-t} \frac{1-t^{2n-2}}{1-t} \Big/ (1+t) \\ &= \frac{(1-t^{2n})(1-t^{2n-2})}{(1-t)(1-t^2)} = (1+t+\dots+t^{2n-1})(1+t^2+\dots+t^{2n-4}). \end{aligned}$$

Thus for $0 \leq i \leq 2n-3$, we have $b_{2i}(X) = \begin{cases} \frac{i+2}{2}, & \text{if } i \text{ is even,} \\ \frac{i+1}{2}, & \text{if } i \text{ is odd.} \end{cases}$

Since $r_X = 2n-1$ and $\dim X = 4n-5$, we have $\dim \mathcal{A}^{-\bar{2}}(X) = b_{2(2n-3)}(X) = \frac{2n-3+1}{2} = n-1$ and $\dim \mathcal{A}^{\bar{2}}(X) = b_{2,2}(X) + b_{2(2n+1)}(X) = b_{2,2}(X) + b_{2(2n-6)}(X) = \frac{2+2}{2} + \frac{2n-6+2}{2} = n \neq \dim \mathcal{A}^{-\bar{2}}$. Hence, $QH^*(X)$ is not semisimple by Theorem 1.2 (2).

Note $r_Y = 2n-2$ and $\dim Y = 4n-6$. By [BP22, Theorem 1.2 and Lemma 3.1], Y is invariant under the natural action of a maximal torus of G , and its torus-fixed loci coincide with the torus-fixed loci of X . Therefore X and Y have the same Euler number. It follows that $b_{2(2n-3)}(Y) = b_{2(2n-3)}(X) + b_{2(2n-2)}(X) = 2n-2$ and $b_{2i}(Y) = b_{2i}(X)$ for $0 \leq i \leq 2n-4$. Thus $\dim \mathcal{A}^1(Y) = b_2(Y) + b_{2(2n-1)}(Y) = b_2(Y) + b_{2(2n-5)}(Y) = 1 + \frac{2n-5+1}{2} = n-1 \neq b_{2(2n-3)}(Y) = \dim \mathcal{A}^{-\bar{1}}$. Hence, $QH^*(Y)$ is not semisimple by Theorem 1.2 (2). $\square$

Remark 4.2. The same argument also works for Y in coadjoint Grassmannians in other non-simply-laced cases, i.e. of type B_n/P_1 , F_4/P_4 , or G_2/P_1 .

4.2. Hyperplane sections of D_n/P_2 . For $n \geq 4$, the (co)adjoint variety D_n/P_2 is the Grassmannian of isotropic planes in $\mathbb{C}^{2n}$ with respect to a non-degenerate symmetric bilinear form. The argument for type C case does not work for type D case. Here we use the monodromy technique, which works not only in type D case.

Let G/P denote a coadjoint Grassmannian of type D or E . Consider the natural projection

$$\pi : \mathcal{Y} := \{(gP, v) \in G/P \times \mathfrak{g} \mid v \in gL\} \longrightarrow \mathfrak{g}$$

where $L \subset \mathfrak{g}$ is the span of root vectors except the lowest root space. Denote by κ the Killing form of $\mathfrak{g}$. Then the fiber $\mathcal{Y}_x$ at $x \in \mathfrak{g}$ can be identified with the hyperplane section defined by the linear equation $\kappa(x, -)$ over $\mathfrak{g}$ under the Plücker embedding $G/P \hookrightarrow \mathbb{P}(\mathfrak{g})$. Moreover, for any x in the set $U = \mathfrak{g}^{rs}$ of regular semisimple elements of $\mathfrak{g}$, the fiber $\mathcal{Y}_x$ is smooth, being a

smooth hyperplane section Y of X . It is well-known that the fundamental group of $\mathfrak{g}^{rs}$ is the braid group of the Weyl group W of G .

Lemma 4.3. *The monodromy action factors through the Weyl group W , and as a W -representation,*

$$H^*(Y) = H^*(Y)_{\text{inv}} \oplus H^*(Y)_{\text{std}}$$

with $H^*(Y)_{\text{std}} \cong \chi_{\text{std}}$ the standard (i.e. reflection) representation of W .

Proof. We will use Springer correspondence in terms of perverse sheaves and Fourier transformation (see e.g. [Gin98]). Let $\mathbb{C}_Z$ be the constant sheaf of a variety Z . Denote by $IC(Z, \chi)$ the intersection complex of the local system χ over Z and we omit χ if $\chi = \mathbb{C}_Z$. By the Decomposition Theorem [BBD], we have

$$\pi_* \mathbb{C}_Y = \bigoplus_{\chi} IC(\mathfrak{g}^{rs}, \chi) \otimes V_{\chi} \oplus (\text{perverse sheaves with smaller support}) \quad (7)$$

where the sum goes over irreducible representations of $\pi_1(\mathfrak{g}^{rs})$, and V_{χ} is the multiplicity space of χ in $H^*(Y)$. Let us consider the analogy ρ of Springer resolutions

$$\rho : \mathcal{Y}^{\perp} := \{(gP, v) \in G/P \times \mathfrak{g} \mid v \in \text{span}(ge_{\theta})\} \longrightarrow \mathfrak{g}$$

where θ is the highest root and e_{θ} is the corresponding root vector. The image of ρ contains only two nilpotent orbits, the zero orbit $\mathbb{O}_0 = \{0\}$ and the minimal G -orbit $\mathbb{O}_{\min} = Ge_{\theta}$. Since ρ is one-to-one over $\mathbb{O}_{\min}$, by the Decomposition Theorem [BBD] again, we have

$$\rho_* \mathbb{C}_{\mathcal{Y}^{\perp}} = IC(\mathbb{O}_{\min}) \oplus IC(\mathbb{O}_0) \otimes V$$

for some coefficients space V . Let us denote by $\mathcal{F}$ the Fourier transformation between $\mathfrak{g}$ and $\mathfrak{g}^* \simeq \mathfrak{g}$, where the isomorphism is given by the Killing form. Since $\mathfrak{g}$ is simply-laced, by the Springer correspondence, we have

$$\mathcal{F}(IC(\mathbb{O}_0)) = IC(\mathfrak{g}^{rs}), \quad \mathcal{F}(IC(\mathbb{O}_{\min})) = IC(\mathfrak{g}^{rs}, \chi_{\text{std}}),$$

see for example [Ju08]. Since Fourier transformation is functorial, similar as [Gin98, Claim 8.4], we have $\mathcal{F}(\rho_* \mathbb{C}_{\mathcal{Y}^{\perp}}) = \pi_* \mathbb{C}_Y$. This implies

$$\pi_* \mathbb{C}_Y = IC(\mathfrak{g}^{rs}, \chi_{\text{std}}) \oplus IC(\mathfrak{g}^{rs}) \otimes V.$$

Comparing with (7), the multiplicity space $V_{\chi} = 0$ unless $\chi = \chi_{\text{std}}$ or χ_{inv} and $\dim V_{\chi} = 1$ for $\chi = \chi_{\text{std}}$. Hence, the decomposition in the statement follows. $\square$

Lemma 4.4. *For any finite irreducible Coxeter group W not of type A , we have*

$$\dim(\text{Sym}^2 \chi_{\text{std}})^W = 1, \quad \dim(\text{Sym}^3 \chi_{\text{std}})^W = 0.$$

Proof. By a theorem of Chevalley, for example [Hum, Section 3.5], we have an algebra isomorphism to the polynomial ring $\bigoplus_{d \geq 0} (\text{Sym}^d \chi_{\text{std}})^W \cong \mathbb{Q}[p_1, p_2, \dots, p_n]$, with p_i homogeneous of degree d_i . The numbers $d_1 \leq d_2 \leq \dots \leq d_n$ are classified, and the statement follows from the fact $d_1 = 2$ and $d_2 \geq 4$ except type A , see [Hum, Section 3.7]. $\square$

Theorem 4.5. *For a smooth hyperplane section Y of a coadjoint Grassmannian X of type D , the quantum cohomology $QH^*(Y)$ is non-semisimple.*

Proof. By Proposition 3.20, the symmetric cubic form

$$(\gamma_1 \star_{q_Y=1} \gamma_2, \gamma_3)_Y \in \mathbb{C}, \quad \gamma_1, \gamma_2, \gamma_3 \in H^{\bullet}(Y)$$

is W -invariant. By Lemma 4.4, when $\gamma_1, \gamma_2, \gamma_3 \in H^*(Y)_{\text{std}}$, we have $(\gamma_1 \star_{q_Y=1} \gamma_2, \gamma_3)_Y = 0$. Since the Poincaré pairing is non-degenerate and W -invariant, so is its restriction to $H^*(Y)_{\text{std}}$ by Lemma 4.4. So we have $\gamma_1 \star_{q_Y=1} \gamma_2 \in H^*(Y)_{\text{inv}}$.

If $\gamma_1, \gamma_2 \in H^*(Y)_{\text{inv}}$ and $\gamma_3 \in H^*(Y)_{\text{std}}$, then we have $(\gamma_1 \star_{q_Y=1} \gamma_3, \gamma_2)_Y = (\gamma_1 \star_{q_Y=1} \gamma_2, \gamma_3)_Y = 0$. Again, since the Poincaré pairing is non-degenerate and W -invariant, so is its restriction to $H^*(Y)_{\text{inv}}$. So we have $\gamma_1 \star_{q_Y=1} \gamma_3 \in H^*(Y)_{\text{std}}$.

Notice that $\dim Y = 2r_Y$, and r_Y is even. Moreover,

$$H^{2i}(Y) = H^{2i}(Y)_{\text{inv}} \text{ unless } i = r_Y.$$

By the discussion above, we can construct a subalgebra $\mathcal{B}$ of $\mathcal{A}$ with a $\mathbb{Z}_2$ -grading by

$$\mathcal{B}^{\bar{0}} = H^0(Y) \oplus H^{2r_Y}(Y)_{\text{inv}} \oplus H^{4r_Y}(Y),$$

$$\mathcal{B}^{\bar{1}} = H^{r_Y}(Y) \oplus H^{3r_Y}(Y) \oplus H^{2r_Y}(Y)_{\text{std}}.$$

Since

$$\dim H^{2r_Y}(Y)_{\text{std}} = \dim H^{2r_Y}(Y)_{\text{inv}} = \text{rank } G,$$

$$\dim H^0(Y) = \dim H^{4r_Y}(Y) = 1,$$

$$\dim H^{r_Y}(Y) = \dim H^{3r_Y}(Y) > 1,$$

we have $\dim \mathcal{B}^{\bar{0}} < \dim \mathcal{B}^{\bar{1}}$. Hence, $\mathcal{B}$ contains a nonzero nilpotent element by Lemma 2.1. $\square$

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